OPTIMAL AND ONE-COMPLEMENTED SUBSPACES

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Abstract. Let X be a real Banach space and let $V \subset X$ be a closed linear subspace. In [3, Prop. 5] it has been proven that if X is strictly convex, reflexive and smooth and V is an optimal subset of X then V is one-complemented in X. In this note we would like to extend this result to non-smooth Banach spaces. In particular, we show that any existence subspace of c, c_o and l_1 is one-complemented. Also some results concerning non-smooth Musielak-Orlicz sequence spaces equipped with the Luxemburg norm will be presented.

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0. Introduction.

Let X be a Banach space and let $C \subset X$ be a non-empty set. A continuous mapping $P : X \to C$ is called *a projection onto* C, whenever $P|_C = Id$, that is $P^2 = P$. Setting

$$Min(C) = \{z \in X : \text{ for every } c \in C, x \in X, \text{ if } \|z - c\| \geq \|x - c\| \text{ then } x = z\},$$

we say that $C \subset X$ is *optimal* if $Min(C) = C$. Observe that for any $C \subset X$, $C \subset Min(C)$. This notion has been introduced by Beauzamy and Maurey in [3], where basic properties concerning optimal sets can be found.

A set $C \subset X$ is called *an existence set of best coapproximation* (*existence set* for brevity), if for any $x \in X$, $R_C(x) \neq \emptyset$, where

$$R_C(x) = \{d \in C : \|d - c\| \leq \|x - c\| \text{ for any } c \in C\}.$$

It is clear that any existence set is an optimal set. The converse, in general, is not true. However, by [3, Prop. 2] if X is one-complemented in X^{**} and strictly convex, then any optimal subset of X is an existence set in X, which, in particular, holds true for strictly convex spaces X, such that $X = Z^*$ for some Banach space Z.

Existence and optimal sets have been studied by many authors from different points of view, mainly in the context of approximation theory (see e.g. [1 - 4], [6 - 13], [18], [24], [26], [27]).

Recall that a closed subspace V of a Banach space X is called *one-complemented* if there exists a linear projection of norm one from X onto V. Also there is a large number of papers concerning one-complemented subspaces (see e.g. a survey paper [25] and a recent paper [16]). It is obvious that any one-complemented subspace is an existence set. The converse, in general, is not true. By a deep result of Lindenstrauss, [20] there exists a Banach space X and V a linear subspace of X, $\mathrm{codim}(V) = 2$, such that:

a). V is one-complemented in any containing it hyperplane Y of X;

b). V is not one-complemented in X.

This gives you immediately an example of a subspace being an existence set which is not one-complemented. However, in [3] (see also [26, p.132], where the case of contractive retractions onto linear subspaces is considered) the the following result has been proven.

Theorem 0.1. *(see [3, Prop. 5]). Let V be a linear subspace of a smooth, reflexive and strictly convex Banach space. If V is an optimal set then V is one-complemented in X. If X is a smooth Banach space, then any subspace of X which is an existence set is one-complemented. Moreover, in both cases a norm-one projection from X onto V is uniquely determined.*

The aim of this paper is to generalize the above result to the case of some non-smooth, real Banach spaces. This can be treated as a partial answer concerning the question from [3, p.125] about generalizations of [3, Prop. 5] to the non-smooth case. In particular, we show that in c, c_o and l_1 any subspace which is an existence set is one-complemented (see Theorems 2.2, 2.3 and 2.4). Next we demonstrate some results in the case of non-smooth Musielak-Orlicz sequence spaces equipped with the Luxemburg norm.

Now we present some notions and results which will be used in this paper.

In the sequel by $S(X)$ we denote the unit sphere in a Banach space X and by $S(X^*)$ the unit sphere in its dual space. A functional $f \in S(X^*)$ is called a *supporting functional* for $x \in X$, if $f(x) = \|x\|$. Analogously, a point $x \in S(X)$ is called a *norming point* for $f \in X^*$ if $f(x) = \|f\|$.

A point $x \in X$ is called a *smooth point* if it has exactly one supporting functional. A Banach space X is called *smooth* if any $x \in S(X)$ is a smooth point.

By $ext(X)$ we denote the set of all extreme points of $S(X)$. A Banach space X is called *strictly convex* if $ext(X) = S(X)$.

If V is a linear subspace of a Banach space X, by $\mathcal{P}(X, V)$ we will denote the set of all linear, continuous projections from X onto V.

Now we present some introductory facts on Musielak-Orlicz spaces. A function $\phi : \mathbb{R} \to [0, +\infty)$ is said to be an *Orlicz function* if $\phi(0) = 0$, ϕ is even, strictly increasing and convex. By ϕ^* we denote its conjugate function in the sense of Young, that is

$$\phi^*(u) = \sup_{v>0}\{|u|v - \phi(v)\},$$

for $u \in \mathbb{R}$ and we notice that ϕ^* is an extended real-valued convex function. If $\phi(u) = (1/p)u^p$, $1 < p < \infty$, then $\phi^*(u) = (1/p')u^{p'}$, where $1/p+1/p' = 1$. Further, a sequence $\Phi = (\phi_n)$ of Orlicz functions ϕ_n will be called a *Musielak-Orlicz function* whenever $\phi_n(1) = 1$ for every $n \in \mathbb{N}$. By $\Phi^* = (\phi_n^*)$ we will denote its conjugate function.

Let l_0 denote the space of all real-valued sequences. With each Musielak-Orlicz function Φ we can associate a mapping $\rho_\Phi : l_0 \to [0, +\infty]$ defined by

$$\rho_\Phi(x) = \sum_{n=1}^{\infty} \phi_n(|x_n|),$$

where $x = (x_n) \in l_0$. Given a Musielak-Orlicz function Φ, let l_Φ denote the corresponding *Musielak-Orlicz space,* that is

$$l_\Phi = \{x \in l_0 : \lim_{\lambda\to 0} \rho_\Phi(\lambda x) = 0\}. \tag{0.1}$$

If a sequence $\Phi = (\phi_n)$ is constant, that is $\phi_n = \phi$ for every $n \in \mathbb{N}$, then l_Φ is an *Orlicz sequence space* and further it will be denoted by l_ϕ. The space l_Φ equipped with the Luxemburg norm

$$\|x\| = \|x\|_\Phi := \inf\{\lambda > 0 : \rho_\Phi(x/\lambda) \le 1\} \tag{0.2}$$

is a Banach space.

Observe that the assumption $\phi_n(1) = 1$ for every $n \in \mathbb{N}$ is not a real restriction on Musielak-Orlicz function Φ. In fact, for every sequence $\Phi = (\phi_n)$, where ϕ_n are Orlicz functions, there exists a function $\Psi = (\psi_n)$ with $\psi_n(1) = 1$ and such that l_Φ is isometric to l_Ψ. It is enough to take $\psi_n(t) = \phi_n(a_n t)$, where $\phi_n(a_n) = 1$ for every $n \in \mathbb{N}$.

We will also consider here the finite dimensional spaces $l_\Phi^{(m)}$, defined on $\mathbb{R}^m$ analogously as l_Φ. The space $l_\Phi^{(m)}$ can be identified with the subspace of l_Φ consisting of all $x = (x_n) \in l_\Phi$ such that $x_n = 0$ for all $n \ge m + 1$.

An important subspace of l_Φ, called the *subspace of finite elements* and denoted by h_Φ is defined as

$$h_\Phi = \{x \in l_\Phi : \rho_\Phi(\lambda x) < \infty \ \text{ for any } \lambda > 0\}. \tag{0.3}$$

It is well known that h_Φ is a closed separable subspace of l_Φ with the Schauder basis consisting of the standard unit vectors $e_i = (0, \dots, 1_i, 0, \dots)$. It is easy to see that for every $x \in h_\Phi$, $\|x\| = 1$ if and only if $\rho_\Phi(x) = 1$. Moreover, $h_\Phi = l_\Phi$ if and only if either the dimension of l_Φ is finite or Φ satisfies a growth condition called δ_2 [19, 22], that is there exist $K, \delta > 0$ and a nonnegative sequence $(c_n) \subset l_1$ such that for every $n \in \mathbb{N}$ and every $t \ge 0$

$$\phi_n(2t) \le K\phi_n(t) + c_n, \tag{0.4}$$

whenever $\phi_n(t) \le \delta$.

Recall that for every $y \in l_{\Phi^*}$, the functional

$$f_y(x) = \sum_{n=1}^{\infty} x_n y_n, \qquad x = (x_n) \in l_\Phi,$$

is bounded on $(l_\Phi, \| \ \|_\Phi)$ and is called a *regular functional.* We denote by R_Φ the set of all regular functionals on l_Φ. The spaces R_Φ and l_{Φ^*} are order isomorphic [14] and so by usual identification we often write $f_y = y$. We say that f_S is a *singular functional* on l_Φ whenever $f_S(x) = 0$ for every $x \in h_\Phi$. The set of all singular functionals on l_Φ will be denoted by S_Φ. It is well known [14, Lemma 1.1 and Theorem 2.9] that

$$(l_\Phi)^* = R_\Phi \oplus_1 S_\Phi.$$

More precisely, for each $f \in (l_\Phi)^*$ there exist uniquely determined $r(f) \in R_\Phi$ and $s(f) \in S_\Phi$ such that

$$f = r(f) + s(f),$$

and

$$\|f\| = \|r(f)\| + \|s(f)\|.$$

More information on Musielak-Orlicz spaces one can find in [5], [19], [22], [23], [28].

The following description of supporting functionals can be deduced from [14, Lemma 1.7 and Theorem 1.9]. Set for any $i \in \mathbb{N}$, and $x \in l_\Phi$

$$\Delta\Phi_i(x) = [\phi_i^-(x_i), \phi_i^+(x_i)],$$

if $x_i \geq 0$ and

$$\Delta\Phi_i(x) = [\phi_i^+(x_i), \phi_i^-(x_i)],$$

if $x_i < 0$. Here for any $i \in \mathbb{N}$ and $x \in \mathbb{R}$ $\phi_i^+(x)$ ($\phi_i^-(x)$, resp.) denote the right-hand side (the left hand-side, resp.) derivative of ϕ_i at x.

Theorem 0.2. *Let $\Phi = (\phi_n)$ be a Musielak-Orlicz function and let $x = (x_n) \in h_\Phi$, $\|x\|_\Phi = 1$. Assume that $(\phi_n)'(0) = 0$ for any $n \in \mathbb{N}$. Set*

$$A_x = \{i \in \mathbb{N} : x_i \neq 0\}.$$

Then any supporting functional f_x of x in l_Φ (with respect to $\| \cdot \|_\Phi$) is a regular functional. Moreover, for any $y = (y_n) \in l_\Phi$

$$f_x(y) = (\sum_{i \in A_x} d_i y_i)/(\sum_{i \in A_x} d_i x_i),$$

where $d_i \in \Delta\Phi_i(x)$ for any $i \in A_x$ and $\sum_{i \in A_x} d_i x_i < \infty$.

From [14, Th.3.1] we can easily obtain

Theorem 0.3. *$(l_\Phi, \| \cdot \|_\Phi)$ is smooth if and only if Φ satisfies δ_2 condition and for any $i \in \mathbb{N}$ ϕ_i is differentiable in [0,1).*

Also we need a well known

Theorem 0.4. *$(l_\Phi, \| \cdot \|_\Phi)$ is reflexive if and only if Φ and Φ^* satisfy δ_2 condition, that is if and only if $l_\Phi = h_\Phi$ and $l_{\Phi^*} = h_{\Phi^*}$.*

In the sequel we will apply the following results.

Theorem 0.5. *[15, 17] $(h_\Phi, \| \cdot \|_\Phi)$ or $(l_\Phi^{(m)}, \| \cdot \|_\Phi)$ is strictly convex if and only if all functions ϕ_n but one are strictly convex in $[0, 1)$. An Orlicz space $(h_\phi, \| \cdot \|_\Phi)$ or $(l_\phi^{(m)}, \| \cdot \|_\phi)$ for $m \geq 3$ is strictly convex if and only if ϕ is strictly convex on the interval $[0, u_o]$, where $\phi(u_o) = 1/2$.*

Theorem 0.6. *(The Šmulian Theorem) Assume X is a Banach space and let $x \in S(X)$ be a smooth point. If $f_n, g_n \in S(X^*)$ are such that $f_n(x) \to 1$ and $g_n(x) \to 1$, then $f_n - g_n \to 0$ weakly* in X^*.*

Theorem 0.7. *(The Mazur Theorem) Let X be a separable Banach space. Then the set of all smooth points of X is a dense G_δ subset of X.*

1. General results.

We start from

Lemma 1.1. *Let X be a Banach space. Let for $n \in \mathbb{N}$ $x_n \in X$ and let $\|x_n - x\| \to 0$ for some $x \in X$. For each $n \in \mathbb{N}$ fix $f_n \in S(X^*)$ with $f_n(x_n) = \|x_n\|$. Set for $n \in \mathbb{N}$*

$$A_n = cl(\{f_k : k \geq n\}),$$

where the closure is taken with respect to the weak-$*$ topology in X^*. Then for any $f \in A(\{f_n\}) = \bigcap_{n=1}^{\infty} A_n$, $f(x) = \|x\|$ and $\|f\| = 1$, if $x \neq 0$.

Proof. If $x = 0$, then obviously $f(x) = \|x\|$ for any $f \in A(\{f_n\})$. If $x \neq 0$, then $x_n/\|x_n\| \to x/\|x\|$. Hence without loss of generality we can assume that $\|x_n\| = \|x\| = 1$. Let $f \in A(\{f_n\})$. By the Banach-Alaoglu Theorem $\|f\| \leq 1$. Fix $\epsilon > 0$ and $n_o \in \mathbb{N}$ with $\|x_n - x\| \leq \epsilon/2$ for $n \geq n_o$. Also there exists $k \geq n_o$ with $|(f_k - f)x| \leq \epsilon/2$. Note that

$$|1 - f(x)| \leq |f_k(x_k - x)| + |(f_k - f)x|$$

$$\leq \|f_k\|\|x_k - x\| + |(f_k - f)x| \leq \epsilon.$$

Consequently, $f(x) = \|x\| = 1 = \|f\|$, as required.

Lemma 1.2. *Let $V \subset X$ be a linear subspace of a Banach space X. Take $x, y \in X$ satisfying*

$$\|y - v\| \leq \|x - v\|$$

for any $v \in V$. Then for any $v \in V$ there exists $f_v \in S(X^*)$, $f_v(v) = \|v\|$ and $f_v(x - y) = 0$.

Proof. Fix $v \in V$. Without loss of generality, we can assume that $v \neq 0$. Since V is a linear subspace, for any $k \in \mathbb{R}$

$$\|y - kv\| \leq \|x - kv\|.$$

Hence for any $k \neq 0$,

$$\|y/k - v\| \leq \|x/k - v\|.$$

Let $k \in \mathbb{N} \setminus \{0\}$. Choose $f_k \in S(X^*)$ such that

$$f_k(x/k - v) = \|x/k - v\|.$$

Note that

$$f_k(y/k - v) \leq \|y/k - v\| \leq \|x/k - v\|.$$

Hence $f_k(x/k - y/k) \geq 0$, which gives

$$f_k(y-x) \leq 0$$

for any $k \in \mathbb{N} \setminus \{0\}$. Observe that $\|x/k - v - (-v)\| \to 0$. By Lemma 1.1 and the Banach - Alaoglu Theorem there exists $f_+ \in A(\{f_k\})$ such that $f_+(-v) = \|v\|$. Since $f_+ \in A(\{f_k\})$, $f_+(y-x) \leq 0$. For $k \in \mathbb{Z} \setminus \mathbb{N}$, by the above reasoning, $f_k(x/k - y/k) \geq 0$, and consequently

$$f_k(y-x) \geq 0.$$

Again, by Lemma 1.1 and the Banach - Alaoglu Theorem, there exists $f_- \in S(X^*)$, $f_-(-v) = \|v\|$ such that

$$f_-(y-x) \geq 0.$$

If $f_-(y-x) = 0$ or $f_+(y-x) = 0$, the lemma is proved. In the opposite case, there exists $a \in (0,1)$ such that

$$af_-(x-y) + (1-a)f_+(x-y) = 0$$

Set

$$f_v = -(af_- + (1-a)f_+).$$

Obviously, $\|f_v\| = f_v(v/\|v\|) = 1$ and $f_v(y-x) = 0$. The lemma is proved.

Theorem 1.3. *Let X be a real Banach space and let $V \subset X$, be a linear subspace. Assume that V is an existence set and $V \neq \{0\}$. Put*

$$G_V = \{v \in V \setminus \{0\} : \text{ there exists exactly one } f \in S(X^*) : f(v) = \|v\|\}. \tag{1.1}$$

Assume that the norm closure of G_V in X is equal to V. Then there exists exactly one projection $P \in \mathcal{P}(X,V)$ such that $\|P\| = 1$.

Proof. Fix $x \in X$. Since V is an existence set, there exists $y \in V$ such that

$$\|y - v\| \leq \|x - v\|$$

for any $v \in V$. By Lemma 1.2 applied to x and y for any $v \in V$ there exists $f_v \in S(X^*)$, $f_v(v) = \|v\|$, such that

$$f_v(x-y) = 0. \tag{1.2}$$

Now we show that there exists exactly one $y \in V$ satisfying (1.2) for any $v \in V$. Assume, on the contrary, that there exist $y_1, y_2 \in V$, $y_1 \neq y_2$ satisfying (1.2) for any $v \in V$. Since $cl(G_V) = V$, there exists $\{z_n\} \subset G_V$, with

$$\|z_n - (y_1 - y_2)\| \to 0. \tag{1.3}$$

By (1.2), for any $n \in \mathbb{N}$ there exists $f_n^1 \in S(X^*)$ and $f_n^2 \in S(X^*)$, with $f_n^i(z_n) = \|z_n\|$ for $i = 1,2$ such that $f_n^i(x - y_i) = 0$. Since $z_n \in G_V$, $f_n^1 = f_n^2$. Hence for any $n \in \mathbb{N}$

$$f_n^1(y_1 - y_2) = 0. \tag{1.4}$$

Since $\|z_n - (y_1 - y_2)\| \to 0$, and $f_n^1(z_n) = \|z_n\|$, by Lemma 1.1, for any $f \in A(\{f_n^1\})$

$$f(y_1 - y_2) = \|y_1 - y_2\|. \tag{1.5}$$

By (1.4), $f(y_1 - y_2) = 0$, which gives, $y_1 = y_2$; a contradiction.
Now, let for any $x \in X$, Px denote the only element $y \in V$ satisfying (1.2) for any $v \in V$. We show that P is a linear mapping. To do this, fix $x_1, x_2 \in X$ and $v \in G_V$. Note that $f_v(x_1 - Px_1) = 0$ and $f_v(x_2 - Px_2) = 0$, where f_v denote the only supporting functional for v in X^*. Consequently, for any $v \in G_V$, $a_1, a_2 \in \mathbb{R}$,

$$f_v(a_1x_1 + a_2x_2 - (a_1Px_1 + a_2Px_2)) = 0. \tag{1.6}$$

Now assume $v \in V \setminus G_V$. Since the norm closure of G_V is equal to V, there exists $\{z_n\} \subset G_V$, $\|z_n - v\| \to 0$. By Lemma 1.1, the Banach - Alaoglu Theorem and (1.6), there exists $f \in S(X^*)$, $f(v) = \|v\|$, such that

$$f(a_1x_1 + a_2x_2 - (a_1Px_1 + a_2Px_2)) = 0. \tag{1.7}$$

Since for any $x \in X$ there exists exactly one element satisfying (1.2) for any $v \in V$, by (1.6) and (1.7),

$$P(a_1x_1 + a_2x_2) = a_1P(x_1) + a_2P(x_2),$$

which shows that P is a linear mapping. Taking $v = 0$ we get $\|Px\| \le \|x\|$. Since $Pv = v$ for any $v \in V$ and $V \ne \{0\}$, we get $\|P\| = 1$. By the above proof there exists exactly one projection $P \in \mathcal{P}(X, V)$ of norm one. The proof is complete.

Corollary 1.4. *Assume X is a smooth space. If a linear subspace $V \subset X$, $V \ne \{0\}$, is an existence set then V is one-complemented in X.*

Proof. Note that in our case $G_V = V \setminus \{0\}$. Hence the statement follows immediately from Theorem 1.3.

Corollary 1.5. *Assume that X is a strictly convex Banach space such that X is one-complemented in X^{**}. (In particular, X can be a reflexive space or by $X = Z^*$ for some Banach space Z.) If a linear subspace $V \subset X$, $V \ne \{0\}$, is an optimal set satisfying the requirements of Theorem 1.3, then V is one-complemented.*

Proof. By [3, Prop. 2] V is an existence set in X. By Theorem 1.3, V is one-complemented.

Now we present another class of subspaces satisfying the assumptions of Theorem 1.3.

Definition 1.6. *Let X be a Banach space and let $V \subset X$ be a linear subspace. V is called weakly separating if any $g \in ext(V^*)$ has exactly one Hahn-Banach extension $f \in S(X^*)$.*

Theorem 1.7. *Let $V \subset X$, $V \ne \{0\}$ be a separable, weakly separating subspace. Then the norm closure of G_V is equal to V.*

Proof. Since V is separable, by Theorem 0.7, the set

$$G_{V,1} = \{v \in V \setminus \{0\} : \text{ there exists exactly one } g \in S(V^*), g(v) = \|v\|\} \tag{1.8}$$

is dense in V. To finish the proof, it is enough to show that $G_{V,1} = G_V$. Fix $v \in G_{V,1}$ and $g \in S(X^*)$ satisfying $g(v) = \|v\|$. Since $v \in G_{V,1}$, $g \in ext(V^*)$. Now we show that there exists exactly one $f \in S(X^*)$ such that $f(v) = \|v\|$, which means that $v \in G_V$. Indeed, assume there exist $f_1, f_2 \in S(X^*)$, $f_1 \neq f_2$ such that $f_i(v) = \|v\|$ for $i = 1, 2$. Hence $f_i|_V(v) = \|v\|$ for $i = 1, 2$. Since $v \in G_{V,1}$, $f_1|_V = f_2|_V = g$. Since $g \in ext(V^*)$, and V is weakly separating, $f_1 = f_2$; a contradiction. It is obvious, that $G_V \subset G_{V,1}$. The proof is complete.

Now we present some examples of weakly separating subspaces of $C(E)$, where E is a compact set and $C(E)$ is the space of continuous, real-valued functions defined on E equipped with the supremum norm.

Theorem 1.8. *[21] A linear subspace $V \subset C(E)$, is weakly separating if and only if for any $t_1, t_2 \in A_V$, $t_1 \neq t_2$, there exist $v_1, v_2 \in V$ such that*

$$v_1(t_1) \neq v_1(t_2)$$

and

$$v_2(t_1) \neq -v_2(t_2).$$

Here

$$C_V = \{t \in E : \hat{t}|_V \in ext(V^*)\}.$$

Example 1.9. *Let $V \subset C(E)$ be a linear subspace such that $1 \in V$ and $t_1|_V \neq t_2|_V$ for any $t_1, t_2 \in E$. Then V is weakly separating.*

Example 1.10. *Let $V \subset C(E)$ be a Haar subspace of dimension ≥ 2. Then V is weakly separating.*

2. Particular cases.

Now applying Theorem 1.3, we show that in c and c_o any linear subspace $V \neq \{0\}$, which is an existence set must be one-complemented. First we need a well known

Lemma 2.1. *Let X and Y be two Banach spaces and let $T : X \to Y$ be a linear isometry. Let $V \subset X$. Then V is an existence set in X if and only if $T(V)$ is an existence set in $T(X)$. If $V \subset X$, $V \neq \{0\}$, is a linear space then V is one-complemented in X if and only if $T(V)$ is one-complemented in $T(X)$.*

Proof. Since V is an existence set for any $x \in X$ there exists $Qx \in V$ such that for any $v \in V$

$$\|v - Qx\| \leq \|v - x\|.$$

Hence

$$\|T(v) - T(Qx)\| \leq \|T(v) - T(x)\|$$

for any $v \in V$. This means that $T(V)$ is an existence set in $T(X)$. Applying the above reasoning to $T(X)$ and T^{-1} we get the first claim of our lemma.
Now suppose that V is one-complemented subspace of X. Take a projection $P_o \in \mathcal{P}(X, V)$, $\|P_o\| = 1$. Set $P_1 = T \circ P_o \circ T^{-1}$. Then obviously $P_1 \in \mathcal{P}(T(X), T(V))$, and $\|P_1\| = 1$. The converse is obvious.

Theorem 2.2. *Let $V \subset c$, $V \neq \{0\}$, be a linear subspace which is an existence set. Then V is one-complemented in c.*

Proof. Since by the Hahn-Banach theorem any one-dimensional subspace of c is one-complemented, we can assume that $dim(V) \geq 2$. For each $i \in \mathbb{N}$ set

$$C_{i,1} = \{j \in \mathbb{N}, j \neq i, v_i = v_j \text{ for any } v \in V\},$$

$$C_{i,2} = \{j \in \mathbb{N}, j \neq i, v_i = -v_j \text{ for any } v \in V\}$$

and

$$C_i = \{i\} \cup C_{i,1} \cup C_{i,2}.$$

Observe that for any $i, j \in \mathbb{N}$,

$$C_i \cap C_j = \emptyset \text{ or } C_i = C_j. \tag{2.1}$$

Also it is clear that

$$\bigcup_{i=1}^{\infty} C_i = \mathbb{N}.$$

First assume that for any $n \in \mathbb{N}$

$$\bigcup_{i=1}^{n} C_i \neq \mathbb{N}. \tag{2.2}$$

Define

$$i_1 = 1 = min(C_1),$$

$$i_2 = min(\mathbb{N} \setminus C_1),$$

and

$$i_n = min(\mathbb{N} \setminus \bigcup_{j=1}^{n-1} C_{i_j}).$$

Note that by (2.2), the above definition is correct. Put

$$N_1 = \{n \in \mathbb{N} : card(C_{i_n}) < \infty\}.$$

Since $V \subset c$, $N_1 = \mathbb{N}$ or there exists exactly one $n_o \in \mathbb{N}$ such that

$$N_1 = \mathbb{N} \setminus \{n_o\}. \tag{2.3}$$

Set

$$V_1 = \{x \in c : \text{ there exists } v \in V, x_j = v_{i_j} \text{ for } j \in N_1\}. \tag{2.4}$$

Note that for any $x \in V_1$ there exists exactly one $v \in V$ such that (2.4) is satisfied. Let us denote it by Lx. Observe that L is a linear isometry between V_1 and V equal to the identity mapping if $C_i = \{i\}$ for any $i \in \mathbb{N}$.

Now we show that V_1 is an existence set in c. First we assume that $N_1 = \mathbb{N}$. Take $x \in c$. Define a sequence Tx by

$$T(x)_k = \begin{cases} x_j, & \text{if } k \in C_{i_j,1} \cup \{i_j\}, \\ -x_j & \text{if } k \in C_{i_j,2}. \end{cases} \tag{2.5}$$

By (2.1), $T(x)$ is properly defined. If $N_1 = \mathbb{N} \setminus \{n_o\}$ (see (2.3)), we modify our definition for $k \in C_{i_{n_o},i}$, if $\operatorname{card}(C_{i_{n_o},i}) = \infty$, for $i = 1$ or $i = 2$ setting

$$T(x)_k = \lim_n x_n. \tag{2.6}$$

By (2.2), (2.5) and (2.6) T is a linear isometry from c into c. Observe that $T|_{V_1} = L$. Since V is an existence set in c, $V = T(V_1) \subset T(c)$ is an existence set in $T(c)$. By Lemma 2.1, $V_1 = T^{-1}(V)$ is an existence set in c.

Now we consider two cases.

Case I. For any $i \in \mathbb{N}$ there exists $v \in V_1$ and $w \in V_1$ with $v_i \neq lim_n v_n$ and $w_i \neq -lim_n w_n$. We show that V_1 is a weakly separating subspace of c.Note that c is isometric to $C(E)$, where $E = \{0, 1/n : n \in \mathbb{N}\}$ and an isometry $I : c \to C(E)$ is given by:

$$(Ix)(0) = lim_n(x) \text{ and } (Ix)(1/n) = x_n.$$

Hence it is enough to show that $I(V_1)$ is a weakly separating subspace of $C(E)$. To do this, we apply Theorem 1.8. Take any $t_1, t_2 \in C_{V_1}$, (see Th.1.8) $t_1 \neq t_2$. If $t_i \neq 0$ for $i = 1, 2$, by the construction of V_1, there exists $v_1 \in V_1$, such that

$$I(v_1)(t_1) \neq I(v_1)(t_2)$$

and $v_2 \in V_1$ with

$$I(v_2)(t_1) \neq -I(v_2)(t_2).$$

If $t_1 = 0$ or $t_2 = 0$, we can find $v_1 \in V_1$ and $v_2 \in V_1$ satisfying the above conditions, because we consider the Case I. By Theorem 1.8, $I(V_1)$ is a weakly separating subspace of $C(E)$ and consequently V_1 is a weakly separating subspace of c. By Theorems 1.3, 1.7 and Lemma 2.1 $V = T(V_1)$ is one-complemented in $T(c)$. Let $P_1 \in \mathcal{P}(T(c), V)$ be a projection of norm one. Set

$$Q_1 = P_1 \circ T \circ R,$$

where

$$Rx = (x_{i_1}, x_{i_2}, ...)$$

for $x \in c$ and $i_1, i_2, ... \in N_1$. It is clear that $\|Q_1\| \leq 1$. Since $R|_V = T^{-1}|_V = L^{-1}$, Q_1 is a norm-one projection from $\mathcal{P}(c, V)$, which completes our proof in this case.

Case II. There exists $i \in \mathbb{N}$ such that for any $w \in V_1$ $w_i = \lim_n w_n$ or there exists $i \in \mathbb{N}$ such that for any $w \in V_1$ $w_i = -\lim_n w_n$. By definition of V_1, there exists exactly one $i \in \mathbb{N}$ satisfying the above condition. Without loss of generality we can assume that $i = 1$ and that for any $w \in V$, $w_1 = \lim_n w_n$. Let $S : c \to c$ be given by :

$$S(x) = (\lim_n x_n, x). \tag{2.7}$$

Note that $V_1 \subset S(c)$. Set

$$V_2 = \{(v_2, ...) : v = (v_1, v_2, ...) \in V_1\}.$$

It is clear that $V_1 = S(V_2)$. Since V_1 is an existence set in c, and $V_1 \subset S(c)$, V_1 is an existence set in $S(c)$. By Lemma 2.1, V_2 is an existence set in c. Reasoning as in the Case I, we get that V_2 is a weakly separating subspace of c. By Theorems 1.3 and 1.7 there exists $P_2 : c \to V_2$, a linear projection of norm one. Define $R_1 : c \to V_1$ by

$$R_1(x) = (S \circ P_2)(x_2, ...).$$

It is clear that $\|R_1\| = 1$, $R_1(c) \subset V_1$ and $R_1|_{V_1} = id_{V_1}$. Hence V_1 is one-complemented in c. Reasoning as in the Case I, we get that V is one-complemented in c. This completes the proof of our theorem under assumption (2.2).
If (2.2) is not satisfied, without loss of generality we can assume that

$$\mathbb{N} = \bigcup_{j=1}^{n} C_{i_j},$$

where $1 = i_1 < i_2 <, \dots, < i_n$ and $C_{i_j} \cap C_{i_k} = \emptyset$ for $j \neq k$. Let $T : l_\infty^{(n)} \to c$ be defined by (2.5). Note that $V \subset T(l_\infty^{(n)})$. Let $V_1 = T^{-1}(V)$. Reasoning as in the first part of the proof, we get that V_1 is an existence set in $l_\infty^{(n)}$ and V_1 is a weakly separating subspace of $l_\infty^{(n)}$. By Theorems 1.3 and 1.7 there exists a norm-one projection $P_3 \in \mathcal{P}(l_\infty^{(n)}, V_1)$. Set

$$Q_2 = T \circ P_3 \circ R.$$

Here $R : c \to l_\infty^{(n)}$ is given by

$$Rx = (x_{i_1}, ..., x_{i_n}).$$

Since $R|_V = T^{-1}|_V$, $Q_2 \in \mathcal{P}(c, V)$ and $\|Q_2\| = 1$. The proof is complete.

In analogous way we can prove

Theorem 2.3. *Let $V \subset c_o$, $V \neq \{0\}$, be a linear subspace which is an existence set. Then V is one-complemented in c_o.*

Now we consider the case $X = l_1$.

Theorem 2.4. *Let $V \subset l_1$, $V \neq \{0\}$, be a linear subspace, which is an existence set. Then V is one-complemented in l_1.*

Proof. Set for $x \in l_1$

$$A_x = \{n \in \mathbb{N} : x_n = 0\} \tag{2.8}$$

and

$$A_V = \bigcup_{v \in V} A_v.$$

Without loss of generality we can assume that $A_V = \mathbb{N}$. We show that $G_{V,1} = G_V$ (see (1.1) and (1.8)). Since $(l_1)^* = l_\infty$, $v \in G_{V,1}$ if and only if $A_v = A_V = \mathbb{N}$. Hence $v \in G_V$. By Theorems 0.7, 1.3 and 1.7, V is one-complemented in l_1.

Now we consider the case of Musielak-Orlicz sequence spaces equipped with the Luxemburg norm (see (0.2)). For a linear subspace $V \subset l_\Phi$ (see (0.1)) and $n \in \mathbb{N}$ define $I_n \in V^*$ by

$$I_n = (y \to y_n)|_V. \tag{2.9}$$

We start from

Lemma 2.5. *Let $V \subset h_\Phi$ (see (0.3)) be a linear subspace, dim$(V) \geq 2$. Let $v \in G_{V,1}$, (see (1.8)), $\|v\| = 1$ and card$(A_v) \geq 2$. Then $v \in G_{V,1} \setminus G_V$ if and only if there exists $n_o \in \mathbb{N}$ such that $I_{n_o} \neq 0$, card$(\Delta\Phi_{n_o}(v)) > 1$ and v is a norming point for a functional $J_{n_o} = I_{n_o}/\|I_{n_o}\|$.*

Proof. Take $v \in G_{V,1} \setminus G_V$ with $\|v\| = 1$ and card$(A_v) \geq 2$. By Theorem 0.2, there exists $n_o \in A_v$ and $c_{n_o}, d_{n_o} \in \Delta\Phi_{n_o}(v)$, $c_{n_o} \neq d_{n_o}$. Define for $y \in l_\Phi$

$$f_1(y) = (\sum_{n \in A_v, n \neq n_o} d_n y_n + d_{n_o} y_{n_o})/(\sum_{n \in A_v} d_n v_n),$$

and

$$f_2(y) = (\sum_{n \in A_v, n \neq n_o} d_n y_n + c_{n_o} y_{n_o})/(\sum_{n \in A_v, n \neq n_o} d_n v_n + c_{n_o} v_{n_o}),$$

where $d_n \in \Delta\Phi_n(v)$ for any $n \in A_v$, $n \neq n_o$. Note that by Th.0.2, f_1 and f_2 are two different supporting functionals for v. Since $v \in G_{V,1} \setminus G_V$, $f_1|_V = f_2|_V$. We show that $f_1|_V = J_{n_o}$. If not, there exists $w \in V$ such that $f_1(w) \neq 0$ and $I_{n_o}(w) = 0$. Hence

$$f_1(w) = (\sum_{n \in A_v, n \neq n_o} d_n w_n)/(\sum_{n \in A_v} d_n v_n),$$

and

$$f_2(w) = (\sum_{n \in A_v, n \neq n_o} d_n w_n)/(\sum_{n \in A_v, n \neq n_o} d_n v_n + c_{n_o} v_{n_o}).$$

Since $v_{n_o} \neq 0$ and $c_{n_o} \neq d_{n_o}$, $f_1(w) \neq f_2(w)$; a contradiction.
To prove the converse, assume that $v \in G_{V,1}$, $\|v\| = 1$, is a norming point for $I_{n_o} \neq 0$ and card$(\Delta\Phi_{n_o}(v)) > 1$. Fix $c, d \in \Delta\Phi_{n_o}(v)$, $c \neq d$ and $d_n \in \Delta\Phi_n(v)$, for $n \in A_v$, $n \neq n_o$. Set

$$f_1(y) = (\sum_{n \in A_v, n \neq n_o} d_n y_n + d y_{n_o})/(\sum_{n \in A_v, n \neq n_o} d_n v_n + d v_{n_o})$$

and

$$f_2(y) = (\sum_{n \in A_v, n \neq n_o} d_n y_n + c y_{n_o})/(\sum_{n \in A_v, n \neq n_o} d_n v_n + c v_{n_o}).$$

By Th. 0.2, f_1, f_2 are supporting functionals for v in l_Φ. Since $f_1(e_{n_o}) \neq f_2(e_{n_o})$, $v \in G_{V,1} \setminus G_V$, as required.

Theorem 2.6. *Let Φ be a Musielak - Orlicz function. Let $V \subset h_\Phi$, $V \neq \{0\}$, be a linear subspace which is an existence set. Set*

$$N_1 = \{n \in \mathbb{N} : I_n \neq 0, card(\Delta\Phi_n(\|I_n\|)) > 1\} \tag{2.10}$$

and for any $n \in N_1$

$$Z_n = \{v \in S(V) : J_n(v) = 1\}. \tag{2.11}$$

Assume that for any $n \in N_1$, int(Z_n) in $S(V)$ is empty. If $\liminf_n \|I_n\| > 0$ or if there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n are differentiable in $[0, a]$, for $n \geq n_o$, then V is one-complemented in l_Φ.

Proof. Note that, by the Hahn-Banach theorem, any one-dimensional subspace of l_Φ is one-complemented. Hence we can assume that $\dim V \geq 2$. We will apply Theorem 1.3. To do this, it is necessary to show that $cl(G_V) = V$. Since h_Φ is separable, by Th. 0.7, $cl(G_{V,1}) = V$. To end the proof, we demonstrate that any $v \in G_{V,1}$, can be approximated by elements from G_V. Fix $v \in G_{V,1} \setminus G_V$, $\|v\| = 1$. Since $dim(V) \geq 2$, we can assume that $\text{card}(A_v) \geq 2$. By Lemma 2.5, there exists $n_o \in \mathbb{N}$ with $I_{n_o} \neq 0$, $\text{card}(\Delta\Phi_{n_o}(v)) \geq 2$ such that v is a norming point for J_{n_o}. Set

$$S_k = \{w \in S(V) : \|w - v\| \leq 1/k\}. \tag{2.12}$$

Now we show that for any $k \in \mathbb{N}$ there exists $w_k \in G_{V,1} \cap S_k$, such that

$$w_k \notin \bigcup_{j=1}^{k} (Z_j). \tag{2.13}$$

Indeed, if there exists $k \in \mathbb{N}$ such that (2.13) is not satisfied for any $w \in G_{V,1} \cap S_k$, by Th. 0.7,

$$S_k = \bigcup_{j=1}^{k} (Z_j \cap S_k). \tag{2.14}$$

Since l_Φ is a Banach space, and V is closed, S_k is a complete, metric space (with respect to the norm topology). By the Baire Property and (2.14), $\text{int}(Z_j)$ in $S(V)$ is nonempty for some $j \in \{1, ..., k\}$; a contradiction.
Now we show that for k sufficiently large there exists $w_k \in S_k \cap G_V$ satisfying (2.13).
First assume that there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n is differentiable for any $n \geq n_o$. Fix for any $k \in \mathbb{N}$ $w_k \in G_{V,1} \cap S_k$. If there exists a subsequence $\{n_k\}$ such that $w_{n_k} \notin G_V$, by Lemma 2.1, w_{n_k} is a norming point for some J_{m_k}. By (2.13), $m_k \to +\infty$. Note that

$$|1 - J_{m_k}(v)| = |J_{m_k}(v_{n_k} - v)| \leq \|v_{n_k} - v\| \to 0.$$

By Theorem 0.6,

$$I_{m_k}/\|I_{m_k}\| = J_{m_k} \to J_{n_o} = I_{n_o}/\|I_{n_o}\|. \tag{2.15}$$

weakly* in V^*. Consequently, $J_{m_k}(v) \to J_{n_o}(v) = 1$. Since $V \subset h_\Phi$, $lim_k v_k = 0$. By (2.15) $\|I_{m_k}\| \to 0$. Since there exist $a > 0$ and $n_o \in \mathbb{N}$ such that ϕ_n is differentiable for any $n \geq n_o$, N_1 is a finite set. Hence for $k \geq k_o$

$$card(\Delta\Phi_{m_k}(\|I_{m_k}\|)) = 1.$$

By Lemma 2.1, $w_k \in G_V$ for $k \geq k_o$: a contradiction.
If $\liminf \|I_n\| > 0$, we proceed in the same way as above. The proof is complete.

Theorem 2.7. *Let $V \subset l_\Phi^{(m)}$, $V \neq \{0\}$, be a linear subspace which is an existence set. Assume that for any $n \in N_1$, (see (2.10)) $int(Z_n)$ in $S(V)$ is empty. Then V is one-complemented in $l_\Phi^{(m)}$.*

Proof. Goes in the same way as the proof of Theorem 2.6.

Now we present some examples in which $\text{int}(Z_n)$ with respect to $S(V)$ is empty. Recall that a convex function $f : \mathbb{R} \to \mathbb{R}$ is called *strictly convex* if for any $x \neq y$ and $a \in (0,1)$,

$$f(ax + (1-a)y) < af(x) + (1-a)f(y). \tag{2.16}$$

Theorem 2.8. *Let Φ be a Musielak-Orlicz function such that ϕ_k are strictly convex for $k > k_o$. Set for any $F \in S((l_\Phi)^*)$, ($l_\Phi^{(m)}$ resp.)*

$$Z_F = \{x \in S(h_\Phi) : F(x) = 1\}. \tag{2.17}$$

Fix $x \in Z_F$. Then dim(span($Z_F - x$)) $< k_o$.

Proof. Assume on the contrary, that $dim(span(Z_F - x)) \geq k_o$. Hence there exists $x_1, ..., x_{k_o+1} \in Z_F$ such that $y_i = x_{i+1} - x_1$, $i = 1, ..., k_o$ are linearly independent. Note that by definition of Z_F,

$$1 = F(\sum_{j=1}^{k_o+1} x_j/(k_o+1)) = \| \sum_{j=1}^{k_o+1} x_j/(k_o+1))\|_\Phi = \sum_{l=1}^{\infty} \phi_l(\sum_{j=1}^{k_o+1} (x_j/(k_o+1))_l)$$

$$\leq (1/(k_o+1)) \sum_{j=1}^{k_o+1} (\sum_{l=1}^{\infty} \phi_l((x_j)_l) \leq (1/(k_o+1)) \sum_{j=1}^{k_o+1} \rho_\Phi(x_j) = 1. \tag{2.18}$$

Since for $k > k_o$ ϕ_k are strictly convex, by (2.18), $(x_j)_l = (x_1)_l$ for $l > k_o$ and $j = 1, ..., k_o+1$. Hence for $j = 1, ..., k_o$,

$$y_j = x_{j+1} - x_1 = ((x_j - x_1)_1, ..., (x_j - x_1)_{k_o}, 0, ...,).$$

Since y_j, $j = 1, ..., k_o$ are linearly independent,

$$det((y_j)_k)_{j,k=1,...,k_o} \neq 0. \tag{2.19}$$

Because of (2.18), for any $l = 1, ..., k_o$ ϕ_l is linear in some interval $E_l = [c_l, d_l]$, $c_l < d_l$ containing all the coordinates $(x_j)_l$, $j = 1, ..., k_o+1$. Assume that for $l = 1, ..., k_o+1$ and $x \in E_l$,

$$\phi_l(x) = a_l x + b_l.$$

By (2.18) for any $j = 1, ..., k_o+1$,

$$\sum_{l=1}^{k_o} a_l (x_j)_l + b_l = 1 - \sum_{l=k_o+1}^{\infty} \phi_l((x_1)_l). \tag{2.20}$$

Since all the functions ϕ_l are even and strictly increasing in $[0, +\infty)$, $a_l \neq 0$ for $l = 1, ..., k_o$. By (2.20) we get for $j = 1, ..., k_o$,

$$\sum_{l=1}^{k_o} a_l (y_j)_l = 0,$$

which contradicts (2.19). The proof is complete.

From Theorem 2.8 we immediately get

Corollary 2.9. *Let Φ satisfy the assumptions of Theorem 2.8 with some $k_o \geq 1$. Then for any subspace $V \subset h_\Phi$ $dim(V) \geq k_o$ and for any $n \in \mathbb{N}$ $int(Z_n)$ in $S(V)$ is empty. In particular if $k_o \leq 2$ and Φ satisfy the assumptions of Theorem 2.6 any subspace of h_Φ which is an existence set is one-complemented in l_Φ.*

Corollary 2.10. *Let $k_o \leq 2$ and let $V \subset l_\Phi^{(m)}$, $V \neq \{0\}$, be a linear subspace which is an existence set. Then V is one-complemented. Moreover, there exists exactly one projection of norm one from $l_\Phi^{(m)}$, onto V. If $l_\Phi^{(m)}$, is strictly convex (see Th. 0.5) then any subspace of $l_\Phi^{(m)}$, which is an optimal set is one-complemented.*

Proof. Follows immediately from [3, Prop.2] Theorems 2.7 and 2.8.

Corollary 2.11. *Let $k_o \leq 2$ and let $V \subset h_\Phi$, $V \neq \{0\}$, be a linear subspace which is an existence set. If Φ satisfies the requirements of Theorem 2.6, then V is one-complemented. Moreover, there exists exactly one projection of norm one from l_Φ, onto V. If l_Φ, is strictly convex (see Th. 0.5) and reflexive (see Th. 0.4), then any subspace of l_Φ which is an optimal set is one-complemented.*

Proof. By Theorem 0.4 if l_Φ is reflexive then $l_\Phi = h_\Phi$. Then [3, Prop.2], Theorems 2.6 and 2.8 should be applied.

Remark 2.12. *Note that c_o, c and l_1 are not smooth spaces. In general, Musielak-Orlicz or Orlicz spaces satisfying the requirements of Theorem 2.6 and Th.2.7 are also not smooth (compare with Th. 0.3). This shows that Theorem 1.3, which proof is very similar to that of [3, Prop. 5], can be applied to to non-smooth case in many concrete situations.*

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