Residue calculus for c-holomorphic functions

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ABSTRACT. In this paper we introduce Coleff-Herrera residue currents defined by systems of c-holomorphic functions and prove a Lelong-Poincaré and a Cauchy-type formula as well as the Transformation Law for these currents.

1. PRELIMINARIES

In complex analysis one often comes across what is called *weakly holomorphic* functions. These functions appear in a natural way e.g. in problems related to Abel's or Lie-Griffiths Theorem — see [HP]. They are defined and holomorphic on the regular part of a (complex) analytic set and locally bounded near the singularities. However, they are not as *maniable* as one would like them to be.

Among other possible notions of 'holomorphicity' for functions defined on analytic sets there is one which is of greater interest and was introduced by Remmert (see [R]). Let A be an analytic subset of an open set $\Omega \subset \mathbb{C}^m$.

Definition 1.1. ([Wh]) A mapping $f\colon A \to \mathbb{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset $\mathrm{Reg}A$ of regular points is holomorphic. We denote by $\mathcal{O}_c(A, \mathbb{C}^n)$ the ring of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

This happens to be a very good generealization of holomorphic functions onto analytic sets. It is well-known that a mapping defined in an open set is holomorphic if and only if it is continuous and its graph is an analytic

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set (it is then a submanifold). We have a similar result for c-holomorphic mappings (cf. [Wh] 4.5Q), which motivates this generalization:

Theorem 1.2. *A mapping* $f\colon A\to\mathbb{C}^n$ *is c-holomorphic iff it is continuous and its graph* $\Gamma_f:=\{(x,f(x))\mid x\in A\}$ *is an analytic subset of* $\Omega\times\mathbb{C}^n$.

It is worth noting that by a recent result of N. V. Shcherbina [Sh] the pluripolarity of the graph is sufficient (unlike for instance sub- or semianalyticity: $f(x):=|x|$ for $x\in\mathbb{C}$ has semianalytic graph which is not complex analytic). By theorem 1.2 the zero set of a c-holomorphic function is analytic.

Throughout this paper we assume that $A\subset\Omega$ is a pure k-dimensional analytic set in an open set $\Omega\subset\mathbb{C}^m$.

2. Residue currents defined by c-holomorphic functions

Let $f\in\mathcal{O}_c(A)$ be such that it does not vanish identically on any irreducible component of A. The aim of this part is to define, following an idea of A. Yger, a *residue current* which would generalize to the c-holomorphic case the *restricted residue current* of Coleff-Herrera $[A]\wedge\overline{\partial}[1/f]$ (see [CH], [TsY]). Were f holomorphic in Ω, we would have for any $\varphi\in\mathcal{D}_{(k,k-1)}(\Omega)$ by the definition of Coleff and Herrera:

$$\langle[A]\wedge\overline{\partial}[1/f],\varphi\rangle:=\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_{\mathrm{Reg}A\cap\{|f|^2=\varepsilon\}}\frac{\varphi}{f}=$$
$$=-\lim_{\varepsilon\to0^+}\frac{1}{2\pi i}\int_{\mathrm{Reg}A\cap\{|f|^2>\varepsilon\}}\frac{\overline{\partial}\varphi}{f}.$$

The current we obtain is $\overline{\partial}$-closed and supported by $A\cap f^{-1}(0)$. It is a deep result that such a current is well-defined — actually, we are concealing here the problem of the existence of the current $\overline{\partial}[1/f]$, i.e. the $\overline{\partial}$ of the *principal value* current $[1/f](\varphi):=(2\pi i)^{-1}\lim_{\varepsilon\to0^+}\int_{\{|f|^2>\varepsilon\}}\varphi/f$ solving the equation $(2\pi i)f\cdot\mathfrak{t}=1$ in Ω.

We introduce the notation

$$\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A:=\langle[A]\wedge\overline{\partial}[1/f],\varphi\rangle,\quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega).$$

When $A=\Omega$ we simply do not write it in subscript since it does not interfere with anything. Note that the Lelong-Poincaré formula says in particular that if the hypersurface $X=\{g=0\}$ is given by a reduced analytic equation, then

$$(LP)\qquad \langle[X],\varphi\rangle=\mathrm{Res}\begin{bmatrix}dg\wedge\varphi\\ g\end{bmatrix}.$$

Observe that the above equality can be rewritten as $[X]=\overline{\partial}[1/g]\wedge dg$ (since one has $(2\pi i)^{-1}\overline{\partial}\partial\log|g|^2=\overline{\partial}[1/g]\wedge dg$ in the sense of currents).

Now, if f is just c-holomorphic on A we define a residue current of type $(m-k, m-k+1)$ setting

$$(*)\qquad \mathrm{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A := \mathrm{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_f}, \quad \varphi\in\mathcal{D}_{(k,k-1)}(\Omega)$$

where $(z,w)\in\Gamma_f\subset\Omega\times\mathbb{C}$ (i.e. on the right-hand side we compute the residuum relatively to the projection pr_w). In other words, we use the graph (which also is of pure dimension k as is easily seen) to define properly the current $\mathrm{Res}\left[\begin{smallmatrix}\varphi(z)\\ f(z)\end{smallmatrix}\right]_A$. This definition makes sense in that it coincides with the usual one when f is holomorphic as we will see in the proof of the following theorem. Note that there is no problem of support relative to the vertical variable w since we may successfully replace in $(*)$ the form $\varphi(z)$ by $\varphi(z)\cdot\theta(w)$, where $\theta(w)$ is a $\mathscr{C}^\infty$ function with compact support, identically equal to 1 on a ball of radius $r>\max_{z\in\mathrm{supp}\varphi}|f(z)|$.

The Coleff-Herrera residue is also defined for n functions in the proper intersection case. Namely, if $f_1,\dots,f_n\in\mathcal{O}(\Omega)$ are such that $A\cap\bigcap_j f_j^{-1}(0)$ has pure dimension $k-n$, then for any $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$,

$$\mathrm{Res}\begin{bmatrix}\varphi\\ f_1,\dots,f_n\end{bmatrix}_A := \lim_{\delta\to0^+}\left(\frac{1}{2\pi i}\right)^n\int_{\mathrm{Reg}A\cap T_\delta(f)}\frac{\varphi}{f_1\cdot\ldots\cdot f_n},$$

where $T_\delta(f)=\{|f_j|^2=\varepsilon_j(\delta), j=1,\dots,n\}$ and $\varepsilon_1\ll\ldots\ll\varepsilon_n$ are special functions tending to zero with δ (along what is called an *admissible path*:

$$\lim_{\delta\to0^+}\frac{\varepsilon_j(\delta)}{\varepsilon_{j+1}(\delta)^p}=0,\ \forall p\in\mathbb{N},\ j=1,\dots,n-1$$

and the limit is independent of the choice of the admissible path), is a well defined current of type $(m-k, m-k+n)$. For more information see [CH], [TsY]. Note that the actual ordering of $\{f_1,\dots,f_n\}$ is important.

It is quite easy to extend this notion of residual current to the case of a c-holomorphic mapping $f=(f_1,\dots,f_n)\colon A\to\mathbb{C}^n_w$ defining a proper intersection on A, i.e. $f^{-1}(0)$ is of pure dimension $k-n$ (then Γ_f and $\Omega\times\{0\}$ intersect properly in $\mathbb{C}^m\times\mathbb{C}^n$). We put

$$(**)\quad \mathrm{Res}\begin{bmatrix}\varphi(z)\\ f_1(z),\dots,f_n(z)\end{bmatrix}_A := \mathrm{Res}\begin{bmatrix}\varphi(z)\\ w_1,\dots,w_n\end{bmatrix}_{\Gamma_f}, \quad \varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$$

getting a current of type $(m-k, m-k+n)$. It is a natural generalization of the Coleff-Herrera restricted residue current $[A]\wedge\overline{\partial}[1/f_1]\wedge\ldots\wedge\overline{\partial}[1/f_n]$ to the case of c-holomorphic functions since we have the following

Theorem 2.1. *If the function f (respectively the mapping $f=(f_1,\dots,f_n)$) is holomorphic in Ω, then equality $(*)$ (respectively $(**)$) holds.*

Proof. To simplify notation we confine ourselves to showing formula $(*)$, the generalization to $(**)$ of the forthcoming argument being quite obvious. So as to show

$$\operatorname{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A = \operatorname{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_f}, \quad \varphi \in \mathcal{D}_{(k,k-1)}(\Omega),$$

it is sufficient to establish an equality between the integrals (we fix φ)

$$I_\varepsilon := \int_{\mathrm{Reg}A\cap\{|f|^2=\varepsilon\}} \frac{\varphi(z)}{f(z)} \quad \text{and} \quad J_\varepsilon := \int_{\mathrm{Reg}\Gamma_f\cap\{|w|^2=\varepsilon\}} \frac{\varphi(z)}{w}$$

for almost all (cf. Sard's Theorem and the fact that we know the respective limits do exist) $\varepsilon > 0$ arbitrarily small. This equality follows from the Change of Variables Theorem. Indeed, consider the biholomorphism

$$\Theta\colon \Omega\times\mathbb{C} \ni (z,w) \mapsto (z, w+f(z)) \in \Omega\times\mathbb{C}$$

of jacobian 1 and observe that

$$\Theta(\{(z,0) \mid z\in \mathrm{Reg}A\colon |f(z)|^2=\varepsilon\}) = \{(z,f(z)) \mid z\in\mathrm{Reg}A\colon\ |f(z)|^2=\varepsilon\}.$$

Note that in general we have only an inclusion $\Gamma_{f|_{\mathrm{Reg}A}} \subset \mathrm{Reg}\Gamma_f$. However, in view of the fact that the $(2k-1)$-dimensional Hausdorff measure $\mathscr{H}^{2k-1}(\Gamma_{f|_{\mathrm{Sng}A}}) = 0$, we may replace in J_ε the set $\mathrm{Reg}\Gamma_f$ by $\Gamma_{f|_{\mathrm{Reg}A}}$. Then,

$$J_\varepsilon = \int_{(\mathrm{Reg}A\cap\{z\colon |f(z)|^2=\varepsilon\})\times\{0\}} \Theta^*\left(\frac{\varphi(z)}{w}\right).$$

It remains to calculate

$$\Theta^*\left(\frac{\varphi(z)}{w}\right) = \frac{\varphi(z)}{w+f(z)}$$

and since the integration is taken over $\{w=0\}$, we obtain

$$J_\varepsilon = \int_{(\mathrm{Reg}A\cap\{z\colon |f(z)|^2=\varepsilon\})\times\{0\}} \frac{\varphi(z)}{f(z)}.$$

This in turn is equal to I_ε because w is in this case a 'phantom' variable. □

Second proof. It may be interesting to present also a proof based on residue calculus' formulæ. Once again it is sufficient to prove (∗).

First observe that one can 'add' a variable in order to calculate the action of $\operatorname{Res}\begin{bmatrix}\cdot\\ f\end{bmatrix}_A$ on a test form φ. Namely, thanks to Fubini's Theorem and Cauchy's Formula,

$$\operatorname{Res}\begin{bmatrix}\varphi(z)\\ f(z)\end{bmatrix}_A = \operatorname{Res}\begin{bmatrix}\varphi(z)\wedge dw\\ f(z), w\end{bmatrix}_{A\times\mathbb{C}}.$$

Now using the *restricted transformation law* (see [BVY] and 6 hereafter), we have

$$\operatorname{Res}\begin{bmatrix}\varphi(z)\wedge dw\\ f(z), w\end{bmatrix}_{A\times\mathbb{C}} = \operatorname{Res}\begin{bmatrix}\varphi(z)\wedge dw\\ f(z)-w, w\end{bmatrix}_{A\times\mathbb{C}}.$$

Then since $\varphi \in \mathcal{D}_{(k,k-1)}(\Omega)$, we may write $\varphi(z) \wedge dw = d(f(z)-w) \wedge \varphi(z)$ on $A \times \mathbb{C}$ and so

$$\operatorname{Res}\begin{bmatrix} \varphi(z) \wedge dw \\ f(z)-w, w \end{bmatrix}_{A\times\mathbb{C}} = \operatorname{Res}\begin{bmatrix} d(f(z)-w) \wedge \varphi(z) \\ f(z)-w, w \end{bmatrix}_{A\times\mathbb{C}}.$$

Finally we shall just use a restricted form of the Lelong-Poincaré equation (LP) which reads

$$[\Gamma] = [A \times \mathbb{C}] \wedge \frac{1}{\pi} dd^c \log |f(z) - w|,$$

where $[\Gamma]$ is the $(m+1-k, m+1-k)$ integration current over the graph of $f|_A$. Since the equation $w = f(z)$ has a reduced form, there will be obviously no multiplicities attached to $[\Gamma]$. This formula follows from the usual Lelong-Poincaré formula since by the latter

$$\frac{1}{\pi} dd^c \log |f(z) - w| = \overline{\partial}[1/(f(z)-w)] \wedge d(f(z)-w).$$

Note that $\{f(z) = w\} \cap (A \times \mathbb{C})$ is proper in $\Omega \times \mathbb{C}$. The wedge product being associative in the complete intersection case we clearly get

$$[A \times \mathbb{C}] \wedge \overline{\partial}[1/(f(z)-w)] \wedge d(f(z)-w) = \operatorname{Res}\begin{bmatrix} d(f(z)-w) \wedge (\cdot) \\ f(z)-w \end{bmatrix}_{A\times\mathbb{C}}.$$

Now we apply a classical Continuation Theorem (see [De] lemme 3.7) since we have two positive currents of the same type (the one above and $[\Gamma]$), $\overline{\partial}$- and d-closed, supported by $\{(z, f(z)) \mid z \in \operatorname{Reg} A\} \subset \Gamma$ and equal on it (i.e. on forms whose supports are in this set). Thus they must coincide on any test form $\varphi \in \mathcal{D}_{(k,k)}(\Omega \times \mathbb{C})$.

To finish the proof we just need to show that

$$\operatorname{Res}\begin{bmatrix} d(f(z)-w) \wedge \varphi(z) \\ f(z)-w, w \end{bmatrix}_{A\times\mathbb{C}} = \operatorname{Res}\begin{bmatrix} \varphi(z) \\ w \end{bmatrix}_{\Gamma}, \quad \varphi \in \mathcal{D}_{(k,k-1)}(\Omega).$$

But the right-hand side is by definition $[\Gamma] \wedge \overline{\partial}[1/w]$ while we have just seen that

$$[\Gamma] = [A \times \mathbb{C}] \wedge \overline{\partial}[1/(f(z)-w)] \wedge d(f(z)-w).$$

The associativity of the product permits to conclude (cf. the intersection $\{w = 0\} \cap \{f(z) = w\} \cap (A \times \mathbb{C})$ is proper). □

Note. One can also invoke at the end of the second proof another way of defining the restricted residue, namely using the *Mellin transform* (see e.g. [BVY], [TsY]).

The generalization of the above argument to $(**)$ needs a generalized formula (LP) for a proper reduced intersection which is quite obvious (see below).

Roughly speaking, the main idea of all these constructions is that we replace the non-existent df by dw taken on the graph of f (see also the note below, after proposition 3.3).

3. A Lelong-Poincaré formula

The key-point of this part is a more or less known version of the restricted Lelong-Poincaré formula mentioned above. If $f_j \in \mathcal{O}(\Omega)$, $j = 1, \ldots, r$ are such that $\bigcap_1^r f_j^{-1}(0)$ has pure dimension $m - r$, then by [Ts] p. 133,

$$[Z_f] = \mathrm{Res}\begin{bmatrix} df_1 \wedge \ldots \wedge df_r \wedge (\cdot) \\ f_1, \ldots, f_n \end{bmatrix} = \overline{\partial}[1/f_1] \wedge \ldots \wedge \overline{\partial}[1/f_n] \wedge df_1 \wedge \ldots \wedge df_r,$$

where Z_f is the cycle of zeroes of $f = (f_1, \ldots, f_r)$ (computed as the proper intersection cycle $\Gamma_f \cdot (\Omega \times \{0\})$ following Draper [Dr]). Note that there is $Z_f = Z_{f_1} \cdot \ldots \cdot Z_{f_r}$ and the intersection being proper, the product of cycles is associative (see [Ch]). By (LP), for each j, $[Z_{f_j}] = \overline{\partial}\partial u_j$, where we put $u_j := (2\pi i)^{-1} \log |f_j|^2$. Thanks to the results of Bedford-Taylor, the product of a positive ∂- and $\overline{\partial}$-closed current $\mathfrak{t}$ with the dd^c of a locally integrable plurisubharmonic function u is well-defined by $\mathfrak{t} \wedge \overline{\partial}\partial u := \overline{\partial}\partial(u\mathfrak{t})$ and yields once again a positive ∂- and $\overline{\partial}$-closed current (see [De]). Therefore, by induction we obtain

$$\mathfrak{t} \wedge \overline{\partial}\partial u_1 \wedge \ldots \wedge \overline{\partial}\partial u_r = \overline{\partial}\partial(u_r \overline{\partial}\partial(\cdots \overline{\partial}\partial(u_1 \mathfrak{t}) \cdots)).$$

Take $\mathfrak{t} := [A]$, where A is a pure k-dimensional analytic subset of Ω such that $f^{-1}(0) \cap A$ has pure dimension $k - r$. By the version of Lelong-Poincaré formula given in [Ch] p. 216, $\overline{\partial}\partial(u_1 \mathfrak{t}) = [A \cdot Z_{f_1}]$. If now we put $T_1 := A \cdot Z_{f_1}$ and $\mathfrak{t}_1 := [T_1]$, we obtain $\overline{\partial}\partial(u_2 \mathfrak{t}_1) = [T_1 \cdot Z_{f_2}]$ by the same theorem. Iterating this, we get

$$[A] \wedge [Z_{f_1}] \wedge \ldots \wedge [Z_{f_r}] = [A \cdot Z_{f_1} \cdot \ldots \cdot Z_{f_r}] = [A \cdot Z_f],$$

where the left-hand side of the equality is understood in the Bedford-Taylor sense. By [De] (5.5) we know that

$$[Z_{f_1}] \wedge \ldots \wedge [Z_{f_r}] = [Z_f].$$

Therefore,

$$(LP') \qquad [A \cdot Z_f] = \mathrm{Res}\begin{bmatrix} df_1 \wedge \ldots \wedge df_r \wedge (\cdot) \\ f_1, \ldots, f_r \end{bmatrix}_A .$$

This formula leads to three c-holomorphic results. The first one is the c-holomorphic counterpart of the Lelong-Poincaré formula. Let $f \in \mathcal{O}_c(A)$ be such that it does not vanish on any irreducible component of A but has a non-void zero set. Then, by an observation made in [D], the set $f^{-1}(0)$ has pure dimension $k - 1$ and so $\Gamma_f \cap (\Omega \times \{0\})$ is a proper intersection. Thus $Z_f := \Gamma_f \cdot (\Omega \times \{0\})$ is well defined.

Theorem 3.1. *In the introduced setting,*

$$[Z_f] = \frac{1}{2\pi i}[\Gamma_f] \wedge \overline{\partial}\partial \log |w|^2 \quad on \quad \mathcal{D}_{(k-1,k-1)}(\Omega),$$

where w is the variable from the target space.

Proof. It suffices to observe that $(\Omega\times\{0\}) = Z_w$, whence $Z_f = \Gamma_f \cdot Z_w$, and so by (LP'),

$$[Z_f] = \operatorname{Res}\begin{bmatrix} dw\wedge(\cdot) \\ w \end{bmatrix}_{\Gamma_f} = \frac{1}{2\pi i}[\Gamma_f]\wedge\overline{\partial}\partial\log|w|^2,$$

which completes the proof. □

This has a straightforward generalization to the case of several functions:

Theorem 3.2. *Let $f_1,\dots,f_n \in \mathcal{O}_c(A)$ be such that $f^{-1}(0)$ has pure dimension $k-n$ for $f := (f_1,\dots,f_n)$. Then on $\mathcal{D}_{(k-n,k-n)}(\Omega)$ there is*

$$[Z_f] = \operatorname{Res}\begin{bmatrix} dw_1\wedge\ldots\wedge dw_n\wedge(\cdot) \\ w_1,\dots,w_n \end{bmatrix}_{\Gamma_f},$$

where $Z_f := \Gamma_f\cdot(\Omega\times\{0\}^n)$ is the proper intersection cycle of f and w_j are the variables from the target space.

Proof. It is similar to the previous one — we just observe that $[Z_f] = [\Gamma_f\cdot Z_w]$, where Z_w is the cycle of zeroes of the projection onto the target space, $w\colon \Omega\times\mathbb{C}^n \ni (z,w)\mapsto w\in\mathbb{C}^n$. □

If $n=k$, then we can compute the (geometric) multiplicity $m_0(f)$ for $f\in\mathcal{O}_c(A,\mathbb{C}^k)$ with 0 isolated in $f^{-1}(0)$ similarly to the holomorphic case. Recall first that $m_0(f)$ is by definition the number of points in the generic fibre of f which coincides with the proper intersection multiplicity at zero, denoted by $i(\Gamma_f\cdot(\Omega\times\{0\}^k);0)$, of Γ_f and $(\Omega\times\{0\})$.

Corollary 3.3. *Let $f\in\mathcal{O}_c(A,\mathbb{C}^k_w)$ be such that $f^{-1}(0)=\{0\}$. Then*

$$m_0(f)\delta_0 = \operatorname{Res}\begin{bmatrix} dw_1\wedge\ldots\wedge dw_k\wedge(\cdot) \\ w_1,\dots,w_k \end{bmatrix}_{\Gamma_f},$$

where δ_0 is the Dirac's delta at zero.

Proof. Clearly $m_0(f)\delta_0 = [\Gamma_f\cdot(\Omega\times\{0\}^k)]$, since $m_0(f) = i(\Gamma_f\cdot(\Omega\times\{0\}^k);0)$. It remains to apply the previous result. □

Note. In particular we have by 3.3 the equality

$$m_0(f) = \operatorname{Res}\begin{bmatrix} dw_1\wedge\ldots\wedge dw_k \\ w_1,\dots,w_k \end{bmatrix}_{\Gamma_f},$$

generalizing the well-known holomorphic formula $m_0(f) = \operatorname{Res}\begin{bmatrix} \frac{df_1\wedge\ldots\wedge df_m}{f_1,\dots,f_m} \end{bmatrix}$ in the case $A=\Omega$ and $\bigcap_1^m f_j^{-1}(0)=\{0\}$ (see [Ts] ch. II §6).

4. Residue currents with numerators

We are keeping the notations introduced so far and we consider n c-holomorphic functions $f_j\colon A \to \mathbb{C}$ not vanishing identically on any irreducible component of A and such that $f^{-1}(0) \subset A$ has pure dimension $k-n$ (see [D] for considerations on the dimension of zero-sets of c-holomorphic mappings). These play the role of denominators. Let us take a 'numerator' $h \in \mathcal{O}_c(A)$. Our aim is to define a residue current which would be an analogue of the restricted Coleff-Herrera current of type $(m-k, m-k+n)$

$$h \cdot [A] \wedge \overline{\partial}[1/f_1] \wedge \ldots \wedge \overline{\partial}[1/f_n]$$

for $h, f_1, \ldots, f_n$ holomorphic. Once again we follow the idea of A. Yger — we shall make use of the graph.

We introduce a new variable $t \in \mathbb{C}$ and consider the c-holomorphic mapping

$$H\colon \mathbb{C} \times A \ni (t,z) \mapsto (t-h(z), f(z)) \in \mathbb{C}_{w_0} \times \mathbb{C}^n_w.$$

Then we put by definition for $\varphi \in \mathcal{D}_{(k,k-n)}(\Omega)$,

$$(\star) \qquad h(z) \cdot \mathrm{Res}\begin{bmatrix} \varphi(z) \\ f_1(z), \ldots, f_n(z) \end{bmatrix}_A := \mathrm{Res}\begin{bmatrix} t\varphi(z) \wedge dt \\ w_0, w_1, \ldots, w_n \end{bmatrix}_{\Gamma_H}.$$

This coincides in the holomorphic case with the usual definition as we prove in the following

Theorem 4.1. *If $h, f_1, \ldots, f_n$ are holomorphic, then $(\star)$ holds.*

Proof. By assumptions we have $H \in \mathcal{O}(\mathbb{C} \times \Omega)$. Let Λ be the graph of $H|_{\mathbb{C}\times A}$. Put

$$S_\delta := \int_{\mathrm{Reg}A \cap \{|f_j|^2 = \varepsilon_j(\delta), j\}} \frac{h(z)\varphi(z)}{f_1(z) \cdot \ldots \cdot f_n(z)}$$
$$\text{and} \quad R_\delta := \frac{1}{2\pi i} \int_{\mathrm{Reg}\Lambda \cap \{|w_0|^2 = \varepsilon_0(\delta), |w_j|^2 = \varepsilon_j(\delta)\}} \frac{t\varphi(z) \wedge dt}{w_0 w_1 \cdot \ldots \cdot w_n}.$$

Obviously, we have here any admissible path $\varepsilon_0(\delta) \ll \varepsilon_1(\delta) \ll \ldots \ll \varepsilon_n(\delta)$ (we can also 'complete' the admissible path $\varepsilon_1(\delta), \ldots, \varepsilon_n(\delta)$ by taking $\varepsilon_0(\delta) := \exp(-1/\varepsilon_1(\delta))$).

Consider the biholomorphism

$$\Xi\colon \mathbb{C} \times \Omega \times \mathbb{C}^n \ni (t, z, w_0, w) \mapsto (t, z, w_0 + t - h(z), w + f(z)) \in \mathbb{C} \times \Omega \times \mathbb{C}^n.$$

Then

$$\{(t, z, t-h(z), f(z)) \mid z \in \mathrm{Reg}A, |t-h(z)|^2 = \varepsilon_0(\delta), |f_j(z)|^2 = \varepsilon_j(\delta), j\} =$$
$$= \Xi(\{(t, z, 0, 0) \mid z \in \mathrm{Reg}A, |t-h(z)|^2 = \varepsilon_0(\delta), |f_j(z)|^2 = \varepsilon_j(\delta), j\}).$$

As in the proof of theorem 2.1 we notice that calculating R_δ we may confine ourselves to Λ taken over $\mathrm{Reg}A$ (instead of taking $\mathrm{Reg}\Lambda$ — we just 'forget' a set of measure zero, which does not affect in any way the integral). Besides, since Jac$\Xi \equiv 1$,

$$\Xi^*\left(\frac{t\varphi(z)\wedge dt}{w_0w_1\cdot\ldots\cdot w_n}\right)=\frac{t\varphi(z)\wedge dt}{(w_0+t-h(z))(w_1+f_1(z))\cdot\ldots\cdot(w_n+f_n(z))}$$

and the fact that the integral is calculated with $w_0 = w_1 = \ldots = w_n = 0$ leads to the following expression of R_δ:

$$\frac{1}{2\pi i}\int_{\{(t,z,0,0)\colon z\in\mathrm{Reg}A,|t-h(z)|^2=\varepsilon_0(\delta),|f_j(z)|^2=\varepsilon_j(\delta),j\}}\frac{t\varphi(z)\wedge dt}{(t-h(z))f_1(z)\cdot\ldots\cdot f_n(z)}.$$

Now Fubini's Theorem (vd. [Ch]) allows us to rewrite this integral in the form

$$\int_{\mathrm{Reg}A\cap\{z\colon |f_j(z)|^2=\varepsilon_j(\delta),j\}}\left(\frac{1}{2\pi i}\int_{\{t\in\mathbb{C}\colon |t-h(z)|^2=\varepsilon_0(\delta)\}}\frac{tdt}{t-h(z)}\right)\frac{\varphi(z)}{f_1(z)\cdot\ldots\cdot f_n(z)}.$$

Cauchy's Formula yields

$$\frac{1}{2\pi i}\int_{\{t\colon |t-h(z)|^2=\varepsilon_0(\delta)\}}\frac{tdt}{t-h(z)}=h(z),$$

which means in particular that R_δ is independent of $\varepsilon_0(\delta)$. Therefore

$$\lim_{\delta\to 0^+} S_\delta=\lim_{\delta\to 0^+} R_\delta$$

which completes the proof. □

Proposition 4.2. *If $f_1,\ldots,f_n$ are just c-holomorphic but h is holomorphic on A, then*

$$h\cdot\mathrm{Res}\begin{bmatrix}\varphi\\ f_1,\ldots,f_n\end{bmatrix}_A=\mathrm{Res}\begin{bmatrix}h\cdot\varphi\\ f_1,\ldots,f_n\end{bmatrix}_A,\quad \varphi\in\mathcal{D}_{(k,k-n)}(\Omega).$$

Proof. The left-hand side of the required equality is defined by ($\star$). Consider the following biholomorphism

$$\Psi\colon \mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}^n\ni(t,z,w_0,w)\mapsto(t,z,w_0+t-h(z),w)\in\mathbb{C}\times\Omega\times\mathbb{C}\times\mathbb{C}^n.$$

Its Jacobian is identically equal to 1 and we have

$$\Psi(\{(t,z,0,f(z))\mid z\in\mathrm{Reg}A,|t-h(z)|^2=\varepsilon_0(\delta),|w_j|=\varepsilon_j(\delta),j=1,\ldots,n\})$$
$$=\{(t,z,t-h(z),f(z))\mid z\in\mathrm{Reg}A,|t-h(z)|^2=\varepsilon_0(\delta),|w_j|=\varepsilon_j(\delta),j\},$$

where $\varepsilon_0(\delta),\varepsilon_1(\delta),\ldots,\varepsilon_n(\delta)$ is an admissible path. Moreover,

$$\Psi^*\left(\frac{t\varphi(z)\wedge dt}{w_0w_1\cdot\ldots\cdot w_n}\right)=\frac{t\varphi(z)\wedge dt}{(w_0+t-h(z))w_1\cdot\ldots\cdot w_n}.$$

Therefore by the Change of Variables Theorem the integral appearing on the left-hand side of the sought equality becomes (we omit as earlier a set of measure zero)

$$\int_{\{(t,z,0,f(z))\colon z\in\mathrm{Reg}A,|t-h(z)|^2=\varepsilon_0(\delta),|w_j|=\varepsilon_j(\delta),j\}}\frac{t\varphi(z)\wedge dt}{(t-h(z))w_1\cdot\ldots\cdot w_n}.$$

Applying now Fubini's Theorem and the Cauchy Formula we obtain

$$2\pi i \int_{\{(z,0,f(z))\colon\, z\in \mathrm{Reg}A, |w_j|=\varepsilon_j(\delta), j\}} \frac{h(z)\varphi(z)}{w_1\cdot\ldots\cdot w_n}.$$

From this we clearly get the required equality. □

Note. In all these constructions, though the continuity of f is not used *explicite* (it remains hidden somehow), what is used is the analycity of the graph. Actually, all the proofs are based on it and they would not work, tried we to apply them to weakly holomorphic functions.

5. A Cauchy-type formula

If $f\in\mathcal{O}(\Omega)$ and $0\in\Omega\subset\mathbb{C}^m$, then the usual Cauchy's formula may be expressed as follows (cf. Fubini's Theorem):

$$f(0)=\left(\frac{1}{2\pi i}\right)^m \lim_{\delta\to 0^+}\int_{T_\delta(z)}\frac{f(z)dz_1\wedge\cdots\wedge dz_m}{z_1\cdot\ldots\cdot z_m},$$

where $T_\delta(z)$ is the tube defined earlier, taken for $z_1,\ldots,z_m$ and an admissible path. This formula may be more generally written as

$$f(0)=(f\cdot\mathfrak{t})(\theta dz_1\wedge\ldots\wedge dz_m),$$

where θ is a $\mathscr{C}^\infty$ function with compact support, identically equal to 1 in a neighbourhood of zero (we shall not write it any longer, it is 'cosmetics') and $\mathfrak{t}$ is the current (of type $(m,0)$) defined by

$$\mathfrak{t}(\varphi):=\mathrm{Res}\begin{bmatrix}\varphi\\ z_1,\ldots,z_m\end{bmatrix}.$$

This approach cannot be directly transposed to the c-holomorphic case (roughly speaking, the main problem is that there are too many variables $z_1,\ldots,z_m$ for a set of dimension $< m$). Nonetheless, we may proceed in the following way: let as earlier $A\subset\Omega$ be an analytic set containg 0 and of pure dimension k. Suppose that the natural projection π on the first k coordinates realizes the degree (Lelong number) $\deg_0 A$ as the sheet number (multiplicity) of the branched covering $\pi|_A$ (see [Ch]). Then by (LP'), for any $f\in\mathcal{O}(A)$ and all $\xi\in\mathbb{C}^k$ sufficiently small,

$$(*)\qquad \sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\cdot f(\zeta)=\mathrm{Res}\begin{bmatrix}f(z)dz_1\wedge\ldots\wedge dz_k\\ z_1-\xi_1,\ldots,z_k-\xi_k\end{bmatrix}_A,$$

where $\mu_\zeta(\pi|_A)$ is the multiplicity of $\pi|_A$ at the point $\zeta\in A$ (see [Ch] for these notions). More generally, we have the following proposition:

Proposition 5.1. *In the introduced setting,*

$$\sum_{\zeta\in\pi^{-1}(\xi)\cap A}\mu_\zeta(\pi|_A)\delta_\zeta=\mathrm{Res}\begin{bmatrix}(\cdot)\wedge dz_1\wedge\ldots\wedge dz_k\\ z_1-\xi_1,\ldots,z_k-\xi_k\end{bmatrix}_A,$$

where δ_ζ are Dirac's functions. In particular, for any function $f \in \mathcal{O}(\Omega)$,

$$\deg_0 A \cdot f(0) = \mathrm{Res}\begin{bmatrix} f(z)dz_1 \wedge \ldots \wedge dz_k \\ z_1, \ldots, z_k \end{bmatrix}_A .$$

Proof. Fix ξ and take $h(z) := (z_1 - \xi_1, \ldots, z_k - \xi_k)$ for $z \in \mathbb{C}^m$. Then the cycle Z_h is well defined and equal to $\{\xi\} \times \mathbb{C}^{m-k}$ (with no multiplicities). This intersects A properly at the points $\zeta \in A$ for which $\pi(\zeta) = \xi$ and the multiplicities attached to these points correspond to the multiplicities $\mu_\zeta(\pi|_A)$ (see [Ch]). Now, by (LP') and a similar argument to the one used in the proof of 3.3 we obtain

$$[A \cdot Z_h] = \mathrm{Res}\begin{bmatrix} (\cdot) \wedge dz_1 \wedge \ldots \wedge dz_k \\ z_1 - \xi_1, \ldots, z_k - \xi_k \end{bmatrix}_A = \sum_{\zeta \in \pi^{-1}(\xi) \cap A} \mu_\zeta(\pi|_A)\delta_\zeta$$

and the proof is accomplished (to get the assertion for f holomorphic, we just replace f by a compactly supported smooth function equal to f in a small enough neighbourhood $U \times V \subset \mathbb{C}^k \times \mathbb{C}^m$ of the fibre $\pi^{-1}(\xi) \cap A$ chosen so that $A \cap (U \times V)$ does not meet $U \times \partial V$). □

By the way, observe that since $\deg_0 A = i(A \cdot \pi^{-1}(0); 0) = m_0(\pi|_A)$ (π is seen as a function $\Omega \to \mathbb{C}^k$), by theorem 3.3 and in view of the fact that $\pi|_A$ is holomorphic, there is then

$$\deg_0 A = \mathrm{Res}\begin{bmatrix} dw_1 \wedge \ldots \wedge dw_k \\ w_1, \ldots, w_k \end{bmatrix}_{\Gamma_{\pi|A}} = \mathrm{Res}\begin{bmatrix} dz_1 \wedge \ldots \wedge dz_k \\ z_1, \ldots, z_k \end{bmatrix}_A .$$

On the right-hand side of $(*)$ we have the residue $\mathfrak{s} := \mathrm{Res}\begin{bmatrix} \cdot \\ z_1 - \xi_1, \ldots, z_k - \xi_k \end{bmatrix}_A$ (current of type $(m-k, m)$) multiplied by f and computed on the test form $dz_1 \wedge \ldots \wedge dz_k$. This we may try to transpose to the c-holomorphic case. Let now $f \in \mathcal{O}_c(A)$ and $\xi \in \mathbb{C}^k$, then we set

$$(**) \qquad \mathfrak{r}_\xi(\varphi) := \mathrm{Res}\begin{bmatrix} t\varphi(z) \wedge dt \\ w_0, \ldots, w_k \end{bmatrix}_{T_\xi}, \quad \varphi \in \mathcal{D}_{(k,0)}(\Omega)$$

where T_ξ is the graph of the c-holomorphic mapping

$$g_\xi \colon \mathbb{C} \times A \ni (t, z) \mapsto (t - f(z), z_1 - \xi_1, \ldots, z_k - \xi_k) \in \mathbb{C}_{w_0} \times \mathbb{C}^k_w .$$

We obtain a current of type $(m+1, m+2+k)$ and we have to compute $\mathfrak{r}_\xi(dz_1 \wedge \ldots \wedge dz_k)$. It is easy to see that we may replace this current by

$$\tilde{\mathfrak{r}}_\xi(\varphi) := \mathrm{Res}\begin{bmatrix} t\varphi(z) \wedge dt \\ w_0, z_1 - \xi_1 \ldots, z_k - \xi_k \end{bmatrix}_\Gamma, \quad \varphi \in \mathcal{D}_{(k,0)}(\Omega),$$

where Γ is the graph of $\mathbb{C} \times A \ni (t, z) \mapsto t - f(z) \in \mathbb{C}_{w_0}$.

Theorem 5.2. *In the introduced setting, proposition 5.1 holds true for c-holomorphic functions and so in particular for $\xi = 0$ we have*

$$\tilde{\mathfrak{r}}_\xi(dz_1 \wedge \ldots \wedge dz_k) = \mathfrak{r}_\xi(dz_1 \wedge \ldots \wedge dz_k) = \deg_0 A \cdot f(0).$$

Proof. We shall use $\tilde{\mathfrak{r}}_\xi$ and (LP'). Fix ξ and let

$$H(t,z,w_0) := (w_0, z_1 - \xi_1, \ldots, z_k - \xi_k), \quad (t,z,w_0) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}.$$

Clearly, the proper cycle of zeroes $Z_H = \mathbb{C} \times (\{\xi\} \times \mathbb{C}^{m-k}) \times \{0\}$. Observe now that

$$\Gamma \cdot Z_H = \sum_{\zeta \in \pi^{-1}(\xi) \cap A} \mu_\zeta(\pi|_A)\{(f(\zeta), \zeta, 0)\}.$$

If now p denotes the projection $(t,z,w_0) \mapsto t$, then obviously

$$\langle [\Gamma \cdot Z_H], p \rangle = \sum_{\zeta \in \pi^{-1}(\xi) \cap A} \mu_\zeta(\pi|_A)\delta_{(f(\zeta),\zeta,0)}(p) = \sum_{\zeta \in \pi^{-1}(\xi) \cap A} \mu_\zeta(\pi|_A) f(\zeta),$$

and since by (LP'),

$$\langle [\Gamma \cdot Z_H], p \rangle = \tilde{\mathfrak{r}}_\xi(dz_1 \wedge \ldots \wedge dz_k)$$

the proof is completed. □

Remark 5.3. What also may be treated as a c-holomorphic counterpart of a Cauchy-type formula for c-holomorphic functions is the integral dependence relation established in the following easy lemma (cf. [Wh]):

Lemma 5.4. *Suppose that A has pure dimension k. Then a continuous function $f\colon A \to \mathbb{C}$ is c-holomorphic iff for any point $a \in A$ there is a neighbourhood $U \ni a$ and a polynomial $P \in \mathcal{O}(U)[t]$ monic in t (i.e. unitary) and such that $P(x, f(x)) = 0$ for $x \in U \cap A$.*

Proof. If f is c-holomorphic, then for any point $a \in A$ we may choose coordinates in $\mathbb{C}^m$ in such a way that the projection π onto the first k coordinates is a branched covering on A in a neighbourhood $V \times W \subset \mathbb{C}^k \times \mathbb{C}^{m-k}$ of a and $\pi^{-1}(\pi(a)) \cap A \cap (V \times W) = \{a\}$. Then for any point $v \in V$ outside the critical locus σ of $\pi|_A$ there are exactly d different point w^j such that $(v, w^j) \in A$. Then setting $P(v,w,t) := \prod_j (t - f(v, w^j))$ and extending its coefficients analytically through σ by the Riemann Extension Theorem we obtain the required $P \in \mathcal{O}(V \times W)[t]$.

On the other hand, if such a polynomial exists in a neighbourhood U of $a \in \mathrm{Reg}A$, then shrinking U if necessary, we may assume that $U \cap \mathrm{Reg}A$ is biholomorphic to the unit polydisc in $\mathbb{E}^k \subset \mathbb{C}^k$. Thus in fact we reduce ourselves to the case of a continuous function $f\colon \mathbb{E}^k \to \mathbb{C}$ such that $P(x, f(x)) = 0$, $x \in \mathbb{E}^k$, for some monic $P \in \mathcal{O}(\mathbb{E}^k)[t]$. This is well-known that f must be holomorphic. □

Therefore, in the situation under consideration,

$$f(z)^d + a_1(\xi) f(z)^{d-1} + \ldots + a_d(\xi) \equiv 0,$$

in a neighbourhood of zero, with $\pi(z) = \xi$, $d := \deg_0 A$ and $a_j(\xi)$ being the symmetric functions (taking account of the sign) of $f(z^{(j)})$ for $\pi^{-1}(\xi) \cap A = \{z^{(1)}, \ldots, z^{(d)}\}$.

6. Transformation law

The aim of this part is to prove the transformation law in the c-holomorphic case. Assume, as earlier, that A is a pure k-dimensional analytic set in an open set $\Omega \subset \mathbb{C}^m$.

Theorem 6.1. *Assume that $a, f \in \mathcal{O}_c(A)$ are such that neither of them vanishes identically on any irreducible component of A. Then*

$$\mathrm{Res}\begin{bmatrix} \cdot \\ f \end{bmatrix}_A = a \cdot \mathrm{Res}\begin{bmatrix} \cdot \\ af \end{bmatrix}_A.$$

Proof. On the left-hand side of the required equality we have by definition

$$(L)\qquad \mathrm{Res}\begin{bmatrix} \varphi \\ f \end{bmatrix}_A := \mathrm{Res}\begin{bmatrix} \varphi(z) \\ w \end{bmatrix}_{\Gamma_f}, \quad \varphi \in \mathcal{D}_{(k,k-1)}(\Omega), \quad ((z,w) \in \Omega \times \mathbb{C})$$

while on the right-hand side there is

$$(R)\qquad a \cdot \mathrm{Res}\begin{bmatrix} \varphi \\ af \end{bmatrix}_A := \mathrm{Res}\begin{bmatrix} t\varphi(z) \wedge dt \\ v, w \end{bmatrix}_{\Gamma}, \quad \varphi \in \mathcal{D}_{(k,k-1)}(\Omega),$$

where Γ denotes the graph of the c-holomorphic mapping

$$h\colon \mathbb{C} \times A \ni (t,z) \mapsto (t - a(z), a(z)f(z)) \in \mathbb{C}_v \times \mathbb{C}_w.$$

Take now the mapping

$$\Xi\colon \mathbb{C} \times \Omega \times \mathbb{C} \times \mathbb{C} \ni (t,z,v,w) \mapsto (t, z, t-v, vw) \in \mathbb{C} \times \Omega \times \mathbb{C} \times \mathbb{C}$$

whose jacobian is equal to

$$\det\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \mathrm{Id}_z & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & w & v \end{bmatrix} = -v.$$

Thus Ξ is a biholomorphism on the open set $\mathbb{C} \times (\Omega \setminus a^{-1}(0)) \times \mathbb{C}_* \times \mathbb{C}$. Note that we may restrict the integrals to graphs taken over $A^* := \mathrm{Reg}A \setminus a^{-1}(0)$ since we forget only zero measure sets. Keeping the same notation for the restricted graphs we have

$$\Xi(\mathbb{C} \times \Gamma_{(a,f)}) = \Gamma_{(t-a,af)} = \Gamma.$$

Applying now the change of variables formula to the integrals appearing in the definition of (R) we will obtain

$$\int_{\Gamma \cap \{|v|^2 = \varepsilon_1, |w|^2 = \varepsilon_2\}} \frac{t\varphi(z) \wedge dt}{vw} = \int_{\Xi^{-1}(\Gamma \cap \{|v|^2 = \varepsilon_1, |w|^2 = \varepsilon_2\})} \Xi^*\left(\frac{t\varphi(z) \wedge dt}{vw}\right).$$

Now, we calculate

$$\Xi^*\left(\frac{t\varphi(z) \wedge dt}{vw}\right) = \frac{t\varphi(z) \wedge dt}{(t-v)(vw)}$$

and since $\Xi^{-1}(t,z,\tilde v,\tilde w)=(t,z,t-\tilde v,\tilde w/(t-\tilde v))$, there is

$$\begin{aligned}&\Xi^{-1}(\{(t,z,\tilde v,\tilde w)\mid \tilde v=t-a(z),\tilde w=a(z)f(a),|\tilde v|^2=\varepsilon_1,|\tilde w|^2=\varepsilon_2\})=\\&=\{(t,z,t-\tilde v,\tilde w/(t-\tilde v))\mid a(z)=t-\tilde v,\tilde w=a(z)f(z),|\tilde v|^2=\varepsilon_1,|\tilde w|^2=\varepsilon_2\}=\\&=\{(t,z,v,w)\mid a(z)=v,vw=a(z)f(z),|t-v|^2=\varepsilon_1,|vw|^2=\varepsilon_2\}=\\&=\{(t,z,v,w)\mid a(z)=v,w=f(z),|t-v|^2=\varepsilon_1,|vw|^2=\varepsilon_2\}.\end{aligned}$$

Fubini's Theorem together with Cauchy's formula applied to $tdt/(t-v)$ yields for (R),

$$2\pi i\int_{\Gamma_{(a,f)}\cap\{|vw|^2=\varepsilon_2\}}\frac{v\varphi(z)}{vw}.$$

Taking the limit we obtain from (R) the residue

$$\mathrm{Res}\begin{bmatrix}v\varphi(z)\\ vw\end{bmatrix}_{\Gamma_{(a,f)}}=\mathrm{Res}\begin{bmatrix}\varphi(z)\\ w\end{bmatrix}_{\Gamma_{(a,f)}},$$

the equality coming from the restricted holomorphic transformation law (see [BVY]). Since in the integrals from the right-hand side the variable v is now a 'phantom' one, we may forget it getting just (L) as required. $\square$

We turn now to proving a more general version of this theorem. To achieve this aim we shall need the following lemma proposed by A. Yger:

Lemma 6.2. *Assume that $f=(f_1,\dots,f_n)\in\mathcal{O}_c(A,\mathbb{C}^n)$ is such that $f^{-1}(0)$ has pure dimension $k-n$. Let $a_1,\dots,a_l\in\mathcal{O}_c(A)$. Then for any polynomial $Q\in\mathbb{C}[t_1,\dots,t_l]$ we have the following equality between currents of type $(m-k,m-k+n)$: for any test form $\varphi(z)$,*

$$Q(a_1,\dots,a_l)\cdot\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f_1,\dots,f_n\end{bmatrix}_A=\mathrm{Res}\begin{bmatrix}Q(t_1,\dots,t_l)\varphi(z)\wedge dt_1\wedge\dots\wedge dt_l\\ v_1,\dots,v_l,w_1,\dots,w_n\end{bmatrix}_\Gamma,$$

where Γ is the graph of $\gamma(t_1,\dots,t_l,z)=(t_1-a_1(z),\dots,t_l-a_l(z),f(z))$ defined and c-holomorphic on $\mathbb{C}^l_t\times A$ with values in $\mathbb{C}^l_v\times\mathbb{C}^n_w$.

Proof. By definition, for $\varphi\in\mathcal{D}_{(k,k-n)}(\Omega)$, there is

$$Q(a_1,\dots,a_l)\cdot\mathrm{Res}\begin{bmatrix}\varphi(z)\\ f_1,\dots,f_n\end{bmatrix}_A=\mathrm{Res}\begin{bmatrix}t_0\varphi(z)\wedge dt_0\\ w_0,w_1,\dots,w_n\end{bmatrix}_\Lambda,$$

where Λ is the graph of $(t_0,z)\mapsto(t_0-Q(a_1(z),\dots,a_l(z)),f(z))$. To prove the assertion we will compute in two different ways the residue

$$\mathfrak{t}(\varphi):=\mathrm{Res}\begin{bmatrix}Q(t_1,\dots,t_l)\varphi(z)\wedge dt_1\wedge\dots\wedge dt_l\wedge dt_0\\ v_1,\dots,v_l,w_0,w_1,\dots,w_n\end{bmatrix}_\Upsilon,$$

where Υ is the graph of

$$(t_0,t_1,\dots,t_l,z)\mapsto(t_1-a_1(z),\dots,t_l-a_l(z),t_0-Q(a_1(z),\dots,a_l(z)),f(z)).$$

Put $a(z) = (a_1(z), \dots, a_l(z))$ and $dt := dt_1 \wedge \dots \wedge dt_l$. The integrals appearing in the definition of $\mathfrak{t}(\varphi)$ are computed over the set

$$E := \{(t_0, t, z, v, w_0, w) \colon v = t - a(z), w_0 = t_0 - Q(a(z)), w = f(z),\\ |v_0|^2 = \eta_0, |v_\iota|^2 = \eta_\iota, |w_0|^2 = \varepsilon_0, |w_j|^2 = \varepsilon_j\},$$

(given by an admissible path $\eta_0 \ll \dots \ll \eta_n \ll \varepsilon_0 \ll \dots \ll \varepsilon_n$) and are of the form

$$\int_E \frac{Q(t)\varphi(z) \wedge dt \wedge dt_0}{v_1 \cdot \ldots v_l \cdot w_1 \cdot \ldots \cdot w_n \cdot w_0} =$$
$$= \int_{E_1} \left(\int_{E_2^z} \frac{dt_0}{t_0 - Q(a(z))} \right) \frac{Q(t)\varphi(z) \wedge dt}{v_1 \cdot \ldots \cdot v_l w_1 \cdot \ldots \cdot w_n} =$$
$$= 2\pi i \int_{E_1} \frac{Q(t)\varphi(z) \wedge dt}{v_1 \cdot \ldots \cdot v_l w_1 \cdot \ldots \cdot w_n},$$

where $E_1 := \{(t, z, v, w) \colon v = t - a(z), w = f(z), |v_\iota|^2 = \eta_\iota, |w_j|^2 = \varepsilon_j\}$ and on $E_2^z := \{(t_0, w_0) \colon w_0 = t_0 - Q(a(z)), |w_0|^2 = \varepsilon_0\}$ we computed the index (independent of z). Therefore

$$\mathfrak{t}(\varphi) = \operatorname{Res} \begin{bmatrix} Q(t_1, \dots, t_l)\varphi(z) dt_1 \wedge \dots \wedge dt_l \\ v_1, \dots, v_l, w_1, \dots, w_n \end{bmatrix}_\Gamma .$$

Let us find an other expression for this current. First observe that in the expression of $\mathfrak{t}(\varphi)$ we may write t_0 instead of $Q(t_1, \dots, t_l)$. Indeed, on Υ we have $t_0 - w_0 = Q(a(z))$ and $t - w = a(z)$. Remember that the residue is annihilated by the ideal. Thus, since Q is a polynomial, we may first replace in $\mathfrak{t}(\varphi)$ the factor $Q(t)$ by $Q(t - w)$. This in turn is equal to $t_0 - w_0$ on Υ and since w_0 is in the ideal, we get the assertion.

If we repeat now the previous argument extracting this time, by means of Fubini's Theorem ($Q(t_1, \dots, t_l)$ does not bother us any longer), all the integrals

$$\int_{\{(t_j, v_j) \colon v_j = t_j - a_j(z), |v_j|^2 = \eta_j\}} \frac{dt_j}{t_j - a_j(z)} = 2\pi i,$$

and so we get

$$\mathfrak{t}(\varphi) = \operatorname{Res} \begin{bmatrix} t_0 \varphi(z) \wedge dt_0 \\ w_0, w_1, \dots, w_n \end{bmatrix}_\Lambda .$$

This completes the proof. □

Theorem 6.3. *Assume that $f_1, \dots, f_n, g_1, \dots, g_n \in \mathcal{O}_c(A)$ are such that $\bigcap_j f_j^{-1}(0)$ and $\bigcap_j g_j^{-1}(0)$ have pure dimension $k - n$. If there exist functions $a_{\iota j} \in \mathcal{O}_c(A)$, $\iota, j = 1, \dots, n$ such that $g_j = \sum_\iota a_{\iota j} f_\iota$ for all j, then*

$$\operatorname{Res} \begin{bmatrix} \varphi \\ f_1, \dots, f_n \end{bmatrix}_A = \Delta \cdot \operatorname{Res} \begin{bmatrix} \varphi \\ g_1, \dots, g_n \end{bmatrix}_A , \quad \varphi \in \mathcal{D}_{(k,k-n)}(\Omega),$$

where $\Delta := \det[a_{\iota j}]_{\iota, j} \in \mathcal{O}_c(A)$.

Proof. For simplicity sake we shall restrict ourselves to the case $n = 2$, the main idea being the same in the general one. Thanks to the preceding lemma we only need to show that for any $\varphi \in \mathcal{D}_{(k,k-2)}(\Omega)$,

$$\operatorname{Res}\begin{bmatrix}\varphi\\ f_1, f_2\end{bmatrix}_A = \operatorname{Res}\begin{bmatrix}(t_{11}t_{22} - t_{12}t_{21})\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22}\\ v_{11}, v_{12}, v_{21}, v_{22}, w_1, w_2\end{bmatrix}_\Gamma,$$

where Γ is the graph of $\gamma(t_{11}, t_{12}, t_{21}, t_{22}, z) = ((t_{\iota j} - a_{\iota j}(z))_{\iota j}, g_1(z), g_2(z))$.

In the integrals approximating the residue on the right-hand side we change the variables in the following way: we leave the z_j, the $t_{\iota j}$ and the $v_{\iota j}$ untouched changing only

$$(w_1, w_2) \quad \text{to} \quad (u_1, u_2) \quad \text{such that} \quad \begin{cases} w_1 = u_1 t_{11} + u_2 t_{12}\\ w_2 = u_1 t_{21} + u_2 t_{22}.\end{cases}$$

The integrals become (we forget only a zero-measure set not affecting them)

$$\int_E \frac{(t_{11}t_{22} - t_{12}t_{21})\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22}}{v_{11}v_{12}v_{21}v_{22}(u_1t_{11} + u_2t_{12})(u_1t_{21} + u_2t_{22})}$$

computed over

$$E := \{((t_{\iota j})_{\iota j}, z, (v_{\iota j})_{\iota j}, u_1, u_2) \mid |v_{\iota j}|^2 = \varepsilon_{\iota j}, |u_1t_{11} + u_2t_{12}|^2 = \varepsilon_1, \\ |u_1t_{21} + u_2t_{22}|^2 = \varepsilon_2\}.$$

Note that this is a subset of the graph Γ' of $((t_{\iota j} - a_{\iota j}(z))_{\iota j}, f_1(z), f_2(z))$. Applying now the restricted transformation law to the residue obtained in this way, we have

$$\operatorname{Res}\begin{bmatrix}(t_{11}t_{22} - t_{12}t_{21})\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22}\\ v_{11}, v_{12}, v_{21}, v_{22}, (u_1t_{11} + u_2t_{12}), (u_1t_{21} + u_2t_{22})\end{bmatrix}_{\Gamma'} = \\ = \operatorname{Res}\begin{bmatrix}\varphi(z) \wedge dt_{11} \wedge dt_{12} \wedge dt_{21} \wedge dt_{22}\\ v_{11}, v_{12}, v_{21}, v_{22}, u_1, u_2\end{bmatrix}_{\Gamma'}.$$

Finally, applying Fubini's Theorem and the index formula we easily check (as in the previous theorem) that the latter is equal to

$$\operatorname{Res}\begin{bmatrix}\varphi(z)\\ u_1, u_2\end{bmatrix}_{\Gamma_{(f_1,f_2)}} = \operatorname{Res}\begin{bmatrix}\varphi(z)\\ f_1, f_2\end{bmatrix}_A$$

which ends the proof. □

Final remark

The idea of using the graph and the coordinates functions on it to compute the residue could be perhaps useful when looking for a desingularization-free proof of the existence of the Coleff-Herrera residue currents. At least, the approach involving the graphs carries over the problem of desingularization from functions to sets. This may turn out to be simpler in use, in some sense.

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