CHALMERS-METCALF OPERATOR AND UNIQUENESS OF MINIMAL PROJECTIONS

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Abstract. The aim of this paper is to investigate Chalmers-Metcalf operator (introduced in [CM2]) and examine its connection to the uniqueness of minimal projections. In section 1 we show that Chalmers-Metcalf operator can be inscribed in the general approximation scheme. We treat this operator as a functional given by Theorem 1.3 and establish the correspondence in Theorems 1.9 and 1.10 (we actually reprove Theorem 0.2). This gives us a possibility to obtain series of interesting properties of this operator (Theorems 1.14, 1.15 and 1.17) which will be used in sections 2 and 3. The main theorem of this paper is Theorem 2.2. It is further development of the ideas contained in [SK3] and it relates uniqueness of minimal projections to the invertibility of Chalmers-Metcalf operator. It is worth mentioning that to a given minimal projection (even unique) we may find many different Chalmers-Metcalf operators, some of them invertible some not - see Theorem 2.5. Additionally we provide examples of application Theorem 2.2 - see Theorems 2.4, 2.5, 2.7, 2.8, 2.9, 2.10. The main application is in section 3 - we have proved that minimal projections onto symmetric subspaces in smooth Banach spaces are unique (Theorem 3.2). This leads in Section 4 to the solution of the problem of uniqueness of classical Rademacher projections in $L_p[0,1]$ for $1 < p < \infty$.

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0. Introduction

A projection is taken to mean any bounded linear operator P which carries a Banach space X onto its linear subspace V in such a way that it acts as identity on V. The set of all projections from X onto V we denote by $\mathcal{P}(X, V)$.

Not every subspace of a given Banach space is the range of bounded projection, for example there is no continuous projection from $\mathcal{B}[0, 1]$ onto $\mathcal{C}[0, 1]$. Subspaces which are the range of bounded projections are called complemented subspaces and are crucial in the study of Banach spaces.

Projections plays an important role in numerical analysis, the error of approximation of element x by Px ,i.e., the quantity $\|x - Px\|$, can be estimated by means of the elementary inequality

$$\|x - Px\| \leq \|Id - P\| \cdot dist(x, V) \leq (1 + \|P\|) \cdot dist(x, V).$$

Here $dist(x, V)$ denotes the infimum of $\|x - v\|$ as v ranges over the subspace V. This signifies the consideration of a related problem - making $\|P\|$ small. We are therefore led to make the following definition. A projection $P_0 \in \mathcal{P}(X, V)$ is called **minimal** if

$$\|P_0\| = \lambda(V, X) = \inf\{\|P\| : P \in \mathcal{P}(X, V)\}.$$

The constant $\lambda(V, X)$ is called the **relative projection constant**. If P minimalizes $\|Id - P\|$ then it is called co-minimal.

There is also a significant connection between projections and the Functional Analysis. We can consider a minimal projection as the extension of the Hahn-Banach theorem, since using a particular minimal projection we can linearly extend any functional $v^* \in V^*$ to X^* (setting $x^* = v^* \circ P$), or equivalently we can speak of a linear extension of the operator $Id_V : V \to V$ to the operator $P : X \to V$. And the uniqueness of a minimal projection results in the uniqueness of this extension.

Observe that any projection with norm one is automatically minimal, though in general, a given subspace will not be the range of a projection of norm 1. First problem is to find out whether considered subspace is complemented, but even if a considered subspace is complemented there could be no minimal projection (as infimum does not have to be attained), for example see [BC]. In many cases the existence of a minimal projection is known a priori (see [IS]), for example this is the case when the subspace is finite-dimensional or finite-codimensional. Even in such cases minimal projections will be difficult to find. As a dramatic evidence of the difficult nature of such problems, one may cite the fact that the minimal projections of $\mathcal{C}([0, 1])$ onto the subspace of polynomials of degree > 5 remain unknown. There are rather few situations in which minimal projections are known explicitly or are characterized by some interesting property. Still rarer is the situation in which the minimal projection is know to be unique.

Minimal projections and its properties have been studied by many authors. See, for example, [CM1, CM2, CF, CMO, CP, CS, HV, KTJ, LA1, LA2, LE1, LE2, LE3, LE4, LI, OL, RU].

One of the main tools to study minimal projections is so-called Chalmers-Metcalf operator. We can define it as follows.

Below we assume that X is a normed space and V is its finite-dimensional subspace.

Definition 0.1. A pair $(x, y) \in S(X^{\star\star}) \times S(X^{\star})$ will be called an **extremal pair** for $P \in \mathcal{P}(X, V)$ iff $y(P^{\star\star}x) = \|P\|$, where $P^{\star\star} : X^{\star\star} \to V$ is the second adjoint extension of P to $X^{\star\star}$ ($S(X)$ denotes here a unit sphere of X). Let $\mathcal{E}(P)$ be the set of all extremal pairs for P.

To each $(x, y) \in \mathcal{E}(P)$ we associate the rank-one operator $y \otimes x$ from X to $X^{\star\star}$ given by $(y \otimes x)(z) = y(z) \cdot x$ for $z \in X$.

Theorem 0.2 (Chalmers-Metcalf) [CM2, Theorem 1]**.** *A projection $P \in \mathcal{P}(X, V)$ has a minimal norm if and only if the closed convex hull of $\{y\otimes x\}_{(x,y)\in\mathcal{E}(P)}$ contains an operator E_P for which V is an invariant subspace.*

The operator E_P is called Chalmers-Metcalf operator and is given by the formula

$$E_P = \int\limits_{\mathcal{E}(P)} y \otimes x \, d\mu(x, y) : X \to X^{\star\star},$$

where μ is a probabilistic Borel measure on $\mathcal{E}(P)$.

Chalmers-Metcalf theorem have many applications especially in case $X = L_1$, see ,e.g., [CL1, CL2, CL3, CM1, CM2, FMW, LE2, LE3, LE4, SK3, SK4].

The aim of this paper is to relate some properties of this operator to the uniqueness of minimal projections. We will study several examples of minimal projections and as a result we will prove uniqueness of Rademacher projections in L_p.

1. Chalmers-Metcalf operator as a consequence of best approximation scheme

Using Choquet theorem we will connect Chalmers-Metcalf operator to classical theorems of best approximation. We will state some known and basic theorems without proof. Here and in the following paragraphs we will assume that V, a subspace onto we are projecting space X, is finite-dimensional.

Theorem 1.1. *A projection $P_0 \in \mathcal{P}(X, V)$ is minimal if and only if 0 is a best approximation to P_0 in $\mathcal{L}_V(X, V)$ (i.e., $0 \in P_{\mathcal{L}_V(X,V)}(P_0)$).*

Now, from the general theorem

Theorem 1.2. *Let Y be a subspace of a Banach space X. Take $x \in X \setminus \overline{Y}$. Then $y \in P_Y(x)$ if and only if there is a functional $f \in S_{X^*}$ such that $f/_Y = 0$ and $f(x - y) = dist(x, Y) = \|x - y\|$.*

Applying Theorem 1.1 we can deduct

Theorem 1.3. *A projection $P_0 \in \mathcal{P}(X, V)$ is minimal if and only if there is a functional $f \in S_{\mathcal{L}^*(X,V)}$ such that $f/_{\mathcal{L}_V(X,V)} = 0$ and $f(P_0) = \|P_0\|$.*

Using Theorem 1.3 and the following Choquet theorem

Theorem 1.4 (Choquet [PH, Section 3]**).** *Suppose that K is a metrizable compact convex subset of a locally convex space X, and that x_0 is an element of K. Then there is a probability (borel) measure μ on K which represents x_0 and is supported by the extreme points of K.*

In case of not metrizable subsets we can only have

Theorem 1.5 (Choquet-Bishop-de Leeuw [PH, Section 4]**).** *Suppose that K is a compact convex subset of a locally convex space X, and that x_0 is an element of K. Then there is a probability (borel) measure μ on K which represents x_0 and which vanishes on every Baire subset of K which is disjoint from the set of extreme points of K.*

The main difficulty is that the set of extreme points of K need not form a Borel set. If we drop the requirement that μ is a Borel measure then for each point $x_0 \in K$ there is a nonnegative measure μ on the $\sigma-$ring of subsets of K generated by $extK$ and the Baire sets such that $\mu(X) = 1$, μ represents x_0 and $\mu(extK) = 1$ (see [PH, Section 4]).

The following theorem gives the characterization of extreme points of unit sphere in $K^*(X, V)$.

Theorem 1.6 (Collins-Ruess [LE1]). *Let $\mathcal{K}(X, V)$ denotes the ideal of all compact operators from X to V. Then*

$$extS(\mathcal{K}(X, V)^*) = extS(X^{**}) \otimes extS(V^*), \tag{1.1}$$

where $x \otimes y \in extS(X^{**}) \otimes extS(V^*)$ is considered as a functional in $\mathcal{K}(X, V)^*$. I.e., $\mathcal{K}(X, V)$ is isometric with $\mathcal{L}_e(X^{**}, V)$ the space of all (weak-)–(weakly) continuous operators and*

$$(x^{**} \otimes y)L = y(\widetilde{L}x^{**}), \tag{1.2}$$

where $\widetilde{L}$ is the image of L in this isometry. If V is finite-dimensional the above isometry can be easily described, if $L = \sum x_i^* \otimes v_i$ then $\widetilde{L}(f) = \sum f(x_i^*) \cdot v_i$ (that is $\widetilde{L}$ is just the bidual operator to L).

When $x \in X^{**}$ and $v \in V^*$ then the notation $x \otimes y$ may be interpreted as a functional on $K(X, V)$ in the sense of (1.2) but also it may be interpreted as a rank one operator from V onto X^{**} as follows $x \otimes y(v) = y(v) \cdot x$. As a result in proofs of Theorems 1.9 and 1.10 the same notation will have two different meanings, we hope that this does not cause troubles to the reader but actually helps to understand the correspondence.

Now Proposition 1.3 [PH] says that if K is a metrizable, compact convex subset of a topological vector space, then the extreme points of K form G_δ set. Separability is connected to metrizability. Theorem 6.16 [RU] says that if X is a separable topological vector space then any subset K of X^* which is weak-* compact is also metrizable in the weak-* topology. Now using the following Lemma

Lemma 1.7. *If X^* is separable Banach space then $\mathcal{K}(X, V)$ is separable.*

Proof. Since V is finite-dimensional and X^* is separable we can take v_n and f_m dense respectively in V and X^*. It is easy to show that the operators

$$K_{n,m} : x \to f_m(x)\, v_n$$

are dense in the space of one-dimensional operators of $\mathcal{K}(X, V)$. This completes the proof.

By Banach-Alaoglu theorem we may now conclude that

Corollary 1.8. *If X^* is separable then $S(\mathcal{K}^*(X,V))$ is metrizable and weak-* compact.*

If P is a projection from X onto finite-dimensional subspace V then since P is a compact operator it has a norming functional, i.e., there is a functional $y \in S(V^*)$ such that $\|y \circ P\| = \|P\|$ (see Theorem III.2.1 [OL]). If X is reflexive then any functional attains its norm, therefore there is a extremal pair for P, i.e. there is $(x,y) \in S(X) \times S(V^*)$ such that $yPx = \|P\|$ (see Definition 0.1). If X is not reflexive that in general it is not true. For example classical Fourier projection does not attain its norm in $\mathcal{C}[0,2\pi]$. But any functional attains its norm in X^{**}, hence we can always find a extremal pair for P in $S(X^{**}) \times S(V^*)$.

Theorem 1.9. *To each f from Theorem 1.3 there corresponds E_P*

Proof. In case of X^* being separable, due to Corollary 1.8, we may use the Choquet Theorem in version of Theorem 1.4. Otherwise we have to use Theorem 1.5.

Assume that P_0 is a minimal projection in $\mathcal{P}(X,V)$. Applying Theorem 1.3 we find a functional $f \in S_{\mathcal{L}^*(X,V)}$ such that $f/_{\mathcal{L}_V(X,V)} = 0$ and $f(P_0) = \|P_0\|$. But if V is finite-dimensional then all linear operators onto V are compact (i.e., $\mathcal{L}(X,V) = \mathcal{K}(X,V)$).

Now applying the Choquet Theorem for K being a unit sphere of $\mathcal{K}(X,V)^*$ and $x_0 = f$ we obtain

$$f = (w) \int_{extS(\mathcal{K}(X,V)^*)} x \, d\mu(x) : K(X,V)^* \to \mathbb{K} \tag{1.3}$$

Using Theorem 1.6 the above can be written as

$$f = (w) \int_{extS(X^{**}) \otimes extS(V^*)} x \otimes y \, d\mu(x \otimes y). \tag{1.4}$$

But since $f(P_0) = \|P_0\|$, in view of (1.2) we have

$$\begin{aligned} \|P\| = f(P) &= \int_{extS(X^{**}) \otimes extS(V^*)} (x \otimes y)(P) \, d\mu(x \otimes y) \\ &= \int_{extS(X^{**}) \otimes extS(V^*)} y(P^{**}x) \, d\mu(x \otimes y) \\ &\le \int_{extS(X^{**}) \otimes extS(V^*)} \|P\| \, d\mu(x \otimes y) = \|P\|. \end{aligned} \tag{1.5}$$

This means that for $x \otimes y$ appearing in the integral (1.4) $y(P^{**}x) = \|P\|$ μ-almost everywhere, therefore $(x,y) \in \mathcal{E}(P)$ μ-almost everywhere. Hence (1.4) can be written as

$$f = (w) \int_{\mathcal{E}(P)} x \otimes y \, d\mu(x,y) : K(X,V)^* \to \mathbb{K}. \tag{1.6}$$

Let us define an operator (where y, x are the same as in (1.6) though $x \otimes y$ is viewed this time as an operator $V \to X^{**}$)

$$E_P = \int\limits_{\mathcal{E}(P)} x \otimes y \, d\mu(x,y) : V \to X^{**}, \tag{1.7}$$

Now if P_0 is a projection then $P = P_0 + v^* \otimes v$ ($v \in V$ and $v^* \in V^\perp = \{v^* \in X^* : v^*/V = 0\}$) is also a projection. Hence

$$f(v^* \otimes v) = 0, \quad \text{for any } v \in V \text{ and } v^* \in V^\perp.$$

As a result

$$\begin{aligned} 0 = f(v^* \otimes v) &= \int_{\mathcal{E}(P)} x \otimes y(v^* \otimes v)\, d\mu(x,y) \\ &= \int_{\mathcal{E}(P)} v^*(x) \cdot y(v)\, d\mu(x,y) \\ &= v^* \left(\int_{\mathcal{E}(P)} y(v) \cdot x\, d\mu(x,y) \right) \\ &= v^*(E_P(v)), \end{aligned}$$

for any $v \in V, v^* \in V^\perp$. Therefore $E_P(v) \in V$ for any $v \in V$. Now we just extend functionals $y's$ to X^* by the Hahn-Banach Theorem and we obtain a Chalmers-Metcalf operator. □

Theorem 1.10. *To each E_P there corresponds f from Theorem 1.3.*

Proof. Assume that for a projection P we are given a Chalmers-Metcalf operator E_P ($E_P(V) \subset V$) given by the formula

$$E_P = \int_{\mathcal{E}(P)} x \otimes y\, d\mu(x,y) : X \to X^{**}, \tag{1.8}$$

We define f as following ($x \otimes y$ are the same as in (1.8) though are now viewed as a functional on $K(X,V)^*$)

$$f = (w) \int_{\mathcal{E}(P)} x \otimes y\, d\mu(x,y) : K(X,V)^* \to \mathbb{K}. \tag{1.9}$$

Now, for any $L \in K(X,V)$ we have

$$|f(L)| = |\int_{\mathcal{E}(P)} x \otimes y(L)\, d\mu(x,y)| = |\int_{\mathcal{E}(P)} yL^{**}x\, d\mu(x,y)| \le \|L^{**}\| = \|L\|, \tag{1.10}$$

and by the very same computations $f(P) = \|P\|$. That means that f defined by (1.9) is a norm-one functional on $K(X,V) = L(X,V)$.

Observe also that for any $v \in V, v^* \in V^\perp$, since $E_P(V) \subset V$ (which implies $v^*(E_P(v)) = 0$)

$$\begin{aligned} f(v^* \otimes v) &= \int_{\mathcal{E}(P)} x \otimes y(v^* \otimes v)\, d\mu(x,y) \\ &= \int_{\mathcal{E}(P)} v^*(x) \cdot y(v)\, d\mu(x,y) \\ &= v^* \left(\int_{\mathcal{E}(P)} y(v) \cdot x\, d\mu(x,y) \right) \\ &= v^*(E_P(v)) = 0. \end{aligned} \tag{1.11}$$

Since $\{P - Q : Q \in \mathcal{P}(X, V)\} = span\{v^* \otimes v : v \in V, v^* \in V^{\perp}\}$ and since P is a minimal projection we see that for any projection $Q \in \mathcal{P}(X, V)$ we have

$$f(Q) = f(P) = \|P\| \leq \|Q\|. \tag{1.12}$$

It remains to prove that f vanishes on $L_V(X, V)$. Take any $L \in L_V(X, V)$. Observe that P and $P - L$ are both projections from X onto V. Hence using (1.12)

$$f(L) = f(P - (P - L)) = f(P) - f(P - L) = 0.$$

As a result f is a desired functional from Theorem 1.3. □

In case when X is finite-dimensional the correspondence between the functional from Theorem 1.3 and Chalmers-Metcalf operator is given by trace duality. Trace duality has been used in [KTJ] for estimating the absolute projection constant of finite-dimensional spaces. Before we remind some basics notion of trace duality we establish theorem saying that if X is finite-dimensional then the measure μ giving f or E_P has finite support.

Theorem 1.11. *Assume that X is finite-dimensional. Then we can find Chalmers Metcalf operator such that measure μ is supported on at most $(dimX \cdot dimV + 1)$ points.*

Proof. Take a functional f from $S_{L^*(X,V)}$. Since $L^*(X, V)$ is finite-dimensional by Caratheodory's theorem f can be written as a convex combination of $dim(L^*(X, V)) + 1$ extreme points of $S_{L*(X,V)}$. Now we are reasoning as in the proof of Theorem 1.9 to show that to f there corresponds Chalmers-Metcalf operator. □

We will need the following facts (see [WO] for details) concerning trace duality.

Let X, V be finite-dimensional Banach spaces. Then the nuclear norm in $\mathcal{L}(X, V)$ is defined by

$$\nu(L) = \inf\{\sum_{i=1}^{n} \|x_i^*\| \cdot \|y_i\| : \ L = \sum_{i=1}^{n} x_i^*(\cdot)\, y_i\}. \tag{1.13}$$

Then we have the following trace duality $(\mathcal{L}(V, X), \nu) = \mathcal{L}(X, V)^*$ where " = " means linearly isometric and this isometry is given by

$$\mathcal{L}(V, X) \ni L \mapsto (K \to tr(L \circ K)) \in \mathcal{L}(X, V)^*. \tag{1.14}$$

Additionally, $\mathcal{L}(V, X)$ endowed with the nuclear ν norm forms a Banach ideal operator.

Theorem 1.12. *If X is finite-dimensional then correspondence between functionals from Theorem 1.3 and Chalmers-Metcalf operators is given by trace duality.*

Proof. Take Chalmers-Metcalf operator E_P $(E_P(V) \subset V)$ given by

$$E_P = \sum_{\mathcal{E}(P)} \alpha_i x_i \otimes y_i : X \to X, \tag{1.15}$$

where $\alpha_i > 0$ and $\sum_i \alpha_i = 1$.

Since (x,y) is an extremal pair for P (i.e., $y(Px) = \|P\|$) and trace of the operator does not depend on its particular representation we have

$$tr(E_P/_V \circ P) = tr\left(\sum_{\mathcal{E}(P)} \alpha_i x_i \otimes (y_i \circ P)\right) = \\ = \sum_{\mathcal{E}(P)} \alpha_i y_i(Px_i) = \sum_{\mathcal{E}(P)} \alpha_i \|P\| = \|P\|. \quad (1.16)$$

Assume that $e_1, ..., e_n$ is a basis for V. Since $E_P(V) \subset V$, we can represent E_P as follows

$$E_P/_V(x) = \sum_{k=1,\dots,n} e_k^\star(x) e_k. \quad (1.17)$$

Since the trace of operator does not depend on a particular representation then and for any projection $R : X \to V$ we get

$$tr\,(E_P/_V \circ R) = \sum_{k=1,\dots,n+1} e_k^\star(Re_k) = \sum_{k=1,\dots,n+1} e_k^\star(e_k) = tr\,(E_P/_V)\,. \quad (1.18)$$

Therefore for any projection $R : X \to V$ we get

$$tr(E_P \circ R) = tr\,(E_P/_V \circ R) = \|P\|. \quad (1.19)$$

Now, by $(E_P/_V)^{tr}$ denote the functional on $L(X,V)$ corresponding to $E_P/_V$ by trace duality (see (1.14) - we also restrict functionals y_i to V). And let f be a functional corresponding to Chalmers-Metcalf operator as in Theorems 1.9 and 1.10. Then $f = \sum_{\mathcal{E}(P)} \alpha_i x_i \otimes y_i : L(X,V)^* \to \mathbb{K}$. Take any $L \in L(X,V)$, we have

$$f(L) = \sum_{\mathcal{E}(P)} \alpha_i y_i L(x_i)$$

while

$$(E_P/_V)^{tr}(L) = tr\,(E_P/_V \circ L) = \sum_{\mathcal{E}(P)} \alpha_i y_i(Lx_i).$$

Therefore $(E_P/_V)^{tr}$ coincides with a functional f derived from E_P in Theorem 1.10. □

Lemma 1.13. *Let P and Q are two minimal projections. Let E_P be a Chalmers-Metcalf operator for P and μ be the measure it represents. Then*

$$\mathcal{E}(P) \cap supp\,\mu \subset \mathcal{E}(Q) \qquad \mu\text{-almost everywhere.}$$

Proof. Since $\|Q\| = \|P\|$ and by (1.12),

$$\begin{aligned}\|Q\| = E_P(Q) &= \int_{\mathcal{E}(P)} x \otimes y(Q)\, d\mu(x,y) \\ &= \int_{\mathcal{E}(P)} y(Q^{**}x)\, d\mu(x,y) \qquad (1.20)\\ &\le \int_{\mathcal{E}(P)} \|Q\|\, d\mu(x,y) = \|Q\|.\end{aligned}$$

Which means that for $(x,y) \in \mathcal{E}(P)$ we have $y(Q^{**}x) = \|Q\|$ μ-almost everywhere, therefore $\mathcal{E}(P) \cap supp\,\mu \subset \mathcal{E}(Q)$ μ-almost everywhere. □

Theorem 1.14. *Let P and Q are two minimal projections. Let E_P be a Chalmers-Metcalf operator for P. Then E_P is also a Chalmers-Metcalf operator for Q, i.e, any Chalmers-Metcalf operator is "good" for all minimal projections.*

Proof. This is a straightforward consequence of Lemma 1.13. □

Theorem 1.15. *For a given projection P a set of Chalmers-Metcalf operators E_P is convex.*

Proof. This is a straightforward consequence of Theorem 1.10 and the obvious fact that the convex combination of functionals from Theorem 1.3 is a functional from Theorem 1.3. □

Remark 1.16. *In general Lemma 1.13 in version $\mathcal{E}(P) \subset \mathcal{E}(Q)$ is not true.*

As an example consider $l_1(\mathbb{R}^2)$ and let $V = \{(0,t) : t \in \mathbb{R}\}$. Then norm of a minimal projection from $l_1(\mathbb{R}^2)$ onto V is one. The following two projections have norm one (therefore are minimal)

$$\begin{aligned} P_0(a,b) &= (0, b-a), \\ P_1(a,b) &= (0, b+a). \end{aligned} \tag{1.21}$$

Actually, it is easy to check that any minimal projection from $l_1(\mathbb{R}^2)$ onto V have to be a convex combination of P_0 and P_1. Now one may check that

$$\begin{aligned} \mathcal{E}(P_0) =& \{(y,x) : y = (b,1), x = (a, a+1), \text{ where } a \in [-1,0], b \in [-1,1]\} \\ & \cup \{(y,x) : y = (b,-1), x = (a, a-1), \text{ where } a \in [0,1], b \in [-1,1]\}, \end{aligned}$$

while

$$\begin{aligned} \mathcal{E}(P_1) =& \{(y,x) : y = (b,1), x = (a, -a+1), \text{ where } a \in [0,1], b \in [-1,1]\} \\ & \cup \{(y,x) : y = (b,-1), x = (a, -a-1), \text{ where } a \in [-1,0], b \in [-1,1]\}. \end{aligned}$$

This shows that neither $\mathcal{E}(P_0) \subset \mathcal{E}(P_1)$ nor $\mathcal{E}(P_1) \subset \mathcal{E}(P_0)$.

However,

Theorem 1.17. *Let X be a finite-dimensional space. Then a set*

$$\bigcap \{\mathcal{E}(P) : \ P \text{ is a minimal projection } \}$$

is not empty. Moreover

$$supp\, \mu \subset \bigcap \{\mathcal{E}(P) : \ P \text{ is a minimal projection } \}$$

Proof. We proceed as in Lemma 1.13. Examining (1.20), since the measure μ has finite support, we see that if for some $(x,y) \in \mathcal{E}(P)$ we would have $y(Qx) < \|Q\|$ then we would have also strict inequality in (1.20) which is not possible. □

Theorem 1.1 with a Kolmogorov criterion for best approximation leads to another characterization of minimal projections

Theorem 1.18 (Kolmogorov type criterion). *A projection $P_0 \in \mathcal{P}(X,V)$ is minimal if and only if for any $L \in \mathcal{L}_V(X,V)$ there exists $y \in ext(L^*(X,V))$ and $y(P_0) = \|P_0\|$ such that*

$$(Re\, y)(L) \leq 0.$$

Theorem 1.11 in [OL] led to next characterization

Theorem 1.19 [OL, Theorem III.2.2]**.** *A projection $P_0 \in \mathcal{P}(X,V)$ is minimal if and only if for any $P \in \mathcal{P}(X,V)$ there exists $y \in crit^* P_0$ such that*

$$\|Re(y \circ P)\| \geq \|P_0\|$$

We may slightly improve Theorem 1.12 using Chalmers-Metcalf operator.

Theorem 1.20. *Let E be a Chalmers-Metcalf operator for a set of minimal projections from finite-dimensional space X onto its subspace V. Assume that this operator is represented by the measure μ_E. Then a projection $P_0 \in \mathcal{P}(X,V)$ is minimal if and only if for any $P \in \mathcal{P}(X,V)$ there exists $y \in supp\,(\mu_E)$ such that*

$$\|(y \circ P)\| \geq \|P_0\|$$

Proof. " $\Leftarrow$ " Take any $P \in \mathcal{P}(X,V)$, now $\|P\| \geq \|(y \circ P)\| \geq \|P_0\|$. Hence P_0 is minimal.

" $\Rightarrow$ " Assume, for contrary, that P_0 is a minimal projection and there is projection P such that for any $y \in supp\,(\mu_E)$ we have $\|y \circ P\| < \|P_0\|$. Using Theorem 1.10 we can compute

$$\|P_0\| = E(P) = \sum_{\mathcal{E}(P)} \alpha_i\, y(Px) < \sum_{\mathcal{E}(P)} \alpha_i\, \|P_0\| = \|P_0\|,$$

a contrary. □

2. Chalmers-Metcalf operator and uniqueness of minimal projections

Definition 2.1. Let X be a Banach space X. An element $x \in X$ is called a *smooth point* if it has a unique supporting functional f_x. If every x from unit sphere is a smooth point, then X is called a *smooth* space.

For some basics facts of smoothness as well as some interesting applications see, e.g., [HHZ].

Theorem 2.2. *Assume X is a smooth Banach space and V is its finite-dimensional subspace. Assume that for a minimal projection P there is a Chalmers-Metcalf operator E_P such that E_P/V is invertible. Then P is the unique minimal projection.*

Proof. Since $P \in \mathcal{P}(X,V)$ is a minimal projection, by Chalmers-Metcalf theorem (Theorem 0.2) there is an operator E_P such that $E_p(V) \subset V$, given by the formula

$$E_P = \int_{\mathcal{E}(P)} x \otimes y \, d\mu(x,y) : X \to X^{**}, \tag{2.1}$$

where μ is a probabilistic borel mesure on $\mathcal{E}(P)$. Assume now that there is another minimal projection $Q \in \mathcal{P}(X, V)$.

Now we will see when smoothness comes in handy Take $(x, y) \in \mathcal{E}(P) \cap supp\mu$. By Lemma 1.13 $(x, y) \in \mathcal{E}(Q)$ (μ almost everywhere). We will prove that

$$y \circ P = y \circ Q \quad \text{as a functionals in } X^{***} \ (\mu \text{ almost everywhere}) \tag{2.2}$$

Indeed, $y(P^{**}x) = \|P\|$ and $y(Q^{**}x) = \|Q\|$ implies

$$\frac{y \circ P^{**}}{\|P^{**}\|}(x) = 1 \quad \text{and} \quad \frac{y \circ Q^{**}}{\|Q^{**}\|}(x) = 1. \tag{2.3}$$

But $\|v \circ Q^{**}\| = \|v \circ P^{**}\| = \|Q^{**}\| = \|P^{**}\|$ and (2.3) gives two supporting functionals for x in X^{**}. But X is smooth as a result X^{*} is strictly convex and therefore X^{**} is also smooth. Therefore these two functionals have to be the same, i.e.,

$$\frac{y \circ P^{**}}{\|P^{**}\|} = \frac{y \circ Q^{**}}{\|Q^{**}\|}. \tag{2.4}$$

Applying $\|Q^{**}\| = \|Q\| = \|P\| = \|P^{**}\|$ we get $y \circ P^{**} = y \circ Q^{**}$, hence (2.2) is proved.

Now if $P \neq Q$ then also $P^{**} \neq Q^{**}$ and there is $x_0 \in X^{**}$ such that

$$w_0 := P^{**}(x_0) - Q^{**}(x_0) \neq 0. \tag{2.5}$$

From Lemma 2, for μ-almost every $(x, y) \in \mathcal{E}(P)$ we have

$$\begin{aligned} y(w_0) &= y(P^{**}(x_0) - Q^{**}(x_0)) \\ &= y(P^{**}(x_0)) - y(Q^{**}(x_0)) = (y \circ P^{**} - y \circ Q^{**})(x_0) = 0. \end{aligned} \tag{2.6}$$

Considering now (2.1) , using (2.6) we have

$$E_P(w_0) = \int_{\mathcal{E}(P)} y(w_0) \cdot x \, d\mu(x, y) = 0, \tag{2.7}$$

a contradiction with invertibility of $E_P/_V$. $\square$

Remark 2.3. *It is possible that to one minimal projection there corresponds two or more different Chalmers-Metcalf operators. What is more it is possible that some of them are invertible and some of them not.*

We will construct the desired projection in series of theorems below

Theorem 2.4. *Let $f = (1, 1, ..., 1) \in \ell_q^n$ be a representation of a functional. Then $P : \ell_p^n \to kerf$ given by*

$$P = Id - \tfrac{1}{n}(1, 1, ..., 1) \otimes (1, 1, ..., 1) \tag{2.8}$$

is the unique minimal projection for any $1 < p < \infty$. Moreover there is E_P such that

$$E_P/_V = \frac{\|P\|}{n-1} Id_V. \tag{2.9}$$

Proof. Since P is acting on finite-dimensional space it has a norming pair $(x, y) \in \mathcal{E}(P)$. Now for any $\sigma \in S_n$ a permutation of a set $\{1, 2, ..., n\}$ define the isometries

$$L_\sigma(x_1, x_2, ..., x_n) = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(n)}) \tag{2.10}$$

Observe that P commutes with L_σ. Hence if $(x, y) \in \mathcal{E}(P)$ then $(L_\sigma x, L_\sigma y) \in \mathcal{E}(P)$. Therefore we can construct E_P as follows

$$E_P = \frac{1}{n!} \sum_{\sigma \in S_n} L_\sigma x \otimes L_\sigma y. \tag{2.11}$$

Now compute $E_P(1, -1, 0, 0, ..., 0)$,

$$\begin{aligned} n! \cdot E_P(1, -1, 0, 0, ..., 0) &= \sum_{\sigma \in S_n} L_\sigma y(1, -1, 0, 0, ..., 0) \cdot L_\sigma x \\ &= \sum_{\sigma \in S_n} (y_{\sigma(1)} - y_{\sigma(2)}) \cdot L_\sigma x \\ &= \sum_{\sigma \in S_n} y_{\sigma(1)} \cdot L_\sigma x - \sum_{\sigma \in S_n} y_{\sigma(2)} \cdot L_\sigma x \\ &= \sum_{k=1}^n y_k \sum_{\sigma:\sigma(1)=k} L_\sigma x - \sum_{k=1}^n y_k \sum_{\sigma:\sigma(2)=k} L_\sigma x \\ &= \sum_{k=1}^n y_k \left(\sum_{\sigma:\sigma(1)=k} L_\sigma x - \sum_{\sigma:\sigma(2)=k} L_\sigma x \right) \end{aligned} \tag{2.12}$$

Now observe that

$$\sum_{\sigma:\sigma(1)=k} L_\sigma x = (n-1)!(x_k, \sum_{i \neq k} x_i, \sum_{i \neq k} x_i, ..., \sum_{i \neq k} x_i) \tag{2.13}$$

and

$$\sum_{\sigma:\sigma(2)=k} L_\sigma x = (n-1)!(\sum_{i \neq k} x_i, x_k, \sum_{i \neq k} x_i, ..., \sum_{i \neq k} x_i). \tag{2.14}$$

Hence

$$\begin{aligned} n \cdot E_P(1, -1, 0, 0, ..., 0) &= \sum_{k=1}^n y_k \left(\sum_{\sigma:\sigma(1)=k} L_\sigma x - \sum_{\sigma:\sigma(2)=k} L_\sigma x \right) \\ &= \sum_{k=1}^n y_k \cdot (x_k - \sum_{i \neq k} x_i, -(x_k - \sum_{i \neq k} x_i), 0, ..., 0) \\ &= \left(\sum_{k=1}^n y_k \cdot \left(x_k - \sum_{i \neq k} x_i \right) \right) \cdot (1, -1, 0, ..., 0). \end{aligned} \tag{2.15}$$

In the same manner, for any $j = 2, ..., n$ we obtain

$$n \cdot E_P(e_1 - e_j) = \left(\sum_{k=1}^n y_k \cdot \left(x_k - \sum_{i \neq k} x_i \right) \right) \cdot (e_1 - e_j). \tag{2.16}$$

And hence $\{e_1 - e_k\}$ is a basis for $V = ker(1,1,...,1)$ we proved that

$$E_P\big/_V = c \cdot Id_V. \tag{2.17}$$

Applying computations made in section 1 (see (1.13) and (1.14)) and since $tr(Id_V) = dimV = n-1$ we obtain $c = \frac{\|P\|}{n-1}$. Hence E_P defined by (2.11) is a Chalmers-Metcalf operator for P and therefore P is minimal. By the results of this section P is the unique minimal projection (actually the uniqueness follows from the fact that in ℓ_p^n $(1 < p < \infty)$ minimal projections onto hyperplanes are unique - see Theorem I.1.3 [OL]). $\square$

Theorem 2.5. *Take*

$$f_1 = (\underbrace{1,1,...,1}_{n},\underbrace{0,0,...,0}_{n}) \in \ell_q^{2n}$$

and

$$f_2 = (\underbrace{0,0,...,0}_{n},\underbrace{1,1,...,1}_{n}) \in \ell_q^{2n}.$$

Then $Q : \ell_p^n \to V = kerf_1 \cap kerf_2$ *given by*

$$Q = Id - \tfrac{1}{n} f_1 \otimes f_1 - \tfrac{1}{n} f_2 \otimes f_2 \tag{2.18}$$

is the unique minimal projection for any $1 < p < \infty$. *Moreover there are* E_Q *and* $\widetilde{E_Q}$ *both Chalmers-Metcalf operators such that*

$$E_Q\big/_V \text{ is invertible} \quad \text{and} \quad \widetilde{E_Q}\Big/_V \text{ is not invertible.}$$

Proof. Let P be a projection as in Theorem 2.4 (see (2.8)). It can be easily seen that

$$Q(x_1,...,x_n,x_{n+1},...,x_{2n}) = (P(x_1,...,x_n), P(x_{n+1},...,x_{2n})). \tag{2.19}$$

Now we will show that $\|Q\| = \|P\|$, indeed

$$\begin{aligned} \|Q(x_1,...,x_{2n})\|_p^p &= \sum_{i=1}^{n} |x_i - (\sum_{i=1}^{n} x_i)|^p + \sum_{i=n+1}^{2n} |x_i - (\sum_{i=n+1}^{2n} x_i)|^p \\ &\le \|P\|_p^p (\sum_{i=1}^{n} |x_i|^p) + \|P\|_p^p (\sum_{i=n+1}^{2n} |x_i|^p) \\ &= \|P\|_p^p \cdot \|(x_1,...,x_{2n})\|_p^p \end{aligned} \tag{2.20}$$

From the above computation we deduce $\|Q\| = \|P\|$ and

$$\begin{gathered} (x_1,...,x_{2n}) \text{ is a norming point for } Q \\ \Updownarrow \\ (x_1,...,x_n) = 0 \text{ or } \frac{(x_1,...,x_n)}{\|(x_1,...,x_n)\|} \text{ is a norming point for } P \\ \text{and} \\ (x_{n+1},...,x_{2n}) = 0 \text{ or } \frac{(x_{n+1},...,x_{2n})}{\|(x_{n+1},...,x_{2n})\|} \text{ is a norming point for } P. \end{gathered} \tag{2.21}$$

Let $E_P = \frac{1}{k}\sum_{i=1}^k x_i \otimes y_i$ be a Chalmers-Metcalf operator for P given by Theorem 2.4. By the above statement (2.21) $((x_i,0),(y_i,0)) \in \mathcal{E}(Q)$, and we may define

$$\widetilde{E_Q} = \frac{1}{k}\sum_{i=1}^k (x_i,0)\otimes(y_i,0). \tag{2.22}$$

It is easy to check that

$$\widetilde{E_Q}(z_1,...,z_{2n}) = (E_P(z_1,...,z_n),0), \tag{2.23}$$

Hence by (2.9) if $(v_1,...,v_n) \in ker f_1 \cap ker f_2$ then

$$\widetilde{E_Q}(v_1,...,v_{2n}) = \frac{\|Q\|}{n}(v_1,...,v_n,0) \in ker f_1 \cap ker f_2. \tag{2.24}$$

So $\widetilde{E_Q}$ is a Chalmers-Metcalf operator for Q and $\widetilde{E_Q}/V$ is not invertible. Now in the same manner as $\widetilde{E_Q}$ we may define $\widehat{E_Q}$ by

$$\widehat{E_Q} = \frac{1}{k}\sum_{i=1}^k (0,x_i)\otimes(0,x_i). \tag{2.25}$$

As previous if $(v_1,...,v_n) \in ker f_1 \cap ker f_2$ then

$$\widehat{E_Q}(v_1,...,v_{2n}) = \frac{\|Q\|}{n}(0,v_{n+1},...,v_{2n}) \in ker f_1 \cap ker f_2. \tag{2.26}$$

Now consider

$$E_Q := (\widetilde{E_Q} + \widehat{E_Q})/2. \tag{2.27}$$

By Theorem 1.15 the above defined E_Q is a Chalmers-Metcalf operator for Q. Using (2.24) and (2.26) we have

$$E_Q/V = \frac{\|Q\|}{2n}\cdot Id_V.$$

Hence E_Q is an invertible Chalmers-Metcalf operator. As a result Q is the unique minimal projection. □

Remark 2.6. *It is clear that we can extend the above example to show that the set of Chalmers-Metcalf operators may contain arbitrary finite number of independent elements.*

Theorem 2.7 [CS]**.** *Assume that X is a Banach space and V is its finite-dimensional subspace. Let P be a minimal projection of norm one. Then P is the unique minimal projection.*

Proof. Since P is a projection on finite-dimensional subspace V it has a representation

$$P = \sum_{i=1}^n f_i \otimes v_i,$$

where $f_i(v_j) = \delta_{ij}$.

Observe that since $f_i(Pv_i) = f_i(v_i) = 1 = \|P\|$ we have $(v_i, f_i) \in \mathcal{E}(P)$. We can now define

$$E_P := \frac{1}{n}\sum_{i=1}^n f_i \otimes v_i = \frac{1}{n}P$$

Since $E_P/V = \frac{1}{n}Id/V$ it is a Chalmers-Metcalf operator and it is invertible, since by Theorem 2.2 P is the unique minimal projection.

Theorem 2.8 [SK3]**.** *Take Cheney-Light projection $Q : L_p[m\times n] \to L_p[m]+L_p[n]$ given by the formula*

$$Qe_{rs}(i,j) = \begin{cases} \frac{n+m-1}{nm} & i=r, j=s \\ \frac{m-1}{nm} & i\neq r, j=s \\ \frac{n-1}{nm} & i=r, j\neq s \\ \frac{-1}{nm} & i\neq r, j\neq s \end{cases},$$

where $e_{rs}(i,j) = \delta_{ri}\delta_{sj}$. Then there is a Chalmers-Metcalf operator for this projection equal $\frac{\|Q\|}{n+m-1}\cdot Id$. Its invertibility implies now that Q is the unique minimal projection for $1 < p < \infty$. It is worth mentioning that for $p = 1$ and $p = \infty$ it has been proved that this projection is not unique [SK3].

Theorem 2.9 [LE2]**.** *Suppose that X, Y are separable Banach spaces and V, W are finite-dimensional subspaces of X and Y respectively. Assume that $P : X \to V$, and $Q : Y \to W$ are minimal projections and let E_P and E_Q be the corresponding Chalmers-Metcalf operators. Consider the tensor product $X \otimes Y$ with some reasonable, uniform crossnorm. Then the projection*

$$P \otimes Q : X \otimes Y \to V \otimes W$$

is minimal and its Chalmers-Metcalf operator may be given by

$$E_{P\otimes Q} = E_P \otimes E_Q.$$

Theorem 2.10. *If both $E_P/_V : V \to V$ and $E_Q/_W : W \to W$ are invertible then $E_{P\otimes Q}/_{V\otimes W} : V \otimes W \to V \otimes W$ is invertible.*

Proof. We need to show that $ker\{E_{P\otimes Q}/_{V\otimes W}\} = 0$. To do this take $0 \neq x \in V \otimes W$, we may write x in the form $x = \sum_{i=1}^{n} v_i \otimes w_i$, where w_i are linearly independent. Now assume for the contrary that $E_{P\otimes Q}(x) = 0$, that is $\sum_{i=1}^{n} E_P(v_i) \otimes E_Q(w_i) = 0$. This implies

$$\sum_{i=1}^{n} f(E_P(v_i)) \otimes E_Q(w_i) = 0, \quad \text{for any } f \in V^*. \tag{2.28}$$

But since $x \neq 0$, the dimension of $span\{v_i\}$ is greater then zero and since E_P in invertible the dimension of $span\{E_P(v_i)\}$ is greater then zero. Hence by (2.28) $E_Q(w_i)$ are linearly dependent and as E_Q is invertible this would imply that w_i are linearly dependent, contrary. □

The above theorem gives us, with the assumptions that the considered tensor norm is smooth, the uniqueness of tensor product of projections which Chalmers-Metcalf operators are invertible.

3. Symmetric case

Recall that a space is called symmetric if there is a basis $e_1, ..., e_n$ such that for any permutation σ of $\{1, ..., n\}$

$$\|\pm\alpha_1\, e_{\sigma(1)} \pm \alpha_2\, e_{\sigma(2)} \pm ... \pm \alpha_n\, e_{\sigma(n)}\| = \|\alpha_1\, e_1 + ... + \alpha_n\, e_n\|. \tag{3.1}$$

For $\epsilon = (\epsilon_1, ..., \epsilon_n) \in \{-1, 1\}^n$ and $\sigma \in S_n$ we will denote by $T_{\sigma,\epsilon}$ the isometry given by

$$T_{\sigma,\epsilon}(\alpha_1\, e_1 + ... + \alpha_n\, e_n) = \epsilon_1\, \alpha_1\, e_{\sigma(1)} + ... + \epsilon_n\, \alpha_n\, e_{\sigma(n)}. \tag{3.2}$$

First we will prove some preliminary result

Lemma 3.1. *If X is a symmetric space and $L : X \to X$ is an operator which commutes with group $\{T_{\sigma,\epsilon}\}$ (which means that for any σ and ϵ $L \circ T_{\sigma,\epsilon} = T_{\sigma,\epsilon} \circ L$) then L has to be necessary of the form $L = c \cdot Id_X$, where c is a constant.*

Proof. Assume that L is given in basis by $L(e_i) = \sum_{j=1}^{n} a_{ij}\, e_j$. Take S_i an isometry which left all the base elements intact except the e_i and $S_i(e_i) = -e_i$. Then

$$S_i \circ L(e_i) = S_i(\sum_{j=1}^{n} a_{ij}\, e_j) = \sum_{j \neq i} a_{ij}\, e_j - a_{ii}\, e_i, \tag{3.3}$$

while

$$L \circ S_i(e_i) = L(-e_i) = -L(e_i) = \sum_{j=1}^{n} -a_{ij}\, e_j. \tag{3.4}$$

Comparing (3.3) and (3.4) yields $a_{ij} = 0$ for $i \neq j$ and as results L has to have the form $L(e_i) = a_{ii}\, e_i$.

Now, consider P_{kl}, an isometry which left all the base elements intact except e_k, e_l and $P_{kl}(e_k) = e_l, P_{kl}(e_l) = e_k$. Then

$$P_{kl} \circ L(e_k) = P_{kl}(a_{kk}\, e_k) = a_{kk}\, e_l, \tag{3.5}$$

while

$$L \circ P_{kl}(e_k) = L(e_l) = a_{ll}\, e_l. \tag{3.6}$$

Hence $a_{ii} = a_{jj}$, therefore $L = c \cdot Id_X$. □

Theorem 3.2. *Assume X is finite-dimensional and V is its symmetric subspace. If X is additionally a smooth space then there is only one minimal projection in $\mathcal{P}(X, V)$.*

Proof. Since X is finite-dimensional then by the classical result of Izbell and Semadeni [IS] there is a minimal projection in $\mathcal{P}(X, V)$. Take $P \in \mathcal{P}(X, V)$ a minimal projection, by Theorem 1.12 and (1.19) there is an operator $E : V \to X$ such that

$$\begin{gathered} E(V) \subset V, \quad \nu(E/V) = 1 \\ \text{and} \\ tr(E \circ Q) = tr(E/V) = \|Q\| \text{ for any projection } Q : X \to V \end{gathered} \tag{3.7}$$

Let G be the group of isometries $\{T_{\sigma,\epsilon}\}$ given by (3.2). Observe that $T_{\sigma,\epsilon}^{-1} = T_{\sigma^{-1},-\epsilon}$. Put

$$L_0 = \frac{1}{|G|} \sum_{g=(\sigma,\epsilon) \in G} T_g \circ E/_V \circ T_g^{-1} : V \to X. \tag{3.8}$$

Since $E/_V(V) \subset V$ the above definition is correct.

Because $(\mathcal{L}(V,X),\nu)$ is a Banach operator ideal, T_g are isometries and by (3.7)

$$\begin{aligned}
\nu(L_0) = {} & \nu\Big(\frac{1}{|G|}\sum_{g\in G} T_g\circ E/_V\circ T_g^{-1}\Big)\\
& \le \frac{1}{|G|}\sum_{g\in G}\nu(T_g\circ E/_V\circ T_g^{-1})\\
& \le \frac{1}{|G|}\sum_{g\in G}\|T_g\|\cdot\nu(E/_V)\cdot\|T_g^{-1}\|\\
& = \frac{1}{|G|}\sum_{g\in G}\nu(E/_V)\\
& = \nu(E/_V) = 1.
\end{aligned}\tag{3.9}$$

Now, by the trace duality we will show that $\nu(L_0)=1$. Since $L_0(V)\subset V$ therefore $L_0(x)=\sum_{k=1,\dots,n} f_k^*(x)\,e_k$ and for any projection $R: X\to V$ we have

$$tr(L_0)=\sum_{k=1,\dots,n} f_k^*(e_k)=\sum_{k=1,\dots,n} f_k^*(Re_k)=tr(L_0\circ R).\tag{3.10}$$

Now using $tr(A\circ B)=tr(B\circ A)$, we have

$$\begin{aligned}
tr(T_g\circ E/_V\circ T_g^{-1}) &= tr(T_g\circ(E/_V\circ T_g^{-1}))\\
&= tr((E/_V\circ T_g^{-1})\circ T_g)\\
&= tr(E/_V).
\end{aligned}\tag{3.11}$$

Combining (3.10), (3.11) and (3.7) yields

$$\begin{aligned}
tr(L_0\circ P) &= tr(L_0)\\
&= tr\Big(\frac{1}{|G|}\sum_{g\in G} T_g\circ E/_V\circ T_g^{-1}\Big)\\
&= \frac{1}{|G|}\sum_{g\in G} tr(T_g\circ E/_V\circ T_g^{-1})\\
&= \frac{1}{|G|}\sum_{g\in G} tr(E/_V)\\
&= \frac{1}{|G|}\sum_{g\in G} tr(E/_V\circ P)\\
&= \frac{1}{|G|}\sum_{g\in G}\|P\|\\
&= \|P\|
\end{aligned}\tag{3.12}$$

Therefore by trace duality

$$\nu(L_0)=1,\tag{3.13}$$

and as a result L_0 can be treated as a functional in $\mathcal{L}(X,V)^*$ of norm one such that $L_0(R) = \|P\|$. Therefore L_0 in view of Theorems 1.9, 1.10, 1.12 it can be viewed as a Chalmers-Metcalf operator, i.e.,

$$L_0 = \int\limits_{\mathcal{E}(P)} m \otimes n \, d\lambda(m,n) : X \to X.$$

But from the definition of L_0(see (3.8)), L_0 commutes with group $\{T_{\sigma,\epsilon}\}$. Hence by Lemma 3.1 $\widetilde{L_0}/V = c \cdot Id_V$ Applying (3.12)

$$\|P\| = tr(L_0) = tr(c \cdot Id_V) = c \cdot tr(Id_V) = c \cdot dimV,$$

we get $c = \frac{\|P\|}{dimV}$. Applying the Theorem 2.2 yields the result. □

Remark 3.3. *It is easy to see that Theorem 3.2 holds also true for a subspaces V with enough symmetries. Recall that space X is a space with enough symmetries if the only operators which commute with all isometries on X are of the form constant times identity operator.*

4. Rademacher projections

The well-known Rademacher functions, $r_0, r_1, ...,$ defined by $r_j(t) = (-1)^{[2^j t]}$ for $0 \le t \le 1$ plays the central role in many areas of analysis ($[\cdot]$ denotes the integer part of the argument).

For further investigations we will need a notion of **dyadic group**

We shall denote the set of **dyadic rationals** in the unit interval $[0,1)$ by Q. In particular, each element of Q has a form $p/2^n$ for some $p, n \in \mathbb{N}, 0 \le p < 2^n$.

Definition 4.1 (dyadic group on the interval $[0,1]$). Any $x \in [0,1]$ may be written in the form

$$x = \sum_{k=0}^{\infty} x_k \, 2^{-(k+1)}, \tag{4.1}$$

where each $x_k = 0$ or 1. For each $x \in [0,1] \setminus Q$ there is only one expression of this form. We shall call it the **dyadic expansion** of x. When $x \in Q$ there are two expression of this form, one which terminates in 0's and one which terminates in 1's. By the dyadic expansion of $x \in Q$ we shall mean the one which terminates in 0's. Notice that $1 \notin Q$ so the dyadic expansion of $x = 1$ terminates in 1's.

Now we can define the **dyadic addition** of two numbers x, y by

$$x \oplus y = \sum_{k=0}^{\infty} |\, x_k - \beta_y| \, 2^{-(k+1)}. \tag{4.2}$$

Observe that $x \oplus x = 0$ therefore $x \oplus y = x \ominus y$.

Theorem 4.2 [SWS]. *The following holds true*

$$r_n(x \oplus y) = r_n(x) r_n(y) \qquad \textit{for } x \oplus y \notin Q \tag{4.3}$$

Definition 4.3 [SWS] (dyadic group as operators). Define the operators T_y (for $y \in [0,1)$) as following

$$T_y : L_p[0,1] \ni f \mapsto T_y(f) = \{x \mapsto f(x \oplus y)\} \in L_p[0,1], \tag{4.4}$$

It can be easily seen that the above operators are isometries.

For many other interesting facts concerning both Walsh and Rademacher functions the reader is referred to [SWS].

Denote by

$$Rad_n = span\{r_0, r_1, ..., r_{n-1}\} \tag{4.5}$$

the space spanned by the fist n Rademacher functions. The Rademacher projection is defined by

$$R_n = \sum_{i=0,\dots,n-1} r_i \otimes r_i : L_p[0,1] \to Rad_n, \tag{4.6}$$

We can write the above projection as

$$R_n(f) = \sum_{i=0,\dots,n-1} \Big(\int_0^1 r_i(t) f(t)\, dt\Big) r_i, \tag{4.7}$$

or using Dirichlet kernel $D_n^r = r_0 + r_1 + ... + r_{n-1}$ and orthogonality of Rademacher functions as

$$(R_n\, f)(s) = \int_0^1 f(t)\, D_n^r(t \oplus s)\, dt, \tag{4.8}$$

or equivalently

$$R_n(f) = f * D_n^r, \tag{4.9}$$

Rademacher projections are minimal, we can prove that as follows.

Theorem 4.4. *The Rademacher projection R_n is a minimal projection in the set of all projections from $L_p[0,1]$ onto Rad_n.*

Proof. Since Rad_n are finite-dimensional they have norming pairs, i.e., there exists $(a,b) \in S(L_p) \times S(L_q)$ such that $b\, R_n(a) = \|R_n\|$. Observe that dyadic group T_y commutes with R_n, indeed

$$\begin{aligned}
T_y(R_n\, f)(s) &= T_y\Big(\int_0^1 f(t)\, D_n^r(t \oplus s)\, dt\Big) \\
&= \int_0^1 f(t)\, D_n^r(t \oplus s \oplus y)\, dt = (\text{ change of variables}) = \\
&= \int_0^1 f(t \oplus y)\, D_n^r(t \oplus s)\, dt \\
&= \int_0^1 T_y(f)(t)\, D_n^r(t \oplus s)\, dt \\
&= R_n(T_y\, f)(s).
\end{aligned}$$

As a consequence $(T_y b)\, R_n((T_y a)) = \|R_n\|$. We can therefore define an operator

$$E_{R_n} = \int_0^1 T_y(b) \otimes T_y(a)\, dy.$$

Now, we will prove that the above is a Chalmers-Metcalf operator for R_n. We need to show $E_{R_n}(Rad_n) \subset Rad_n$ We can see that as follows

$$\begin{aligned}
E_{R_n}(r_i)(s) &= \int_0^1 (T_y b)(r_i) \cdot (T_y a)(s)\, dy \\
&= \int_0^1 ((T_y b)(t) \cdot r_i(t)\, dt) \cdot (T_y a)(s)\, dy \\
&= \int_0^1 (\int_0^1 b(t \oplus y) \cdot r_i(t)\, dt) \cdot (T_y a)(s)\, dy \\
&= \int_0^1 (\int_0^1 b(t) \cdot r_i(t \oplus y)\, dt) \cdot (T_y a)(s)\, dy \\
&= \int_0^1 (\int_0^1 b(t) \cdot r_i(t) \cdot r_i(y)\, dt) \cdot (T_y a)(s)\, dy \\
&= \int_0^1 (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot r_i(y) \cdot (T_y a)(s)\, dy \\
&= (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot (\int_0^1 r_i(y) \cdot (T_y a)(s)\, dy) \\
&= (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot (\int_0^1 r_i(y) \cdot a(s \oplus y)\, dy) \\
&= (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot (\int_0^1 r_i(y \oplus s) \cdot a(y)\, dy) \\
&= (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot (\int_0^1 r_i(y) \cdot w_i(s) \cdot a(y)\, dy) \\
&= (\int_0^1 b(t) \cdot r_i(t)\, dt) \cdot (\int_0^1 r_i(y) \cdot a(y)\, dy) \cdot r_i(s)
\end{aligned}$$

Therefore

$$E_{R_n}(r_i) = < r_i, d > \cdot < r_i, c > \cdot r_i,$$

which shows that $E_{R_n}(Rad_n) \subset Rad_n$. □

Now, using Theorem 3.2, since $L_p[0,1]$ are smooth spaces for $p \in (1, \infty)$, we would like to obtain uniqueness of Rademacher projections. First we need to check that the norm of Rademacher projections is strictly greater then one (otherwise it would follow from fairly easy Cohen-Sulivan result - Theorem 2.6).

Remark 4.5. *Assume $n \geq 3$. Then $\|R_n\|_p > 1$, for $p \neq 2$.*

Proof. Let $w_3(t) := r_1(t) \cdot r_2(t)$. Consider the following function

$$f = \frac{3^{q/p} + 1}{4^{1/q}(3 + 3^q)^{1/p}} \cdot (r_0 + r_1 + r_2) + \frac{3^{q/p} - 3}{4^{1/q}(3 + 3^q)^{1/p}} \cdot w_3.$$

One can check that $\|f\|_p = 1$. Observe that $\int_0^1 w_3(t) \cdot r_n(t)\, dt = 0$, for any n. Therefore

$$R_n(f) = \frac{3^{q/p} + 1}{4^{1/q}(3 + 3^q)^{1/p}} \cdot (r_0 + r_1 + r_2),$$

and as a result

$$\|R_n(f)\|_p = \frac{3^{q/p}+1}{4(3+3^q)^{1/p}} \cdot (3+3^p)^{1/q} = \frac{1}{4}(1+3^{p/q})^{1/p}(1+3^{q/p})^{1/q}.$$

It is easy to check that for $p \neq 2$ the latter is strictly greater then 1. $\square$

Now we are ready to prove the main theorem in this section

Theorem 4.6. *Rademacher projection* $R_n : L_p[0,1] \to span\{r_0, ..., r_{n-1}\}$, *for* $p \in (1, \infty)$, *given by*

$$R_n(f) = \sum_{i=0,\dots,n-1} \Big(\int_0^1 r_i(t) f(t)\, dt\Big) r_i, \tag{4.10}$$

is the unique minimal projection.

Proof. Let S_n denote the space of simple functions of intervals $I_{l,2^k} = [\frac{l}{2^n}, \frac{l+1}{2^n}]$, where $l \in \{0, 1, ..., 2^n - 1\}$. Therefore for a simple function $A \in S$ we have

$$A = \sum_{l=0}^{2^n-1} A_l \, \chi_{I_{l,2^n}}. \tag{4.11}$$

Additionally we have

$$Rad_n \subset S_n. \tag{4.12}$$

Consider now the projection

$$\widetilde{R_n} = R_n / S_n : S_n \to Rad_n \tag{4.13}$$

Observe that $span\{r_0, ..., r_{n-1}\}$ is a symmetric space in $L_p[0,1]$. Therefore following Theorem 3.2 there is a Chalmers-Metcalf operator

$$E_{\widetilde{R_n}} = \int\limits_{\mathcal{E}(\widetilde{R_n})} y \otimes x \; d\mu(x,y) : S_n \to S_n, \tag{4.14}$$

such that

$$E_{\widetilde{R_n}} / Rad_n = \frac{\|\widetilde{R_n}\|}{n} Id_{Rad_n}. \tag{4.15}$$

Let consider now the projection $L_n : L_p[0,1] \to S_n$

$$L_n(f) = \sum_{l=0}^{2^n-1} \Big(\int_{I_{l,2^n}} f(x)\, dx\Big) \chi_{I_{l,2^n}}. \tag{4.16}$$

Also observe that $L_n/S_n = Id$ and since $r_i \in S_n$ we have also

$$r_n(f) = r_n(L_n f).$$

Take z a norming point for R_n, then

$$\begin{aligned}\|R_n\| = \|R_n(f)\| &= \|\sum_{i=0,\dots,n-1} r_i(f) r_i\| = \|\sum_{i=0,\dots,n-1} r_i(L_n f) r_i\| \\ &= \|R_n(L_n f)\| = \|\widetilde{R_n}(L_n f)\| \leq \|\widetilde{R_n}\| \cdot L_n\| \leq \|\widetilde{R_n}\|\end{aligned}$$

Hence, since $\widetilde{R_n} = R_n/S_n$ we have

$$\|R_n\| = \|\widetilde{R_n}\| \tag{4.17}$$

Let $\widetilde{y}$ are Hahn-Banach extension of functionals y's. It is clear now that if $(x, y) \in \mathcal{E}(\widetilde{R_n})$ then $(x, \widetilde{y}) \in \mathcal{E}(R_n)$. Hence

$$E_{R_n} = \int_{\mathcal{E}(R_n)} \widetilde{y} \otimes x \; d\mu(x, y) : L_p[0, 1] \to L_p[0, 1], \tag{4.18}$$

is a Chalmers-Metcalf operator for R_n and since

$$E_{R_n}/Rad_n = E_{\widetilde{R_n}}/Rad_n = \frac{\|\widetilde{R_n}\|}{n} Id_{Rad_n} \tag{4.19}$$

it is invertible, thus by Theorem 2.2 $R_n : L_p[0, 1] \to Rad_n$ is the unique minimal projection. □

The following theorem shows that smoothness can not be omitted in Theorem 4.6.

Theorem 4.7 (Chalmers-Metcalf [CM1]**).** *The Rademacher projection $R_n : L_1[0, 1] \to span\{r_0, ..., r_{n-1}\}$, given by* (4.10) *is not unique if n is odd.*

Using (4.18), (4.19) in view of Theorems 2.9 and 2.10 we immediately obtain the following theorem.

Theorem 4.8. *Let $1 < p < \infty$. The tensor product of Rademacher projections $R = R_n \otimes R_m : L_p[0, 1]^2 \to Rad_n \otimes Rad_m$ is the unique minimal projection.*

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