CANCELLATION THEOREM FOR ALGEBRAIC LINE BUNDLES

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PREPRINT IMUJ 2005/13

Abstract

Let $\mathbb{K}$ be an algebraically closed field of characteristic 0. In this paper we consider the following cancellation problem. Let X be an affine variety and E an algebraic line bundle on X. Suppose that a variety Y and an isomorphism $Y \times \mathbb{K}^m \cong E \times \mathbb{K}^m$ are given. Is it true that $Y \cong E$? We give an affirmative solution if X is either non-$\mathbb{K}$-uniruled or it is unirational and has a non-uniruled component at infinity.

1 Introduction

We shall work in the category of algebraic varieties over an algebraically closed field $\mathbb{K}$ of characteristic 0. By a variety we will usually mean an irreducible algebraic variety.

A variety X has the *cancellation property* if it satisfies the condition: if Y is a variety such that for some $m \geq 0$ there is an isomorphism $X \times \mathbb{K}^m \cong Y \times \mathbb{K}^m$ then $X \cong Y$. Furthermore, we say that a variety X has the *strong cancellation property* if every isomorphism $f : X \times \mathbb{K}^m \to Y \times \mathbb{K}^m$ satisfies the condition: for each $x \in X$ there exists $y \in Y$ such that $f(\{x\} \times \mathbb{K}^m) = \{y\} \times \mathbb{K}^m$ (clearly, then f induces the isomorphism between X and Y).

It is well-known that affine curves have the cancellation property (a more general result was proved by Abhyankar, Eakin and Heinzer in [1]), but surfaces may not have this property (see Danielewski [3]).

Zariski's Cancellation Problem asks if $\mathbb{K}^n$ has the cancellation property. The affirmative answer for $n = 2$ is due to Fujita [6] and Miyanishi-Sugie [15]. For $n > 2$ this problem remains open.

A variety of non-negative logarithmic Kodaira dimension has the strong cancellation property, which was proved by Iitaka and Fujita in [8]. This property has also an affine variety which is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled component at infinity (see [4]).

In this paper we extend the last result, namely, we will prove the following

Theorem 1. *Suppose that an affine variety X is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled component at infinity. Then*
(i) X has the strong cancellation property.
(ii) any algebraic line bundle on X has the cancellation property.

We use the following terminology.

A variety X of positive dimension n is said to be *uniruled* ($\mathbb{K}$-*uniruled*) if there exists a variety W of dimension $n-1$ and a dominant rational map $W \times \mathbb{P}^1(\mathbb{K}) \dashrightarrow X$ (a dominant morphism $W \times \mathbb{K} \to X$). A reducible variety is said to be *uniruled* ($\mathbb{K}$-*uniruled*) if all its irreducible components are uniruled ($\mathbb{K}$-uniruled).

A variety X of dimension n is said to be *unirational* if there exists a dominant rational map $\mathbb{P}^n \dashrightarrow X$.

A projective variety which contains a variety X as an open subset is called a *compactification* of X.

An affine variety X has *a non-uniruled component at infinity* if there exists a compactification $\overline{X}$ of X such that some component of the set $\overline{X} \setminus X$ is non-uniruled (it is well-known that for any compactification $\overline{X}$ of X the set $\overline{X} \setminus X$ is of pure dimension $\dim X - 1$).

In the sequel π_X denotes the projection $X \times \mathbb{K}^m \ni (x,t) \mapsto x \in X$.

2 Proof of Theorem 1.

Lemma 1. *Let X be an affine variety. Suppose that there exists a variety Y and a dominant morphism $f : Y \times \mathbb{K}^m \to X$ such that $f(\{b\} \times \mathbb{K}^m)$ has positive dimension for some $b \in Y$. Then X is $\mathbb{K}$-uniruled. Furthermore, if Y is unirational then X has only uniruled components at infinity.*

Proof. It suffices to give the proof for $m = 1$ (to see this observe that there is a line L in $\mathbb{K}^m$ such that $f(\{b\} \times L)$ has positive dimension, so changing coordinates we may assume that there exists a dominant morphism $(Y \times \mathbb{K}^{m-1}) \times \mathbb{K} \to X$ not contracting of $\{b'\} \times \mathbb{K}$ for some $b' \in Y \times \mathbb{K}^{m-1}$).

The proof is by induction on $r := \dim Y$. Let $n := \dim X$. If $r = n - 1$ then X is $\mathbb{K}$-uniruled by definition. Furthermore, if Y is unirational with a dominant rational map $g : \mathbb{P}^r \dashrightarrow Y$ then we have a dominat morphism $\operatorname{res} g \times \mathrm{id}_{\mathbb{K}} : U \times \mathbb{K} \to Y \times \mathbb{K}$, where U is the domain of g. Consequently we have a dominant morphism $U \times \mathbb{K} \to X$. Since $\mathbb{P}^r \times \mathbb{P}^1$ is a smooth compactification of $U \times \mathbb{K}$ and $(\mathbb{P}^r \times \mathbb{P}^1) \setminus (U \times \mathbb{K})$ is uniruled, for every compactification $\overline{X}$ of X the set $\overline{X} \setminus X$ is uniruled by [11, th.4].

Suppose $r \geq n$. Let $Z = \{y \in Y : \dim f(\{y\} \times \mathbb{K}) = 0\}$. This is a closed subset of Y, because if X is contained in $\mathbb{K}^N$ and $f = (f_1, ..., f_N)$ then $Z = \bigcap_{i=1,...,N} \bigcap_{s,t \in \mathbb{K}} \{y \in Y : f_i(y, s) - f_i(y, t) = 0\}$. In particular, if V is any non-empty open subset in Y then we may always assume that $b \in V$. Let X_0 be an open subset of X such that $\dim f^{-1}(x) = r+1-n$ for each $x \in X_0$. We may assume that $f(\{b\} \times \mathbb{K})$ meets X_0. Choose $a \in f(\{b\} \times \mathbb{K}) \cap X_0$ and a hyperplane section H of X passing through a and not containing $f(\{b\} \times \mathbb{K})$. Clearly, $f^{-1}(H)$ is of pure dimension r. Let S be a component of $f^{-1}(H)$ such that $S \cap f^{-1}(a) \cap (\{b\} \times \mathbb{K}) \neq \emptyset$. Put $\tilde{f} = \operatorname{res} f : S \to H$ and $\tilde{\pi} = \operatorname{res} \pi_Y : S \to Y$. These morphisms are dominant, since $\dim \tilde{f}^{-1}(a) = r + 1 - n$ and $\dim \tilde{\pi}^{-1}(b) = 0$.

First we show that X is $\mathbb{K}$-uniruled. Choose $c \in S \cap f^{-1}(a) \cap (\{b\} \times \mathbb{K})$ and a hypersurface H' contained in S such that $c \in H'$ and $\dim(H' \cap \tilde{f}^{-1}(a)) < \dim \tilde{f}^{-1}(a)$. Then $\operatorname{res} f : H' \to H$ is dominant. Consequently $\operatorname{res} f : \tilde{\pi}(H') \times \mathbb{K} \to X$ is dominant, since $H \cup f(\{y\} \times \mathbb{K})$ is contained in the closure of $f(\tilde{\pi}(H') \times \mathbb{K})$. This proves that X is $\mathbb{K}$-uniruled.

Suppose now that Y is unirational. Let Y_0 be a non-singular open subset of Y such that $\dim \tilde{\pi}^{-1}(y) = 0$ and $\{y\} \times \mathbb{K}$ is not contained in $f^{-1}(H)$ for each $y \in Y_0$. We may assume that there is an open subset U in $\mathbb{P}^r$ and a finite morphism $U \to Y_0$, and that $b \in Y_0$. Let us choose an unirational hypersurface H'' contained in Y_0 such that $b \in H''$ and $\dim(H'' \cap \tilde{\pi}(\tilde{f}^{-1}(a))) < \dim \tilde{f}^{-1}(a)$. Since H'' is locally principal in Y_0 (we have assumed that Y_0 is smooth), $\tilde{\pi}^{-1}(H'')$ is of pure dimension $r - 1$ and $\dim(\tilde{\pi}^{-1}(H'') \cap \tilde{f}^{-1}(a)) < \dim \tilde{f}^{-1}(a)$. Hence $\operatorname{res} f : \tilde{\pi}^{-1}(H'') \to H$ is dominant. So similarly as above, $\operatorname{res} f : H'' \times \mathbb{K} \to X$ is dominant. This concludes the proof. □

Now we come to the following

Problem 1. *Let R be a ring (commutative with identity). Suppose that A is an R-algebra and there is an R-isomorphism of polynomial rings $R[T_1, ...T_{n+1}] \cong A[T_1, ..T_n]$. Is it true that A is R-isomorphic to $R[T_1]$?*

This problem was considered in [1], [2], [9] and [13]. Generally the answer is negative. Asanuma gave a counterexample in [2] if $\operatorname{char} R > 0$ (see also section 4). In the next section we prove that if R is a coordinate ring of a smooth affine variety then the answer is affirmative. Although this is a consequence of more general results from papers mentioned above, we will give a short geometric proof.

In the proof of Theorem 1 we shall use the following

Theorem 2. (Hamann, [9].) *If R is a $\mathbb{Q}$-algebra then Problem 1 has an affirmative solution.*

We will need also an elementary

Lemma 2. *Let X be a variety and $p_i : E_i \to X$ be an algebraic line bundle on X for $i = 1, 2$. Then $E_1 \cong E_2$ as line bundles on X provided that there exists $m \geq 0$ and an isomorphism $f : E_1 \times \mathbb{K}^m \to E_2 \times \mathbb{K}^m$ such that the following diagram is commutative*

$$\begin{array}{ccc} E_1 \times \mathbb{K}^m & \xrightarrow{\ f\ } & E_2 \times \mathbb{K}^m \\ {\scriptstyle \pi_{E_1}}\downarrow & & \downarrow{\scriptstyle \pi_{E_2}} \\ E_1 & & E_2 \\ & {\scriptstyle p_1}\searrow \qquad \swarrow{\scriptstyle p_2} & \\ & X & \end{array}$$

Proof. Suppose that E_i is given by an open cover $\{U_\alpha\}$ of X and by transition functions $g^i_{\alpha,\beta} : U_\alpha \cap U_\beta \to \mathbb{K}^*$ for $i = 1, 2$. Clearly, we can identify $E_i \times \mathbb{K}^m$ with the direct sum of E_i and the trivial bundle $X \times \mathbb{K}^m$. Hence

$$G^i_{\alpha,\beta} = \begin{pmatrix} g^i_{\alpha,\beta} & 0 \\ 0 & I_m \end{pmatrix}$$

are transition functions for $E_i \times \mathbb{K}^m$ on $U_\alpha \cap U_\beta$, where I_m is the identity in $\mathrm{GL}(\mathbb{K}^m)$. Obviously, f induces the family of polynomial isomorphisms of trivial bundles $f_\alpha : U_\alpha \times \mathbb{K}^{m+1} \to U_\alpha \times \mathbb{K}^{m+1}$ such that for each $u \in U_\alpha \cap U_\beta$,

$$f_\alpha(u, \cdot\,) G^1_{\alpha,\beta}(u) = G^2_{\alpha,\beta}(u) f_\beta(u, \cdot\,).$$

Denote by $Jf_\alpha(u)$ the Jacobi matrix of $f_\alpha(u, \cdot\,)$ for $u \in U_\alpha$, and by h_α the Jacobian of f_α, i.e., $h_\alpha : U_\alpha \ni u \mapsto \det Jf_\alpha(u, \cdot\,) \in \mathbb{K}^*$. Then for $u \in U_\alpha \cap U_\beta$,

$$h_\alpha(u) g^1_{\alpha,\beta}(u) = g^2_{\alpha,\beta}(u) h_\beta(u).$$

Thus the family $\{h_\alpha\}$ determines the isomorphism between E_1 and E_2. This concludes the proof. □

Proof of theorem 1. The following easy observation will be needed: if V_1 and V_2 are affine varieties and $V_1 \times \mathbb{K}^m \cong V_2 \times \mathbb{K}^m$ then V_1 dominates V_2 and conversely. In particular, if V_1 is unirational then so is V_2.

(i) Let $f : Y \times \mathbb{K}^m \to X \times \mathbb{K}^m$ be an isomorphism. Applying Lemma 1 to $\pi_X \circ f$, we find a morphism $g : Y \to X$ such that $\pi_X \circ f = g \circ \pi_Y$. This implies that X has the strong cancellation property.

(ii) Let $p : E \to X$ be an algebraic line bundle on X and $f : Y \times \mathbb{K}^m \to E \times \mathbb{K}^m$ an isomorphism. Again, by Lemma 1 there exists a morphism $q : Y \to X$ such that the following diagram is commutative

$$\begin{array}{ccc} Y \times \mathbb{K}^m & \xrightarrow{\ f\ } & E \times \mathbb{K}^m \\ & & \downarrow{\scriptstyle \pi_X} \\ \downarrow{\scriptstyle \pi_Y} & & E \\ & & \downarrow{\scriptstyle p} \\ Y & \xrightarrow{\ q\ } & X \end{array}$$

Therefore if E is trivial on an affine open subset U of X then $\operatorname{res} q : q^{-1}(U) \to U$ is a trivial bundle by the geometric version of Theorem 2. Thus $q : Y \to X$ is a line bundle and Lemma 2 concludes the proof. □

Remark 1. *Theorem 1 remains true if we assume that $X \setminus \operatorname{Sing} X$ is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled hypersurface at infinity.*

For any variety X we denote, here and in the sequel, the singular locus by $\operatorname{Sing} X$ and the set of non-singular points by $\operatorname{Reg} X = X \setminus \operatorname{Sing} X$.

A variety X of dimension n has *a non-uniruled hypersurface at infinity* if there exists a compactification $\overline{X}$ of X such that the set $\overline{X} \setminus X$ has a non-uniruled component of dimension $n-1$.

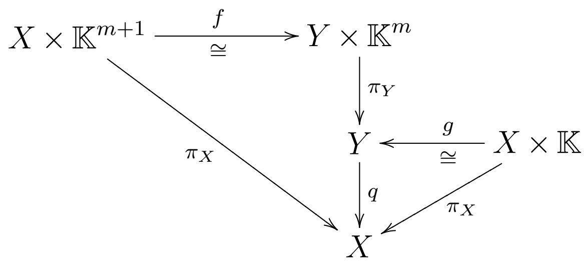
Proof. The above proof works also in this situation, we only need to modify Lemma 1 slightly. Furthermore, the following observation will be needed: if $f : Y \times \mathbb{K}^m \to X \times \mathbb{K}^m$ ($f : Y \times \mathbb{K}^m \to E \times \mathbb{K}^m$, where $p : E \to X$ is a line bundle on X) is an isomorphism then it induces the isomorphism $\operatorname{res} f : \operatorname{Reg} Y \times \mathbb{K}^m \to \operatorname{Reg} X \times \mathbb{K}^m$ ($\operatorname{res} f : \operatorname{Reg} Y \times \mathbb{K}^m \to \operatorname{Reg} E \times \mathbb{K}^m$).

The details are left to the reader. □

3 More about Problem 1

We give now a solution of Problem 1 when R is the coordinate ring of a smooth affine variety over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic.

Proposition 1. *Let X be an affine smooth variety. Let Y be a variety with a morphism $q : Y \to X$ and let $f : X \times \mathbb{K}^{m+1} \to Y \times \mathbb{K}^m$ be an isomorphism such that $\pi_X = q \circ \pi_Y \circ f$. Then there exists an isomorphism $g : X \times \mathbb{K} \to Y$ such that $q \circ g = \pi_X$.*



Proof. First observe that all fibers of q are isomorphic to $\mathbb{K}$. Indeed, f takes $\pi_X^{-1}(x) \cong \mathbb{K}^{m+1}$ onto $q^{-1}(x) \times \mathbb{K}^m$, so $q^{-1}(x) \cong \mathbb{K}$, because affine curves have the cancellation property. Take now the zero section $s_0 : X \ni x \mapsto (x, 0) \in X \times \mathbb{K}^{m+1}$. Then $s : X \ni x \mapsto \pi_Y(f(s_0(x))) \in Y$ is a section of q, i.e., $q \circ s = id_X$. In particular, s is a closed immersion and $\Gamma := s(X)$ meets transversally all fibers of q. We shall show that $q : Y \to X$ is a line bundle.

Since X and Y are smooth, we have isomorphisms $\pi_X^* : \mathrm{Pic}(X) \to \mathrm{Pic}(X \times \mathbb{K}^{m+1})$ and $\pi_Y^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y \times \mathbb{K}^m)$ (see [10, p.134,141]). Hence $q^* : \mathrm{Pic}(X) \to \mathrm{Pic}(Y)$ is an isomorphism too. For a prime divisor Γ on Y there exists a divisor D on X such that Γ and $q^*(D)$ are linearly equivalent. Let $\{U_i\}$ be an open affine cover of X such that $D \cap U_i$ is principal in U_i. Clearly, $q^*(D) \cap q^{-1}(U_i)$ is principal in $q^{-1}(U_i)$, and so is $\Gamma \cap q^{-1}(U_i)$. This implies that the ideal of $\Gamma \cap q^{-1}(U_i)$ is generated in the coordinate ring $\mathbb{K}[q^{-1}(U_i)]$ by some function $F_i \in \mathbb{K}[q^{-1}(U_i)]$. Now F_i restricts to a coordinate function on $\pi^{-1}(x)$ for each $x \in U_i$, because $q^{-1}(x) \cong \mathbb{K}$ and Γ meets $q^{-1}(x)$ at only one point transversally. Therefore the morphism $q^{-1}(U_i) \ni y \mapsto (q(y), F_i(y)) \in U_i \times \mathbb{K}$ is bijective, so it is an isomorphism by Zariski's Main Theorem. Thus we have showed that $q : Y \to X$ is a line bundle. Now Lemma 2 concludes the proof. □

4 Final remarks

We shall begin with

Question 1. Is it true that an affine variety with a non-uniruled component at infinity has the cancellation property?

By Theorem 1 the answer is affirmative if we add the unirationality assumption, as well as for any line bundle on a non-uniruled variety.

It is a good place to mention our result from [5] which is connected with the stable equivalence problem (see also [12])

Let X be a smooth affine variety and let H be a non-uniruled hypersurface in X. Suppose that $f : X \times \mathbb{K}^m \to Y \times \mathbb{K}^m$ is an isomorphism such that $f(H \times \mathbb{K}^m) = H' \times \mathbb{K}^m$, where H' is a hypersurface in a variety Y. Then for each $x \in X$ there exists $y \in Y$ such that $f(\{x\} \times \mathbb{K}^m) = \{y\} \times \mathbb{K}^m$.

Question 2. Suppose that an affine variety X has the strong cancellation property. Does it follow that $X \times \mathbb{K}$ has the cancellation property?

Again, Theorem 1 gives the affirmative answer for some varieties. This question was already considered by Asanuma in [2], where he showed that if $\operatorname{char} \mathbb{K} > 0$ then the answer is no. His example was a rational curve with the coordinate ring $\mathbb{K}[T^n, T^{n+1}]$, where $n > 1$ (this is also a counterexample to Problem 1).

On the other hand if $\operatorname{char} \mathbb{K} = 0$ we have the following

Corollary 1. *If X and Y are affine curves then the surface $X \times Y$ has the cancellation property.*

Proof. The hardest case, when $X \cong Y \cong \mathbb{K}$ follows from [6] and [15].

If X is non-isomorphic to $\mathbb{K}$ then $X \setminus \operatorname{Sing} X$ is non-$\mathbb{K}$-uniruled. (This is a consequence of the following arguments: (1) every smooth affine and $\mathbb{K}$-uniruled curve is isomorphic to $\mathbb{K}$; (2) if C is an affine curve then every non-constant morphism $\mathbb{K} \to C$ is finite and hence surjective.) Thus, by Remark 1, $X \times \mathbb{K}$ has the cancellation property.

The case when neither $X \cong \mathbb{K}$ nor $Y \cong \mathbb{K}$ again follows from Remark 1, since the set $X \times Y \setminus \operatorname{Sing}(X \times Y) = [(\operatorname{Reg} X) \times Y] \cap [X \times \operatorname{Reg} Y)]$ is non-K-uniruled. This concludes the proof. □

We give one more result a proof of which is left as an exercise.

Proposition 2. *Theorem 1 remains true for any affine variety X such that every dominant morphism $X \to X$ is birational (for example, X satisfies this condition if it is of hyperbolic type, i.e., $\overline{\kappa}(X) = \dim X$; see [7, p. 335]).* □

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