Abstract

Let $K$ be an algebraically closed field of characteristic 0. In this paper we consider the following cancellation problem. Let $X$ be an affine variety and $E$ an algebraic line bundle on $X$. Suppose that a variety $Y$ and an isomorphism $Y \times K^m \cong E \times K^m$ are given. Is it true that $Y \cong E$? We give an affirmative solution if $X$ is either non-$K$-uniruled or it is unirational and has a non-uniruled component at infinity.

1 Introduction

We shall work in the category of algebraic varieties over an algebraically closed field $K$ of characteristic 0. By a variety we will usually mean an irreducible algebraic variety.

A variety $X$ has the cancellation property if it satisfies the condition: if $Y$ is a variety such that for some $m \geq 0$ there is an isomorphism $X \times K^m \cong Y \times K^m$ then $X \cong Y$. Furthermore, we say that a variety $X$ has the strong cancellation property if every isomorphism $f : X \times K^m \to Y \times K^m$ satisfies the condition: for each $x \in X$ there exists $y \in Y$ such that $f(\{x\} \times K^m) = \{y\} \times K^m$ (clearly, then $f$ induces the isomorphism between $X$ and $Y$).
It is well-known that affine curves have the cancellation property (a more general result was proved by Abhyankar, Eakin and Heinzer in [1]), but surfaces may not have this property (see Danielewski [3]).

Zariski’s Cancellation Problem asks if $\mathbb{K}^n$ has the cancellation property. The affirmative answer for $n = 2$ is due to Fujita [6] and Miyanishi-Sugie [15]. For $n > 2$ this problem remains open.

A variety of non-negative logarithmic Kodaira dimension has the strong cancellation property, which was proved by Iitaka and Fujita in [8]. This property has also an affine variety which is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled component at infinity (see [4]).

In this paper we extend the last result, namely, we will prove the following

**Theorem 1.** Suppose that an affine variety $X$ is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled component at infinity. Then
(i) $X$ has the strong cancellation property.
(ii) any algebraic line bundle on $X$ has the cancellation property.

We use the following terminology.

A variety $X$ of positive dimension $n$ is said to be uniruled ($\mathbb{K}$-uniruled) if there exists a variety $W$ of dimension $n - 1$ and a dominant rational map $W \times \mathbb{P}^1(\mathbb{K}) \to X$ (a dominant morphism $W \times \mathbb{K} \to X$). A reducible variety is said to be uniruled ($\mathbb{K}$-uniruled) if all its irreducible components are uniruled ($\mathbb{K}$-uniruled).

A variety $X$ of dimension $n$ is said to be unirational if there exists a dominant rational map $\mathbb{P}^n \to X$.

A projective variety which contains a variety $X$ as an open subset is called a compactification of $X$.

An affine variety $X$ has a non-uniruled component at infinity if there exists a compactification $\overline{X}$ of $X$ such that some component of the set $\overline{X} \setminus X$ is non-uniruled (it is well-known that for any compactification $\overline{X}$ of $X$ the set $\overline{X} \setminus X$ is of pure dimension $\dim X - 1$).

In the sequel $\pi_X$ denotes the projection $X \times \mathbb{K}^m \ni (x, t) \mapsto x \in X$.

2 Proof of Theorem 1.

**Lemma 1.** Let $X$ be an affine variety. Suppose that there exists a variety $Y$ and a dominant morphism $f : Y \times \mathbb{K}^m \to X$ such that $f(\{b\} \times \mathbb{K}^m)$ has positive dimension for some $b \in Y$. Then $X$ is $\mathbb{K}$-uniruled. Furthermore, if $Y$ is unirational then $X$ has only uniruled components at infinity.
Proof. It suffices to give the proof for \( m = 1 \) (to see this observe that there is a line \( L \) in \( \mathbb{K}^m \) such that \( f(\{b\} \times L) \) has positive dimension, so changing coordinates we may assume that there exists a dominant morphism \((Y \times \mathbb{K}^{m-1}) \times \mathbb{K} \to X \) not contracting of \( \{b'\} \times \mathbb{K} \) for some \( b' \in Y \times \mathbb{K}^{m-1} \).

The proof is by induction on \( r := \dim Y \). Let \( n := \dim X \). If \( r = n - 1 \) then \( X \) is \( \mathbb{K} \)-uniruled by definition. Furthermore, if \( Y \) is unirational with a dominant rational map \( g : \mathbb{P}^r \to Y \) then we have a dominant morphism \( \text{res} \times \text{id}_\mathbb{K} : U \times \mathbb{K} \to Y \times \mathbb{K} \), where \( U \) is the domain of \( g \). Consequently we have a dominant morphism \( U \times \mathbb{K} \to X \). Since \( \mathbb{P}^r \times \mathbb{P}^1 \) is a smooth compactification of \( U \times \mathbb{K} \) and \( (\mathbb{P}^r \times \mathbb{P}^1) \setminus (U \times \mathbb{K}) \) is uniruled, for every compactification \( X \) of \( X \) the set \( X \setminus X \) is uniruled by [11, th.4].

Suppose \( r \geq n \). Let \( Z = \{ y \in Y : \dim f(\{y\} \times \mathbb{K}) = 0 \} \). This is a closed subset of \( Y \), because if \( X \) is contained in \( \mathbb{K}^N \) and \( f = (f_1, ..., f_N) \) then \( Z = \cap_{i=1,...,N} \cap_{s,t \in \mathbb{K}} \{ y \in Y : f_i(y, s) - f_i(y, t) = 0 \} \). In particular, if \( V \) is any non-empty open subset in \( Y \) then we may always assume that \( b \in V \). Let \( X_0 \) be an open subset of \( X \) such that \( \dim f^{-1}(x) = r + 1 - n \) for each \( x \in X_0 \). We may assume that \( f(\{b\} \times \mathbb{K}) \) meets \( X_0 \). Choose \( a \in f(\{b\} \times \mathbb{K}) \cap X_0 \) and a hyperplane section \( H \) of \( X \) passing through \( a \) and not containing \( f(\{b\} \times \mathbb{K}) \). Clearly, \( f^{-1}(H) \) is of pure dimension \( r \). Let \( S \) be a component of \( f^{-1}(H) \) such that \( S \cap f^{-1}(a) \cap (\{b\} \times \mathbb{K}) \neq \emptyset \). Put \( \tilde{f} = \text{res} f : S \to H \) and \( \tilde{\pi} = \text{res} \pi_Y : S \to Y \). These morphisms are dominant, since \( \dim \tilde{f}^{-1}(a) = r + 1 - n \) and \( \dim \tilde{\pi}^{-1}(b) = 0 \).

First we show that \( X \) is \( \mathbb{K} \)-uniruled. Choose \( c \in S \cap f^{-1}(a) \cap (\{b\} \times \mathbb{K}) \) and a hypersurface \( H' \) contained in \( S \) such that \( c \in H' \) and \( \dim(H' \cap f^{-1}(a)) < \dim \tilde{f}^{-1}(a) \). Then \( \text{res} f : H' \to H \) is dominant. Consequently \( \text{res} f : \tilde{\pi}(H') \times \mathbb{K} \to X \) is dominant, since \( H \cup f(\{y\} \times \mathbb{K}) \) is contained in the closure of \( f(\tilde{\pi}(H') \times \mathbb{K}) \). This proves that \( X \) is \( \mathbb{K} \)-uniruled.

Suppose now that \( Y \) is unirational. Let \( Y_0 \) be a non-singular open subset of \( Y \) such that \( \dim \tilde{\pi}^{-1}(y) = 0 \) and \( \{y\} \times \mathbb{K} \) is not contained in \( f^{-1}(H) \) for each \( y \in Y_0 \). We may assume that there is an open subset \( U \) in \( \mathbb{P}^r \) and a finite morphism \( U \to Y_0 \), and that \( b \in Y_0 \). Let us choose an unirational hypersurface \( H'' \) contained in \( Y_0 \) such that \( b \in H'' \) and \( \dim(H'' \cap \tilde{\pi}(f^{-1}(a))) < \dim \tilde{f}^{-1}(a) \). Since \( H'' \) is locally principal in \( Y_0 \) (we have assumed that \( Y_0 \) is smooth), \( \tilde{\pi}^{-1}(H'') \) is of pure dimension \( r - 1 \) and \( \dim(\tilde{\pi}^{-1}(H'') \cap \tilde{f}^{-1}(a)) < \dim \tilde{f}^{-1}(a) \). Hence \( \text{res} f : \tilde{\pi}^{-1}(H'') \to H \) is dominant. So similarly as above, \( \text{res} f : H'' \times \mathbb{K} \to X \) is dominant. This concludes the proof.

Now we come to the following

**Problem 1.** Let \( R \) be a ring (commutative with identity). Suppose that \( A \) is an \( R \)-algebra and there is an \( R \)-isomorphism of polynomial rings \( R[T_1, ..., T_n+1] \cong A[T_1, ..., T_n] \). Is it true that \( A \) is \( R \)-isomorphic to \( R[T_1] \)?

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This problem was considered in [1], [2], [9] and [13]. Generally the answer is negative. Asanuma gave a counterexample in [2] if char $R > 0$ (see also section 4). In the next section we prove that if $R$ is a coordinate ring of a smooth affine variety then the answer is affirmative. Although this is a consequence of more general results from papers mentioned above, we will give a short geometric proof.

In the proof of Theorem 1 we shall use the following

**Theorem 2.** (Hamann, [9].) If $R$ is a $\mathbb{Q}$-algebra then Problem 1 has an affirmative solution.

We will need also an elementary

**Lemma 2.** Let $X$ be a variety and $p_i : E_i \to X$ be an algebraic line bundle on $X$ for $i = 1, 2$. Then $E_1 \cong E_2$ as line bundles on $X$ provided that there exists $m \geq 0$ and an isomorphism $f : E_1 \times \mathbb{K}^m \to E_2 \times \mathbb{K}^m$ such that the following diagram is commutative

\[
\begin{array}{ccc}
E_1 \times \mathbb{K}^m & \xrightarrow{f} & E_2 \times \mathbb{K}^m \\
\pi_{E_1} \downarrow & & \downarrow \pi_{E_2} \\
E_1 & \xrightarrow{p_1} & X \\
& \downarrow \downarrow & \\
& \downarrow p_2 & \\
& E_2 & \leftarrow \leftarrow \\
\end{array}
\]

**Proof.** Suppose that $E_i$ is given by an open cover $\{U_\alpha\}$ of $X$ and by transition functions $g^i_{\alpha, \beta} : U_\alpha \cap U_\beta \to \mathbb{K}^*$ for $i = 1, 2$. Clearly, we can identify $E_i \times \mathbb{K}^m$ with the direct sum of $E_i$ and the trivial bundle $X \times \mathbb{K}^m$. Hence

\[
G^i_{\alpha, \beta} = \begin{pmatrix} g^i_{\alpha, \beta} & 0 \\ 0 & I_m \end{pmatrix}
\]

are transition functions for $E_i \times \mathbb{K}^m$ on $U_\alpha \cap U_\beta$, where $I_m$ is the identity in $\text{GL}(\mathbb{K}^m)$. Obviously, $f$ induces the family of polynomial isomorphisms of trivial bundles $f_\alpha : U_\alpha \times \mathbb{K}^{m+1} \to U_\alpha \times \mathbb{K}^{m+1}$ such that for each $u \in U_\alpha \cap U_\beta$,

\[
f_\alpha(u, \cdot)G^{\alpha, \beta}_1(u) = G^{\alpha, \beta}_2(u)f_\beta(u, \cdot).
\]

Denote by $Jf_\alpha(u)$ the Jacobi matrix of $f_\alpha(u, \cdot)$ for $u \in U_\alpha$, and by $h_\alpha$ the Jacobian of $f_\alpha$, i.e., $h_\alpha : U_\alpha \ni u \mapsto \text{det} Jf_\alpha(u, \cdot) \in \mathbb{K}^*$. Then for $u \in U_\alpha \cap U_\beta$,

\[
h_\alpha(u)g^{\alpha, \beta}_1(u) = g^{\alpha, \beta}_2(u)h_\beta(u).
\]

Thus the family $\{h_\alpha\}$ determines the isomorphism between $E_1$ and $E_2$. This concludes the proof. \qed
Proof of Theorem 1. The following easy observation will be needed: if $V_1$ and $V_2$ are affine varieties and $V_1 \times \mathbb{K}^m \cong V_2 \times \mathbb{K}^m$ then $V_1$ dominates $V_2$ and conversely. In particular, if $V_1$ is unirational then so is $V_2$.

(i) Let $f : Y \times \mathbb{K}^m \to X \times \mathbb{K}^m$ be an isomorphism. Applying Lemma 1 to $\pi_X \circ f$, we find a morphism $g : Y \to X$ such that $\pi_X \circ f = g \circ \pi_Y$. This implies that $X$ has the strong cancellation property.

(ii) Let $p : E \to X$ be an algebraic line bundle on $X$ and $f : Y \times \mathbb{K}^m \to E \times \mathbb{K}^m$ an isomorphism. Again, by Lemma 1 there exists a morphism $q : Y \to X$ such that the following diagram is commutative

\[\begin{array}{ccc}
Y \times \mathbb{K}^m & \xrightarrow{f} & E \times \mathbb{K}^m \\
\pi_Y & & \pi_X \\
\downarrow & & \downarrow p \\
Y & \xrightarrow{q} & X
\end{array}\]

Therefore if $E$ is trivial on an affine open subset $U$ of $X$ then $\text{res } q : q^{-1}(U) \to U$ is a trivial bundle by the geometric version of Theorem 2. Thus $q : Y \to X$ is a line bundle and Lemma 2 concludes the proof. \(\square\)

Remark 1. Theorem 1 remains true if we assume that $X \setminus \text{Sing } X$ is either non-$\mathbb{K}$-uniruled or it is unirational of dimension greater than 1 and has a non-uniruled hypersurface at infinity.

For any variety $X$ we denote, here and in the sequel, the singular locus by $\text{Sing } X$ and the set of non-singular points by $\text{Reg } X = X \setminus \text{Sing } X$.

A variety $X$ of dimension $n$ has a non-uniruled hypersurface at infinity if there exists a compactification $\overline{X}$ of $X$ such that the set $\overline{X} \setminus X$ has a non-uniruled component of dimension $n - 1$.

Proof. The above proof works also in this situation, we only need to modify Lemma 1 slightly. Furthermore, the following observation will be needed: if $f : Y \times \mathbb{K}^m \to X \times \mathbb{K}^m$ (where $p : E \to X$ is a line bundle on $X$) is an isomorphism then it induces the isomorphism $\text{res } f : \text{Reg } Y \times \mathbb{K}^m \to \text{Reg } X \times \mathbb{K}^m$. The details are left to the reader. \(\square\)

3 More about Problem 1

We give now a solution of Problem 1 when $R$ is the coordinate ring of a smooth affine variety over an algebraically closed field $\mathbb{K}$ of arbitrary characteristic.

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Proposition 1. Let $X$ be an affine smooth variety. Let $Y$ be a variety with a morphism $q : Y \to X$ and let $f : X \times \mathbb{K}^{m+1} \to Y \times \mathbb{K}^m$ be an isomorphism such that $\pi_X = q \circ \pi_Y \circ f$. Then there exists an isomorphism $g : X \times \mathbb{K} \to Y$ such that $q \circ g = \pi_X$.

Proof. First observe that all fibers of $q$ are isomorphic to $\mathbb{K}$. Indeed, $f$ takes $\pi_X^{-1}(x) \cong \mathbb{K}^{m+1}$ onto $q^{-1}(x) \times \mathbb{K}^m$, so $q^{-1}(x) \cong \mathbb{K}$, because affine curves have the cancellation property. Take now the zero section $s_0 : X \ni x \mapsto (x,0) \in X \times \mathbb{K}^{m+1}$. Then $s : X \ni x \mapsto \pi_Y(f(s_0(x))) \in Y$ is a section of $q$, i.e., $q \circ s = id_X$. In particular, $s$ is a closed immersion and $\Gamma := s(X)$ meets transversally all fibers of $q$. We shall show that $q : Y \to X$ is a line bundle.

Since $X$ and $Y$ are smooth, we have isomorphisms $\pi_X^* : \text{Pic}(X) \to \text{Pic}(X \times \mathbb{K}^{m+1})$ and $\pi_Y^* : \text{Pic}(Y) \to \text{Pic}(Y \times \mathbb{K}^m)$ (see [10, p.134,141]). Hence $q^* : \text{Pic}(X) \to \text{Pic}(Y)$ is an isomorphism too. For a prime divisor $\Gamma$ on $Y$ there exists a divisor $D$ on $X$ such that $\Gamma$ and $q^*(D)$ are linearly equivalent. Let $\{U_i\}$ be an open affine cover of $X$ such that $D \cap U_i$ is principal in $U_i$. Clearly, $q^*(D) \cap q^{-1}(U_i)$ is principal in $q^{-1}(U_i)$, and so is $\Gamma \cap q^{-1}(U_i)$. This implies that the ideal of $\Gamma \cap q^{-1}(U_i)$ is generated in the coordinate ring $\mathbb{K}[q^{-1}(U_i)]$ by some function $F_i \in \mathbb{K}[q^{-1}(U_i)]$. Now $F_i$ restricts to a coordinate function on $\pi^{-1}(x)$ for each $x \in U_i$, because $q^{-1}(x) \cong \mathbb{K}$ and $\Gamma$ meets $q^{-1}(x)$ at only one point transversally. Therefore the morphism $q^{-1}(U_i) \ni y \mapsto (q(y), F_i(y)) \in U_i \times \mathbb{K}$ is bijective, so it is an isomorphism by Zariski's Main Theorem. Thus we have showed that $q : Y \to X$ is a line bundle. Now Lemma 2 concludes the proof.

4 Final remarks

We shall begin with

Question 1. Is it true that an affine variety with a non-uniruled component at infinity has the cancellation property?
By Theorem 1 the answer is affirmative if we add the unirationality assumption, as well as for any line bundle on a non-uniruled variety.

It is a good place to mention our result from [5] which is connected with the stable equivalence problem (see also [12])

Let $X$ be a smooth affine variety and let $H$ be a non-uniruled hypersurface in $X$. Suppose that $f : X \times \mathbb{K}^m \to Y \times \mathbb{K}^m$ is an isomorphism such that $f(H \times \mathbb{K}^m) = H' \times \mathbb{K}^m$, where $H'$ is a hypersurface in a variety $Y$. Then for each $x \in X$ there exists $y \in Y$ such that $f(\{x\} \times \mathbb{K}^m) = \{y\} \times \mathbb{K}^m$.

**Question 2.** Suppose that an affine variety $X$ has the strong cancellation property. Does it follow that $X \times \mathbb{K}$ has the cancellation property?

Again, Theorem 1 gives the affirmative answer for some varieties. This question was already considered by Asanuma in [2], where he showed that if $\text{char} \mathbb{K} > 0$ then the answer is no. His example was a rational curve with the coordinate ring $\mathbb{K}[T^m, T^{m+1}]$, where $n > 1$ (this is also a counterexample to Problem 1).

On the other hand if $\text{char} \mathbb{K} = 0$ we have the following

**Corollary 1.** If $X$ and $Y$ are affine curves then the surface $X \times Y$ has the cancellation property.

**Proof.** The hardest case, when $X \cong Y \cong \mathbb{K}$ follows from [6] and [15].

If $X$ is non-isomorphic to $\mathbb{K}$ then $X \setminus \text{Sing}X$ is non-$\mathbb{K}$-uniruled. (This is a consequence of the following arguments: (1) every smooth affine and $\mathbb{K}$-uniruled curve is isomorphic to $\mathbb{K}$; (2) if $C$ is an affine curve then every non-constant morphism $\mathbb{K} \to C$ is finite and hence surjective.) Thus, by Remark 1, $X \times \mathbb{K}$ has the cancellation property.

The case when neither $X \cong \mathbb{K}$ nor $Y \cong \mathbb{K}$ again follows from Remark 1, since the set $X \times Y \setminus \text{Sing}(X \times Y) = [(\text{Reg}X) \times Y] \cap [X \times \text{Reg}Y]$ is non-$\mathbb{K}$-uniruled. This concludes the proof.

We give one more result a proof of which is left as an exercise.

**Proposition 2.** Theorem 1 remains true for any affine variety $X$ such that every dominant morphism $X \to X$ is birational (for example, $X$ satisfies this condition if it is of hyperbolic type, i.e., $\pi(X) = \dim X$; see [7, p. 335]).

**References**


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