Singular points of weakly holomorphic functions

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Abstract. In this paper we are interested in two kinds of singular points of weakly holomorphic functions. Points where a weakly holomorphic function is not holomorphic and points at which it just is not continuous. The latter are closely connected to points of irreducibility of the given analytic set. We investigate the structure of such points proving they form analytically constructible sets. We prove also that non-holomorphicity points of a given weakly or c-holomorphic function form an analytic subset of the singularities.

1. Introduction

Throughout this paper $A \subset \mathbb{C}^m$ is a locally analytic set.

When trying to figure out which complex functions defined on A are the best generalization of the notion of a holomorphic function one comes across two natural notions. The first one is due to R. Remmert.

Definition 1.1. (cf. [Wh]) A mapping $f\colon A \to \mathbb{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset $\mathrm{Reg}A$ of regular points is holomorphic. We denote by $\mathcal{O}_c(A, \mathbb{C}^n)$ the ring of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

A well-known theorem states that a mapping defined in an open set is holomorphic if and only if it is continuous and its graph is an analytic set (it is then a submanifold). We have a similar result for c-holomorphic mappings (cf. [Wh] 4.5Q), which motivates this generalization:

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Theorem 1.2. *A mapping $f\colon A\to\mathbb{C}^n$ is c-holomorphic iff it is continuous and its graph $\Gamma_f:=\{(x,f(x))\mid x\in A\}$ is locally analytic in $\mathbb{C}^m\times\mathbb{C}^n$.*

On the other hand, there is the notion of *weakly holomorphic* functions which usually people are much more acquainted with. They were introduced by H. Cartan.

Definition 1.3. A holomorphic function $f\colon \mathrm{Reg}A\to\mathbb{C}$ is called *weakly holomorphic* if it is locally bounded on A (i.e. for any $a\in A$ there exists a neighbourhood U of a such that f is bounded on $U\cap A$).
A mapping $f\colon \mathrm{Reg}A\to\mathbb{C}^n$ is called weakly holomorphic if all its components are weakly holomorphic. We denote by $\mathcal{O}_w(A,\mathbb{C}^n)$ the ring of weakly holomorphic mappings and put $\mathcal{O}_w(A):=\mathcal{O}_w(A,\mathbb{C})$.

One checks as in [Wh] that if $A=\bigcup A_\iota$ is the decomposition of A into irreducible components, then f has a unique extension onto $\mathrm{Reg}A_\iota$ for each ι (and that works in fact for germs). Then f is weakly holomorphic iff it is such on each irreducible component of A.

It is useful to observe that for locally irreducible sets weak holomorphic and c-holomorphic functions are just the same (see [Wh], *c-holomorphic* stands in fact for *continuous weak holomorphic*).

Recall that an *analytically constructible* subset of some open set $\Omega\subset\mathbb{C}^m$ is a set which can be written locally in Ω in the form $\bigcup_{\iota=1}^{p}\bigcap_{j=1}^{q_j}\{F_{\iota j}*_{\iota j}0\}$, where $*_{\iota,j}\in\{=,\neq\}$ and $F_{\iota j}$ are holomorphic (see [Ł]). In particular the difference of two analytic sets is analytically constructible and the closure of a constructible set is analytic.

In [D2] we proved a weakly holomorphic counterpart of theorem 1.2. It seems strange that it was not stated anywhere till now. It clearly implies theorem 1.2. For convenience sake we recall it with its proof:

Theorem 1.4. ([D2]) *Let $f\colon \mathrm{Reg}A\to\mathbb{C}^n$ be a mapping locally bounded on an analytic subset A of an open set $\Omega\subset\mathbb{C}^m$. The following three conditions are then equivalent:*

(1) *$f\in\mathcal{O}_w(A,\mathbb{C}^n)$;*
(2) *$\overline{\Gamma_f}$ is analytic in $\Omega\times\mathbb{C}^n$;*
(3) *Γ_f is analytically constructible in $\Omega\times\mathbb{C}^n$.*

Proof. First note that we may restrict ourselves to the case $n=1$ since $\Gamma_f=\bigcap_{j=1}^{n}\Gamma_j$, where

$$\Gamma_j:=\{(x,y_1,\ldots,y_{j-1},f_j(x),y_{j+1},\ldots,y_n)\mid x\in\mathrm{Reg}A,y_\iota\in\mathbb{C}\}.$$

We may as well assume that A has pure dimension k (using restrictions to the irreducible components of A) with $0<k<m$ (otherwise, since there are no singularities, there is nothing to do — cf. the Analytic Graph Theorem).

If we have (1)⇔(2), the equivalence (2)⇔(3) is quite immediate. Indeed, if $\overline{\Gamma_f}$ is analytic, then $\Gamma_f=\overline{\Gamma_f}\setminus(\mathrm{Sng}A\times\mathbb{C})$ is the difference of two analytic sets, thence is analytically contructible. On the other hand, if Γ_f is analytically constructible, then its closure is analytic and so $f\in\mathcal{O}_w(A)$.

We turn now to proving (1)⇔(2). The problem being local we may assume that $h\in\mathcal{O}(\Omega)$ is a global universal denominator for A. Having fixed $f\in\mathcal{O}_w(A)$

we can find $g \in \mathcal{O}(\Omega)$ such that $fh = g$ on $\mathrm{Reg}A$. Consider the analytic set $X := \{(z,t) \in A \times \mathbb{C} \mid h(z)t = g(z)\}$. It remains now to observe that the set

$$\begin{aligned}&\Gamma_f \cap \{(z,t) \in \Omega \times \mathbb{C} \mid h(z) \neq 0\} = \\ &= X \cap (\mathrm{Reg}A \times \mathbb{C}) \cap \{(z,t) \in \Omega \times \mathbb{C} \mid h(z) \neq 0\} = \\ &= X \setminus \big[(X \cap (\mathrm{Sng}A \times \mathbb{C})) \cup \{(z,t) \in \Omega \times \mathbb{C} \mid h(z) = 0\}\big]\end{aligned}$$

is dense in Γ_f. Its closure in $\Omega \times \mathbb{C}$ is clearly analytic. □

Note. In the case $A = \Omega$ the theorem above asserts that a necessary and sufficient condition for a locally bounded mapping to be holomorphic is the analytic constructibility of its graph.

Finally, we will call simply *holomorphic* the restrictions to A of holomorphic functions defined in a neighbourhood of A. Their ring will be denoted $\mathcal{O}(A)$. Obviously $\mathcal{O}(A) \subsetneq \mathcal{O}_c(A) \subsetneq \mathcal{O}_w(A)$ (see [Wh]).

2. Proper projections of analytic sets and irreducibility points

In this section we consider the following situation: $X \subset \mathbb{C}^N$ is a locally analytic set and $\pi\colon X \to A$ is a proper holomorphic surjection. Without loss of generality we may assume that $N = m + n$ and π is just the restriction to X of the natural projection onto the first m coordinates. Then its multiplicity is well-defined and locally bounded (see e.g. [Ch]). We fix now a neighbourhood Ω of zero in $\mathbb{C}^m$ and put $U := \Omega \times \mathbb{C}^n$ assuming that X and A are closed in U.

The central result of this section is the following theorem. Recall that a function $f\colon Z \to \mathbb{C}$ is called analytically constructible (where Z is constructible too) if its graph is analytically constructible.

Theorem 2.1. *In the introduced setting the function*

$$\mu_\pi\colon A \ni a \mapsto \#\pi^{-1}(a) \in \mathbb{Z}$$

is analytically constructible.

Proof. Similarly to [T] we start with observing that the analytic constructibility of μ_π is equivalent to the fibres $\mu_\pi^{-1}(p)$ being analytically constructible (indeed, the graph coincides with $\bigcup_{p\in\mathbb{Z}} \mu_\pi^{-1}(p) \times \{p\}$). The problem is clearly a local one.

Let $d := \max\{\#\pi^{-1}(a) \mid a \in A\}$. Consider then the fibred product

$$X^{\{d\}} := \underbrace{X \times_\pi \ldots \times_\pi X}_{d\,times} = \{x \in X^d \mid \pi(x_\iota) = \pi(x_j),\ \iota, j = 1, \ldots, d\}.$$

It is analytic since it coincides with $(\pi \times \ldots \times \pi)^{-1}(\Delta_{A^d})$, where Δ_{A^d} is the diagonal in A^d (i.e. the set of all points $(a, \ldots, a) \in A^d$).

Observe that

$$f_{\iota,j}\colon X^{\{d\}} \ni (x_1, \ldots, x_d) \mapsto (x_\iota - x_j) \in \mathbb{C}^n, \quad 1 \leq \iota < j \leq d,$$

is a finite collection of holomorphic functions and the mapping

$$\rho\colon X^{\{d\}} \ni x \mapsto \pi(x_1) \in A$$

is clearly holomorphic and proper (since π is proper).

Now it is obvious that

$$\{\mu_\pi = 0\} = \Omega \setminus A \quad \text{and} \quad \{\mu_\pi = 1\} = A \setminus \rho\left(\bigcup_{\iota<j}\{f_{\iota,j} \neq, 0\}\right),$$

whence both these sets are analytically constructible. Indeed, due to the properness of ρ, the set $\rho\left(\bigcup_{\iota<j}\{f_{\iota,j} \neq, 0\}\right) = \{\mu_\pi > 1\}$ is constructible by the Chevalley-Remmert Theorem (see [Ł]).

Observe that in the case $d = 1, 2$ there is nothing more to prove (for $d = 1$ leads to $A = \{\mu_\pi = 1\}$ while $d = 2$ yields $\{\mu_\pi = 2\} = A \setminus \{\mu_\pi = 1\}$). We might suppose thus that $d \geq 3$.

It is enough to prove that each set $\{\mu_\pi > p\}$ is analytically constructible, since $\{\mu_\pi = p\} = \{\mu_\pi > p - 1\} \setminus \{\mu_\pi > p\}$.

Clearly enough, we have in general for $p \in \{2, \ldots, d\}$,

$$\{\mu_\pi > p - 1\} = \rho\left(\bigcup_{1\leq\iota_1<\ldots<\iota_p\leq r}\ \bigcap_{1\leq j<\kappa\leq p}\{f_{\iota_j,\iota_\kappa} \neq 0\}\right).$$

The Chevalley-Remmert Theorem ends the proof. □

This theorem has many interesting consequences. The first one concerns the *irreducibility points* of A, i.e. points at which A induces an irreducible germ.

Corollary 2.2. *In the introduced setting let*

$$A^{\prec} := \{a \in \mathrm{Sng}A \mid \textit{the germ } A_a \textit{ is irreducible}\}.$$

Then $A^{\prec}$ is analytically constructible.

Before we prove this corollary, observe that one cannot possibly hope to obtain analycity of $A^{\prec}$ in general, unless $\dim A = 1$:

Example 2.3. Let $X := \{x^2y = z^2\}$ be Whitney's Umbrella and consider $Y := \{z = 0\} \cup \{y^2 = x^3\}$, both sets being taken in $\mathbb{C}^3$. Then

$$X^{\prec} = \{(0, 0)\}, \quad \text{but} \quad Y^{\prec} = \{x = y = 0, z \neq 0\}.$$

The difference in structure of these sets is due to the fact that Y is globally reducible while X is not and in both cases the singularities are along one-dimensional sets.

The corollary above is actually itself a corollary to the following one.

Corollary 2.4. *The function*

$$\mu\colon \mathrm{Sng}A \ni a \mapsto \#\{\textit{irreducible components of } A_a\} \in \mathbb{Z}$$

is analytically constructible.

Proof of corollary 2.2. The problem being local we may suppose that zero belongs to $A^{\prec}$, A is a closed analytic subset of some neighbourhood of zero and $\pi\colon N \to A$ is its normalization, where $N \subset \mathbb{C}^m \times \mathbb{C}^k$ for some $k \geq 1$ (see e.g. [Ł] Local Normalization Theorem). We are exactly in the situation of theorem 2.1 and $\mu = \mu_\pi|_{\mathrm{Sng}A}$, whence $A^{\prec} = \{\mu = 1\}$ is constructible. □

Proof of corollary 2.4. Once again it suffices to consider the local normalizing mapping π and apply theorem 2.1 to $\mu = \mu_\pi|_{\mathrm{Sng}A}$. □

Note. The matter being local, the corollary above has clearly a counterpart on complex analytic manifolds (assumed to be second-countable by definition) since the local maps are biholomorphisms and so they preserve analytic constructibility (i.e. we consider A as an analytic subset of some analytic manifold Ω).

Corollary 2.5. *If A is analytic in the open set (or more generally in the analytic manifold) Ω, then the function $\widehat{\mu}\colon \Omega \to \mathbb{Z}$ defined by*

$$\widehat{\mu}(x) = \begin{cases} \mu(x), & \textit{when } x \in \mathrm{Sng}A; \\ 1, & \textit{when } x \in \mathrm{Reg}A; \\ 0, & \textit{when } x \notin A, \end{cases}$$

is analytically constructible and locally bounded.

Proof. Clearly $\widehat{\mu}^{-1}(0) = \Omega \setminus A$ is analytically constructible. As to $\widehat{\mu}^{-1}(1)$, it is the union of $\mu^{-1}(1)$ and $\mathrm{Reg}A$, and both are analytically constructible. □

Recall now that an *analytic cycle* on some manifold Ω is the formal sum $Z = \sum \alpha_\iota Z_\iota$, where $\alpha_\iota \in \mathbb{Z}$ and $\{Z_\iota\}$ is a locally finite family of pairwise distinct, analytic, irreducible subsets of Ω (see [T]). The analytic set $|Z| := \bigcup Z_\iota$ is called the *support* of the cycle Z. We define also the *degree* of Z by

$$\nu(Z, x) := \sum \alpha_\iota \nu(Z_\iota, x),$$

where $\nu(Z_\iota, x)$ is the classical degree or *Lelong number* of Z_ι at $x \in \Omega$ (if $x \notin Z_\iota$, then $\nu(Z_\iota, x) = 0$).

By [T] (2.1), there exists exactly one analytic cycle Z in Ω such that $\widehat{\mu}$ coincides with $\nu(Z)$, where

$$\nu(Z)\colon \Omega \ni x \mapsto \nu(Z, x) \in \mathbb{Z}$$

is the degree function of Z. A natural question arises here: what kind of information on A does Z bring with, and, besides, what actually *is* Z as a cycle?

Note that in general $A \subsetneq |Z|$ (otherwise, if $A = |Z|$, we obtain $\alpha_\iota = 1$ and so $\widehat{\mu}(x) = \nu(A, x)$ which is not true in general as one can see for instance by considering Whitney's Umbrella).

3. Singular points of weakly holomorphic functions

Fix a function $f \in \mathcal{O}_w(A)$ and put

$$\mathcal{N}(f) := \{a \in \mathrm{Sng}A \mid f \textit{ is not continuous at } a\}.$$

Theorem 3.1. *The set $\mathcal{N}(f)$ is analytically constructible.*

Proof. We know by theorem 1.4 that $\overline{\Gamma_f}$ is analytic. Moreover, for each $a \in \mathrm{Sng}A$ there is $\#(\{a\} \times \mathbb{C}) \cap \overline{\Gamma_f} < +\infty$. Indeed, for any such a point we have a finite decomposition of the germ A_a into irreducible germs, say r of them, and the restriction of f to each of them is continuous at a (see e.g. [Wh]). Thus at a we have at most r possible values for $f(a)$.

Now let π be the restriction to $\overline{\Gamma_f}$ of the natural projection $\mathbb{C}^m \times \mathbb{C} \to \mathbb{C}^m$. Clearly it is a proper mapping (since f is locally bounded) and $\mathcal{N}(f) = \{\mu_\pi > 1\}$ according to the notation introduced in the previous section. Thus, by theorem 2.1, the set $\mathcal{N}(f)$ is analytically constructible. □

Recall that the Zariski tangent space of an analytic germ X at a point $a \in \mathbb{C}^m$ is defined as

$$T_a^{\mathrm{Zar}} X := \bigcap \{\mathrm{Ker}\ d_a\phi \mid \phi \in \mathscr{I}_a(X)\},$$

where $\mathscr{I}_a(X) \subset \mathcal{O}_m$ is the ideal of germs of holomorphic functions vanishing on the germ X (see [Wh]).

There is a nice holomorphicity criterion for c-holomorphic functions, whose short proof one can find in [D1]. The point is that this proof works also in the weakly holomorphic case and we obtain the following

Theorem 3.2. *Let $a \in A \setminus \mathscr{N}(f)$. Then f is holomorphic at a iff*

$$T_{(a,f(a))}^{\mathrm{Zar}} \overline{\Gamma_f} \cap (\{0\} \times \mathbb{C}) = \{0\}.$$

Proof. For simplicity sake assume that $a = f(a) = 0$. If $F \supset f$ is a holomorphic extension of f to a neighbourhood of zero in $\mathbb{C}^m$, then $\varphi(z,t) = t - F(z)$ (here $(z,t) \in \mathbb{C}^m \times \mathbb{C}$) belongs to $\mathscr{I}_0(\overline{\Gamma_f})$. Since φ is a submersion, then Ker $d_0\varphi = T_0\Gamma_F$. Hence

$$T_0^{\mathrm{Zar}} \overline{\Gamma_f} \cap (\{0\} \times \mathbb{C}) \subset T_0\Gamma_F \cap (\{0\} \times \mathbb{C}) = \{0\}.$$

To prove the 'only if' part first note that the assumption leads to $m \geq \dim T_0^{\mathrm{Zar}} \overline{\Gamma_f}$. We know (cf. [Wh]) that there exists a submanifold Γ such that $\Gamma_0 \supset (\overline{\Gamma_f})_0$ and $T_0\Gamma = T_0^{\mathrm{Zar}} \overline{\Gamma_f}$. We apply now Lemma (1.4) from [D1] to find an m-dimensional submanifold $\tilde{\Gamma}$ whose germ at zero contains the germ Γ_0 and whose tangent space at zero meets $\{0\} \times \mathbb{C}$ only at zero.

By the Implicit Function Theorem it is now clear that $\tilde{\Gamma}$ is the germ of the graph of a holomorphic function over $\mathbb{C}^m$. □

Now let us introduce the set

$$\mathscr{O}_w(f) := \{a \in A \mid f \textit{ is not holomorphic at } a\}.$$

It is clearly a closed subset of Sng$A$.

Theorem 3.3. *The set $\mathscr{O}_w(f)$ is an analytic subset of* Sng$A$.

Proof. First observe that $\mathscr{N}(f) \subset \mathscr{O}_w(f)$.

Since $\mathscr{O}_w(f)$ is closed, we only need to show its constructibility at any point $a \in \mathscr{O}_w(f)$. Fix such a point a. There is a finite set $\{b_1, \dots, b_r\} \subset \mathbb{C}$ such that $\overline{\Gamma_f} \cap (\{a\} \times \mathbb{C}) = \{(a, b_1), \dots, (a, b_r)\}$. Take pairwise disjoint discs $D_j \ni b_j$. Let $\pi(z) = (z_1, \dots, z_m)$ denote the natural projection from $\mathbb{C}^m \times \mathbb{C}$ onto $\mathbb{C}^m$. It is proper on $\overline{\Gamma_f}$.

By Cartan's Coherence Theorem (see [Ł]), we can find pairwise disjoint neighbourhoods $U_j = V \times W_j \subset \Omega \times D_j$ of (a, b_j) and some holomorphic functions $g_{j,1}, \dots, g_{j,r_j} \in \mathcal{O}(U_j)$ whose germs at $z \in U_j$ generate the ideal $\mathscr{I}_z(\overline{\Gamma_f})$. Then, for $z \in U_j \cap \overline{\Gamma_f}$,

$$T_z^{\mathrm{Zar}} \overline{\Gamma_f} = \bigcap_{\iota=1}^{r_j} \mathrm{Ker}\ d_z g_{j,\iota}.$$

Indeed, the inclusion '$\subset$' is quite obvious from the definition of the Zariski tangent space. To prove that the converse one holds too, observe that any $g \in \mathscr{I}_z(\overline{\Gamma_f})$ can be written as the sum $\sum_1^{r_j} h_\iota g_{j,\iota}$ with some holomorphic germs h_ι. Then

$d_z g = \sum_1^{r_j}(h_\iota(z) d_z g_{j,\iota} + g_{j,\iota}(z) d_z h_\iota)$ and since $g_{j,\iota}(z) = 0$, there is Ker $d_z g \supset \bigcap_1^{r_j}$ Ker $d_z g_{j,\iota}$.

Observe that $\{0\}\times\mathbb{C} \subset T_z^{\mathrm{Zar}}\overline{\Gamma_f}$ (where $z \in U_j \cap \overline{\Gamma_f}$) is equivalent to $\frac{\partial g_{j,\iota}}{\partial z_{m+1}}(z) = 0$ for all $\iota = 1, \ldots, r_j$. Now,

$$\mathscr{O}_w(f)\cap V = (\mathscr{N}(f)\cap V)\cup\bigcup_{j=1}^{r}\pi\left(\left\{z \in \overline{\Gamma_f}\cap U_j \mid \frac{\partial g_{j,\iota}}{\partial z_{m+1}}(z) = 0, \iota = 1, \ldots, r_j\right\}\right).$$

Indeed, to see that '$\supset$' holds note that if x is a point from the set on the righthand side and $x \notin \mathscr{N}(f) \cap V$, then $f(x)$ is well-defined as the only point over x in $\overline{\Gamma_f}$ and so $(x, f(x))$ belongs to exactly one of the sets U_j. By Theorem 3.2, f cannot be holomorphic at x. On the other hand, if $x \in \mathscr{O}_w(f) \cap V \setminus \mathscr{N}(f)$, then once again x is a continuity point of f at which f is not holomorphic and since $\{(x, f(x))\} = \overline{\Gamma_f} \cap (\{x\} \times \mathbb{C})$, there is exactly one j such that $(x, f(x)) \in U_j$. Then we apply Theorem 3.2.

Finally, Remmert's Proper Mapping Theorem implies that the projections under consideration ara analytic in V, whence $\mathscr{O}_w(f)\cap V$ is analytically constructible in V. □

Corollary 3.4. *The set $A \setminus \bigcap\{\mathscr{O}_w(g) \mid g \in \mathcal{O}_w(A)\}$ coincides with the set of normal points of A and its complement in A is an analytic set.*

Corollary 3.5. *If $f \in \mathcal{O}_c(A)$, then the set*

$$\mathscr{O}_c(f) := \{a \in A \mid f \text{ is not holomorphic at } a\}$$

is analytic in Sng$A$.
In particular, $A \setminus \bigcap\{\mathscr{O}_c(g) \mid g \in \mathcal{O}_c(A)\}$ is the set of weakly normal points of A and its complement in A is analytic.

Obviously, $\mathscr{O}_c(f) = \mathscr{O}_w(f)$, but we would like to distinguish the c-holomorphic case from the weak holomorphic one. In doing so we are aiming at the following proposition.

First fix an open set Ω in which A is closed and put $U := \Omega \setminus \overline{\mathscr{N}(f)}$. Then U is open (we are substracting an analytic subset of Sng$A$). Let $X := A \cap U$. Notice that $X \neq \varnothing$ but there may be Sng$X = \varnothing$.

Proposition 3.6. *In the introduced setting $f|_X \in \mathcal{O}_c(X)$ and the following holds:*

(1) *The set $\mathscr{O}_c(f|_X)$ is analytically constructible in Ω;*
(2) *There is $\mathscr{O}_w(f) = \overline{\mathscr{N}(f)} \cup \mathscr{O}_c(f|_X)$, the union being disjoint.*

Proof. Observe first that

$$\overline{\mathscr{N}(f)} \subset \mathscr{O}_w(f).$$

Indeed, f cannot have a holomorphic extension at a point $a \in \partial\mathscr{N}(f)$, since otherwise this extension would be discontinuous in a sequence of points converging to a.

Clearly enough $f|_X$ is c-holomorphic. Therefore, by theorem 3.5, $\mathscr{O}_c(f|_X)$ is analytic in U. But it is also easy to see that

$$\mathscr{O}_w(f) = \overline{\mathscr{N}(f)} \cup \mathscr{O}_c(f|_X)$$

the union being disjoint by definition. Since $\mathscr{O}_w(f)$ is analytic, then there is $\mathscr{O}_c(f|_X) = \mathscr{O}_w(f) \setminus \overline{\mathscr{N}(f)}$ is the difference of two analytic sets and so is constructible. □

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