The unique minimality of an averaging projection

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Abstract

In this paper we will prove that an averaging projection $P_a\colon K(H) \to Y$, given by a formula $P_a(A) = \frac{A+A^T}{2}$, is the only norm-one projection. Here $K(H)$ is a space of compact operators on a separable real Hilbert space H, and Y is a subspace of $K(H)$ consisting of all symmetric operators.

Key words: compact operator, self-adjoint operators, minimal projection, uniqueness of minimal projection.

1 Introduction

Let X be a normed space over $\mathbb{R}$ and let Y be a linear subspace of X. A bounded linear operator $P\colon X \to Y$ is called a projection if $P\big|_Y = \mathrm{Id}\big|_Y$. The

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set of all projections from X onto Y will be denoted by $\mathrm{P}(X,\ Y)$.
A projection P_0 is called minimal if

$$\|P_0\| = \inf\Big\{\|P\| \quad : \quad P \in \mathrm{P}(X,\ Y)\Big\}. \tag{1.1}$$

The constant

$$\Lambda(X,\ Y) = \inf\Big\{\|P\| \ : \ P \in \mathrm{P}(X,\ Y)\Big\}, \tag{1.2}$$

is called the relative projection constant.
A projection $P_0 \in \mathrm{P}(X,\ Y)$ is called minimal if

$$\|P_0\| = \Lambda(X,\ Y). \tag{1.3}$$

One of the difficult problems in the theory of projections is the unique-minimality of minimal projection. The research concerning this problem has its origin in a famous paper [9] where the unique minimality of the classical Fourier projection F_n (defined on $C_0(2\pi)$) onto the subspace of trigonometric polynomials of degree $\leq n$ has been proved.
Since then, many results concerning the unique minimality of minimal projection has been obtained
(see e.g. [7], [8], [10], [17], [18], [25], [26], [27], [28]). For other results concerning minimal projection
see e.g.[1], [2], [3], [4], [5], [6], [13], [14], [15], [16], [19], [20], [21], [22], [23].
The aim of this paper is to show the unique minimality of an averaging projection

$$P_a(A) = \frac{A + A^T}{2}, \tag{1.4}$$

defined on the space of all compact operators acting on a separable Hilbert space. More precisely, let H denote any real separable Hilbert space and

$$K(H) = \Big\{T \colon H \to H \quad : \quad \text{T is compact and linear}\Big\}, \tag{1.5}$$

$$Y = \Big\{A \in K(H) \quad : \quad A = A^T\Big\}. \tag{1.6}$$

Let $\{e_n\}_{n=1}^{\infty}$ denote an orthonormal basis of H. For $x \in H$ we define the norm by the formula $\|x\| = \sqrt{\langle x, x\rangle}$, where obviously $\langle\cdot,\cdot\rangle$ is a scalar product in H. Hence for each element $x \in H$ there holds well-known Fourier and Parseval formulas

$$x = \sum_{n=1}^{\infty} \langle x,\ e_n\rangle\, e_n, \tag{1.7}$$

$$\|x\|^2 = \sum_{n=1}^{\infty} |\,\langle x,\ e_n\rangle\,|^2. \tag{1.8}$$

In this paper (see section 2) we will prove that an averaging projection $P_a\colon K(H) \to Y$ given by a formula $P_a(A) = \frac{A+A^T}{2}$ is the only norm-one projection. In order to do this we introduce some notations and we state some preliminary results.
With any operator $A \in K(H)$ we will associate an infinite matrix $L(A) = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ given by a formula $a_{ij} = \langle Ae_j, e_i\rangle$.
Let us define $L(A) = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$.
We know that for any A on H we can define the conjugate operator A^T by

$$\langle Ax, y\rangle = \left\langle x, A^T y\right\rangle, \tag{1.9}$$

for any $x,\ y \in H$. We say that operator A is symmetric in H if $A = A^T$. We start with

Lemma 1.1 *For any operator $A = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in K(H)$ and any $i \in \mathbb{N}$*

$$\sum_{j=1}^{\infty} |a_{ij}|^2 < \infty, \tag{1.10}$$

$$\sum_{j=1}^{\infty} |a_{ji}|^2 < \infty\,. \tag{1.11}$$

PROOF. Since for each natural number i we have $Ae_i \in H$ by (1.7)

$$\left\|Ae_i\right\|^2 = \sum_{j=1}^{\infty} |\langle Ae_i,\ e_j\rangle|^2 = \sum_{j=1}^{\infty} |a_{ji}|^2,$$

$$\left\|A^T e_i\right\|^2 = \sum_{j=1}^{\infty} |\left\langle A^T e_i,\ e_j\right\rangle|^2 = \sum_{j=1}^{\infty} |\langle e_i,\ Ae_j\rangle|^2 = \sum_{j=1}^{\infty} |a_{ij}|^2, \tag{1.12}$$

which obviously completes the proof. □
Set

$$M = \left\{(i,\ j) \in \mathbb{N}\times\mathbb{N} \ \ : \ \ i < j\right\} \text{ and} \tag{1.13}$$

$$L = \left\{(i,\ i) \ \ : \ \ i \in \mathbb{N}\right\}. \tag{1.14}$$

For $z = (i,\ j) \in M$ we define a functional $f_z \in (K(H))^*$ by

$$f_{ij}(B) = f_z(B) = b_{ij} - b_{ji}, \tag{1.15}$$

for $L(B) = \{b_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in K(H)$. It is easy to show that f_z is bounded on $K(H)$.
Let

$$Y = \bigcap_{z\in M} \ker f_z. \tag{1.16}$$

Lemma 1.2 *For any $A \in K(H)$ if $A \in Y$, then the operator A is symmetric.*

PROOF. Let $L(A) = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ and $A \in Y$. It is sufficient to show that

$$\langle Ax,\ y\rangle = \langle x,\ Ay\rangle\,, \tag{1.17}$$

for each $x,\ y \in H$. By 1.7 for $x = \sum_{n=1}^{\infty} \langle x,\ e_i\rangle\, e_i$ and $y = \sum_{j=1}^{\infty} \langle y,\ e_j\rangle\, e_j$ we get

$$\begin{aligned}\langle Ax,\ y\rangle = \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x, e_i\rangle \langle y, e_j\rangle \langle Ae_i, e_j\rangle = \\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x, e_i\rangle \langle y, e_j\rangle\, a_{ji} = \\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x, e_i\rangle \langle y, e_j\rangle\, a_{ij} = \\ \sum_{i=1}^{\infty}\sum_{j=1}^{\infty} \langle x, e_i\rangle \langle y, e_j\rangle \langle e_i, Ae_j\rangle = \\ \langle x,\ Ay\rangle\ .\end{aligned} \tag{1.18}$$

□

The following result is a consequence of the Banach-Steinhaus Theorem

Theorem 1.3 *Let $\{T_n\}_{n=1}^{\infty} \subset B(X,\ Y)$, where X and Y are Banach spaces. Suppose that $T_n(x)$ is convergent in Y for each $x \in X$, then the operator $T(x) = \lim_{n\to\infty} T_n(x)$ is bounded.*

PROOF. We will show that T is bounded on X. Since for each $x \in X$ the sequence $\{T_n(x)\}_{n=1}^{\infty}$ is convergent, hence it is bounded on Y. Applying the Banach-Steinhaus Theorem we obtain that there exists a constant $M > 0$, such that

$$\|T_n(x)\|_Y \le M\|x\|_X, \tag{1.19}$$

for $x \in X$ and each n.
Hence

$$\|T(x)\|_Y = \lim_{n\to\infty} \|T_n(x)\|_Y \le M\|x\|_X, \tag{1.20}$$

for each $x \in X$. This obviously concludes the proof of Theorem 1.1.
□

We say that a sequence of bounded operators $\{T_n\}_{n=1}^{\infty}$ is convergent to operator T in a weak operator topology (WOT), if and only if for each $x,\ y \in H$ we have

$$\lim_{n\to\infty} \langle T_n(x), y\rangle = \langle T(x), y\rangle . \tag{1.21}$$

Analogously, the sequence of bounded operators $\{T_n\}_{n=1}^{\infty}$ is convergent to operator T in a strong operator topology (SOT) if and only if for each $x \in H$ we have

$$\lim_{n\to\infty} T_n(x) = T(x). \tag{1.22}$$

2 The unique minimality of an averaging projection in space $l_2(\mathbb{N})$

In this section we will show that the averaging projection P_a is the only norm-one projection.
Let $z = (i,\ j) \in M$. Define an operator B_{ij} on the basis $\{e_l\}_{l=1}^{\infty}$ by

$$B_{ij}(e_l) = \begin{cases} -\frac{1}{2}e_j & \text{if} \quad l = i \\ \frac{1}{2}e_i & \text{if} \quad l = j \\ 0 & \text{if} \quad l \neq i,\ l \neq j, \end{cases} \tag{2.1}$$

for $l \in \mathbb{N}$. It is easy to show that for any $z \in M$, B_z is a bounded operator. Now we prove a general formula for projections belonging to $\mathrm{P}(K(H), Y)$ (Th 2.11). We need some preliminary results.
For any $A = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in K(H)$ and $i \in \mathbb{N}$ define

$$C_i^n(A) = \sum_{j=i+1}^{n} f_{ij}(A)B_{ij}, \tag{2.2}$$

where $f_{ij}(A) = a_{ij} - a_{ji}$.

Theorem 2.1 *For each $x \in H$ and $A \in K(H)$ we have that $\lim_{n\to\infty}\{C_i^n(A)(x)\}_{n=1}^{\infty}$ exists.*

PROOF. It is enough to show that for fixed $A \in K(H),\ x \in H,\ i \in \mathbb{N}$ $\{C_i^n(A)(x)\}_{n=1}^{\infty}$ is a Cauchy sequence in H. Let us define $x = \sum_{l=1}^{\infty} \langle x,\ e_l\rangle\, e_l$. Note that

$$\left\| C_i^n(A)(x) - C_i^m(A)(x) \right\|^2 = \left\| \sum_{j=n+1}^{m} f_{ij}(A) B_{ij}(x) \right\|^2$$
$$= \left\| \sum_{j=n+1}^{m} f_{ij}(A) \left(\langle x,\ e_i \rangle \frac{-e_j}{2} + \langle x,\ e_j \rangle \frac{e_i}{2} \right) \right\|^2$$
$$= | \langle x,\ e_i \rangle |^2 \frac{1}{4} \sum_{j=n+1}^{m} |f_{ij}(A)|^2 + \frac{1}{4} \left| \sum_{j=n+1}^{m} f_{ij}(A) \langle x,\ e_j \rangle \right|^2$$
$$\leq \frac{\|x\|^2}{4} \sum_{j=n+1}^{\infty} |f_{ij}(A)|^2 + \frac{\|x\|^2}{4} \sum_{j=n+1}^{\infty} |f_{ij}(A)|^2$$
$$= \frac{\|x\|^2}{2} \sum_{j=n+1}^{\infty} |f_{ij}(A)|^2 . \tag{2.3}$$

By Lemma 1.1 the sequence $\{C_i^n(A)(x)\}_{n=1}^{\infty}$ is a Cauchy sequence, which completes the proof. □

Denote for $A \in K(H),\ x \in H$ and $i \in \mathbb{N}$

$$C_i(A)(x) = \lim_{n\to\infty} \sum_{j=i+1}^{n} f_{ij}(A) B_{ij}(x) = \sum_{j=i+1}^{\infty} f_{ij}(A) B_{ij}(x). \tag{2.4}$$

By Theorem 1.3 we get that $C_i(A)$ is a bounded operator on H.
Let us define for any $A = \{a_{ij}\}_{(i,j)\in\mathbb{N}\times\mathbb{N}} \in K(H)$ and $n \in \mathbb{N}$

$$S_n(A) = \sum_{i=1}^{n} C_i(A). \tag{2.5}$$

Theorem 2.2 *For each $x \in H,\ A \in K(H)$ we have that $\lim_{n\to\infty} \{S^n(A)(x)\}_{n=1}^{\infty}$ exists.*

PROOF. Define $x = \sum_{l=1}^{\infty} \langle x,\ e_l \rangle e_l \in H$. Analogously, as in the proof for Theorem 2.1, we will show that $S^n(A)(x)$ is a Cauchy sequence on H. Note that

$$\|S_n(A)(x) - S_m(A)(x)\|^2 = \left\| \sum_{i=n+1}^{m} C_i(A)(x) \right\|^2$$
$$= \left\| \sum_{i=n+1}^{m} C_i(A) \left(\sum_{l=1}^{\infty} \langle x,\ e_l \rangle e_l \right) \right\|^2 .$$

Let us calculate

$$\left\| \sum_{i=n+1}^{m} C_i(A) \left(\sum_{l=1}^{\infty} \langle x,\ e_l \rangle e_l \right) \right\|^2 = \left\| \sum_{i=n+1}^{m} \sum_{l=1}^{\infty} \langle x, e_l \rangle C_i(A)(e_l) \right\|^2$$
$$= \left\| \sum_{i=n+1}^{m} \sum_{l=i}^{\infty} \langle x, e_l \rangle C_i(A)(e_l) \right\|^2 .$$

Hence

$$\left\| \sum_{i=n+1}^{m} \left(\langle x,\ e_i \rangle C_i(A)(e_i) + \sum_{l=i+1}^{\infty} \langle x, e_l \rangle C_i(A)(e_l) \right) \right\|^2$$
$$= \left\| \sum_{i=n+1}^{m} \left(\sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle e_j + \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} e_i \right) \right\|^2$$
$$= \sum_{k=1}^{\infty} \left| \left\langle \sum_{i=n+1}^{m} \sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle e_j + \sum_{i=n+1}^{m} \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} e_i,\ e_k \right\rangle \right|^2 .$$

Therefore

$$\sum_{k=1}^{\infty} \left| \sum_{i=n+1}^{m} \sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle \langle e_j, e_k \rangle + \sum_{i=n+1}^{m} \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} \langle e_i, e_k \rangle \right|^2$$
$$= \sum_{k=n+1}^{\infty} \left| \sum_{i=n+1}^{m} \sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle \langle e_j, e_k \rangle + \sum_{i=n+1}^{m} \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} \langle e_i, e_k \rangle \right|^2$$
$$= \sum_{k=n+1}^{m} \left| \sum_{i=n+1}^{m} \sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle \langle e_j, e_k \rangle + \sum_{i=n+1}^{m} \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} \langle e_i, e_k \rangle \right|^2$$
$$+ \sum_{k=m+1}^{\infty} \left| \sum_{i=n+1}^{m} \sum_{j=i+1}^{\infty} \frac{-f_{ij}(A)}{2} \langle x, e_i \rangle \langle e_j, e_k \rangle + \sum_{i=n+1}^{m} \sum_{l=i+1}^{\infty} \langle x, e_l \rangle \frac{f_{il}(A)}{2} \langle e_i, e_k \rangle \right|^2 .$$

Hence

$$\sum_{k=n+1}^{m} \left| \sum_{i=n+1}^{k-1} \frac{-f_{ik}(A)}{2} \langle x, e_i \rangle + \sum_{l=k+1}^{\infty} \langle x, e_l \rangle \frac{f_{kl}(A)}{2} \right|^2$$
$$+ \sum_{k=m+1}^{\infty} \left| \sum_{i=n+1}^{m} \langle x, e_i \rangle \frac{-f_{ik}(A)}{2} \right|^2$$
$$= \sum_{k=n+1}^{m} \left| \sum_{l=n+1}^{\infty} \langle x, e_l \rangle \frac{f_{kl}(A)}{2} \right|^2 + \sum_{k=m+1}^{\infty} \left| \sum_{i=n+1}^{m} \langle x, e_i \rangle \frac{-f_{ik}(A)}{2} \right|^2 . \qquad (2.6)$$

Now we will show that

$$\sum_{k=n+1}^{m}\left|\sum_{l=n+1}^{\infty}\langle x,e_l\rangle\frac{f_{kl}(A)}{2}\right|^2\leq\left\|\frac{A-A^T}{2}\left(\sum_{l=n+1}^{\infty}\langle x,e_l\rangle e_l\right)\right\|^2 \text{ and} \tag{2.7}$$

$$\sum_{k=m+1}^{\infty}\left|\sum_{i=n+1}^{m}\langle x,e_i\rangle\frac{-f_{ik}(A)}{2}\right|^2\leq\left\|\frac{A-A^T}{2}\left(\sum_{i=n+1}^{m}\langle x,e_i\rangle e_i\right)\right\|^2. \tag{2.8}$$

Let us calculate

$$\begin{aligned}\left\|\frac{A-A^T}{2}\left(\sum_{l=n+1}^{\infty}\langle x,e_l\rangle e_l\right)\right\|^2&=\sum_{k=1}^{\infty}\left|\left\langle\frac{A-A^T}{2}\left(\sum_{l=n+1}^{\infty}\langle x,e_l\rangle e_l\right),e_k\right\rangle\right|^2\\&=\sum_{k=1}^{\infty}\left|\sum_{l=n+1}^{\infty}\langle x,e_l\rangle\left\langle\frac{A-A^T}{2}e_l,e_k\right\rangle\right|^2.\end{aligned}$$

Consequently

$$\sum_{k=1}^{\infty}\left|\sum_{l=n+1}^{\infty}\langle x,e_l\rangle\frac{f_{kl}(A)}{2}\right|^2\geq\sum_{k=n+1}^{m}\left|\sum_{l=n+1}^{\infty}\langle x,e_l\rangle\frac{f_{kl}(A)}{2}\right|^2, \tag{2.9}$$

which proves (2.7).
Analogously

$$\left\|\frac{A-A^T}{2}\left(\sum_{i=n+1}^{m}\langle x,e_i\rangle e_i\right)\right\|^2=\sum_{k=1}^{\infty}\left|\left\langle\frac{A-A^T}{2}\left(\sum_{i=n+1}^{m}\langle x,e_i\rangle e_i\right),e_k\right\rangle\right|^2.$$

Hence

$$\begin{aligned}\sum_{k=1}^{\infty}\left|\sum_{i=n+1}^{m}\langle x,e_i\rangle\left\langle\frac{A-A^T}{2}e_i,e_k\right\rangle\right|^2&=\sum_{k=1}^{\infty}\left|\sum_{i=n+1}^{m}\langle x,e_i\rangle\frac{f_{ki}(A)}{2}\right|^2\\&\geq\sum_{k=m+1}^{\infty}\left|\sum_{i=n+1}^{m}\langle x,e_i\rangle\frac{-f_{ik}(A)}{2}\right|^2.\end{aligned} \tag{2.10}$$

Note that

$$\left\|\frac{A-A^T}{2}\left(\sum_{l=n+1}^{\infty}\langle x,e_l\rangle e_l\right)\right\|^2 \le \left\|\frac{A-A^T}{2}\right\|^2 \left\|\sum_{l=n+1}^{\infty}\langle x,e_l\rangle e_l\right\|^2$$

$$= \left\|\frac{A-A^T}{2}\right\|^2 \sum_{l=n+1}^{\infty} |\langle x,e_l\rangle|^2 \qquad (2.11)$$

$$\left\|\frac{A-A^T}{2}\left(\sum_{i=n+1}^{m}\langle x,e_i\rangle e_i\right)\right\|^2 \le \left\|\frac{A-A^T}{2}\right\|^2 \left\|\sum_{i=n+1}^{m}\langle x,e_i\rangle e_i\right\|^2$$

$$= \left\|\frac{A-A^T}{2}\right\|^2 \sum_{i=n+1}^{m} |\langle x,e_i\rangle|^2 . \qquad (2.12)$$

By (2.11), and (2.12) $S_n(A)(x)$ is a Cauchy sequence in H. Thus for any $A \in K(H)$ we have $S(A) \in K(H)$, where

$$S(A) = \lim_{n\to\infty}\sum_{i=1}^{n} C_i(A) = \sum_{i=1}^{\infty} C_i(A) = \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)B_{ij}. \qquad (2.13)$$

By Theorem 1.3 $S(A)$ is bounded on H. □

In order to conclude the proof, it is sufficient to show that the limit of this sequence in SOT topology is a compact operator.

Theorem 2.3 *For each operator $A \in K(H)$ the sequence $\{S^n(A)\}_{n=1}^{\infty}$ is convergent to the operator $\frac{A-A^T}{2}$ in the strong operator topology.*

PROOF. By Theorem 2.2 for each operator $A \in K(H)$, $x \in H$ there exists

$$S(A)(x) = \lim_{n\to\infty}\sum_{i=1}^{n} C_i(A)(x) = \sum_{i=1}^{\infty} C_i(A)(x) =$$

$$\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)B_{ij}(x) . \qquad (2.14)$$

By Theorem 1.3 $S(A)$ is a bounded operator on H. In order to finish the proof it is sufficient to show that

$$L(S(A)) = L\left(\frac{A-A^T}{2}\right) . \qquad (2.15)$$

But for any $l,\ k$

$$\langle S(A)(e_l), e_k\rangle = \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)\langle B_{ij}(e_l), e_k\rangle . \qquad (2.16)$$

Also

$$\langle B_{ij}(e_l), e_k\rangle = \begin{cases} -\frac{1}{2} & \text{if} \quad (i,j) = (l,k) \\ \frac{1}{2} & \text{if} \quad (l,k) = (j,i) \\ 0 & \text{for the remaining} \quad (l,k). \end{cases} \tag{2.17}$$

Hence

$$\langle S(A)(e_l), e_k\rangle = \begin{cases} -\frac{a_{kl}-a_{lk}}{2} & \text{if} \quad l < k \\ \frac{a_{kl}-a_{lk}}{2} & \text{if} \quad l > k \\ 0 & \text{if} \quad l = k, \end{cases} \tag{2.18}$$

which gives

$$\langle S(A)(e_l), e_k\rangle = \left\langle \frac{A - A^T}{2}(e_l), e_k \right\rangle. \tag{2.19}$$

□

We have shown that the sequence of operators $\{S^n(A)\}_{n=1}^{\infty}$ is convergent to $\frac{A-A^T}{2}$ in the strong operator topology. We will show that this sequence is weakly convergent in $K(H)$. First we will present a few well known results. Let $S(V)$ denote the unit sphere in a Banach space V, and let $\text{ext}S(V)$ denote the set of extreme points of $S(V)$.

Theorem 2.4 *(see e. g. [11]) Let $S(K(H)^*)$ be the unit sphere in $(K(H))^*$. Then*

$$extS(K(H)^*) = extS(H) \otimes extS(H), \tag{2.20}$$

where $(x \otimes y)(L) = \langle Lx, y\rangle$.

Let $B(X^*)$ denote the unit ball in X^*.

Theorem 2.5 *(see e. g. [29]) Let X be a separable Banach space. Then the unit ball $B(X^*)$ is metrizable and $\omega*$-compact.*

In particular if H is a separable Hilbert space, then $K(H)$ is separable. The following result will be the main tool in our investigations.

Theorem 2.6 *(Choquet)(see e.g. [28]) Let K be a convex and compact subset of a linear topological space X, such that there exists a sequence $\{f_n\} \in X^*$, which is a total set for K. Then the set of extreme points of K is a Borel set and for each $a \in K$ there is a probabilistic measure v, defined on Borel's subsets of K, such that*

$$v(extK) = 1,$$

$$a = \int_K xd(v) = \int_{extK} xd(v)\,. \tag{2.21}$$

Recall that a set $F \subset X^*$ is total for K if for any $x \in K \setminus \{0\}$ there exists $f \in F$ such that $f(x) \neq 0$.

In particular from Theorem 2.5 and 2.6 one can deduce

Theorem 2.7 *(see e.g [29]) Let H be a real separable Hilbert space. Then for each $f \in B(K(H)^*)$ there exists a probabilistic and Borel measure v determined on Borel's subsets of $B(K(H)^*)$ such that*

$$f(\cdot) = \int_{extS(K(H)^*)} x(\cdot)d(v). \tag{2.22}$$

In the sequel we need

Theorem 2.8 *Let H be a real and separable Hilbert space. Then the sequence L_n converges weakly to L in $K(H)$ if and only if for any $f \in extS(K(H)^*)$ $f(L_n)$ is convergent to $f(L)$.*

PROOF. It is enough to show that if the sequence L_n converges weakly on the set $\mathrm{ext}S(K(H)^*)$, then it is weakly convergent. Fix $f \in B(K^*(H))$. Then by Choquet's theorem

$$f = \int_{\mathrm{ext}S(K(H)^*)} xd(v) = \int_{\mathrm{ext}(S(H)\otimes \mathrm{ext}S(H)} x \otimes yd(v). \tag{2.23}$$

Note that

$$f(L_n) = \int_{\mathrm{ext}(S(H)\otimes \mathrm{ext}S(H)} (x \otimes y)(L_n)d(v). \tag{2.24}$$

We will show that for each $x \in H$ $\{L_n(x)\}$ is weakly convergent in H. If $y \in H = H^*$, then

$$\lim_{n\to\infty} y(L_n(x)) = \lim_{n\to\infty} \langle L_n(x), y\rangle =$$

$$\lim_{n\to\infty} \|x\|\|y\| \left\langle L_n\left(\frac{x}{\|x\|}\right), \frac{y}{\|y\|}\right\rangle = \|x\|\|y\| \left\langle L\left(\frac{x}{\|x\|}\right), \frac{y}{\|y\|}\right\rangle. \tag{2.25}$$

Therefore for each $x \in H$ $\{L_n(x)\}$ is weakly convergent, and consequently bounded. By the Banach-Steinhaus Theorem the sequence $\{\|L_n\|\}$ is bounded. By the Lebesgue Theorem we obtain

$$\lim_{n\to\infty} f(L_n) = \int_{\mathrm{ext}(S(H)\otimes \mathrm{ext}S(H)} \lim_{n\to\infty} (x\otimes y)(L_n) d(v) =$$

$$\int_{\mathrm{ext}(S(H)\otimes \mathrm{ext}S(H)} \lim_{n\to\infty} \langle L_n(x), y\rangle\, d(v) =$$

$$\int_{\mathrm{ext}(S(H)\otimes \mathrm{ext}S(H)} \lim_{n\to\infty} \langle L(x), y\rangle\, d(v) = f(L)\,. \tag{2.26}$$

□

From Theorem 2.8 we get the following

Theorem 2.9 *For any operator $A \in K(H)$ the sequence $\{S^n(A)\}_{n=1}^{\infty}$ is weakly convergent to the operator $\frac{A-A^T}{2}$.*

PROOF. We know that the sequence $S_n(A)$ is SOT-convergent. Therefore, it follows from the Banach-Steinhaus Theorem, that this sequence is bounded. In order to show the weak convergence $S_n(A)$, based on Theorem 2.8, it is enough to show that

$$\lim_{n\to\infty} f(S_n(A)) = f\left(\frac{A-A^T}{2}\right), \tag{2.27}$$

for any $f \in \mathrm{ext}S(K(H)^*)$.
Fix $f \in \mathrm{ext}S(K(H)^*)$. By Theorem 2.4 we know that $f = x\otimes y$ for $x,\ y \in \mathrm{ext}S(H)$. Therefore

$$\lim_{n\to\infty} f(S_n(A)) = \lim_{n\to\infty} (x\otimes y)(S_n(A)) = \lim_{n\to\infty} \langle S_n(A)(x), y\rangle\ . \tag{2.28}$$

Because $S_n(A)$ is SOT-convergent, it is in particular WOT convergent. Hence

$$\lim_{n\to\infty} \langle S_n(A)(x), y\rangle = \left\langle \lim_{n\to\infty} S_n(A)(x), y\right\rangle = \left\langle \frac{A-A^T}{2}(x), y\right\rangle, \tag{2.29}$$

which concludes our proof. □

Remark. By the Mazur Theorem Th.‘ 2.9 is not valid for $A \in B(H)$.
Let us state another simple lemma.

Lemma 2.10 *For each $z = (l,\ k) \in M$ and for each $A \in K(H)$ we have*

$$f_z(S(A)) = f_z(A). \tag{2.30}$$

PROOF. Note that

$$f_z(S(A)) = f_z\left(\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)B_{ij}\right) = \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)f_z(B_{ij}) = f_z(A). \quad (2.31)$$

□

Now let $P \in \mathrm{P}(K(H),\ Y)$. Then applying Lemma 1.1 and Lemma 1.2 we obtain that for each $A \in K(H)$

$$A - S(A) \in Y. \quad (2.32)$$

Therefore $P(A - S(A)) = A - S(A)$. Hence

$$P(A) = A + P(S(A)) - S(A) = A + \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)(P(B_{ij}) - B_{ij}). \quad (2.33)$$

Obviously for each $z,\ w \in M$

$$f_z(P(B_w) - B_w) = f_z(B_w) = \delta_{zw}. \quad (2.34)$$

Hence we have proved the following

Theorem 2.11 *For any projection $P \in P(K(H),\ Y)$ there exists a family of operators $\{F_z\}_{z\in M} \subset K(H)$, which fulfill the following conditions*

a) A sequence $\sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)F_{ij}$ is weakly convergent for each $A \in K(H)$.
b) For each $w,\ z \in M$ we get $f_z(F_w) = \delta_{zw}$.
c) $P(A) = A - \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(A)F_{ij}$, for any $A \in K(H)$.

Theorem 2.12 *The averaging projection P_a has the form*

$$P_a(\cdot) = Id - \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(\cdot)B_{ij}. \quad (2.35)$$

PROOF. By Theorem 2.11 there exists a sequence $\{F_{ij}\}$ satisfying a), b), c) such that

$$P_a(\cdot) = Id - \sum_{i=1}^{\infty}\sum_{j=i+1}^{\infty} f_{ij}(\cdot)F_{ij}. \quad (2.36)$$

Let $F_z = \{f^z_{lk}\}_{(l,k)\in\mathbb{N}\times\mathbb{N}}$. Since $P_a(A) = P_a(A^T)$ we obtain for fixed $z = (i,\ j) \in M$

$$0 = P_a(F_{ij}) = P_a(F_{ij}^T) =$$
$$F_{ij}^T - \sum_{l=1}^{\infty}\sum_{k=l+1}^{\infty} f_{lk}(F_{ij}^T)F_{lk}\,. \tag{2.37}$$

Since for $A \in K(H)$ $f_{lk}(A^T) = -f_{lk}(A)$, therefore

$$F_{ij}^T + F_{ij} = 0. \tag{2.38}$$

Hence

$$f_{lk}^z + f_{kl}^z = 0, \tag{2.39}$$

for each $l,\ k \in \mathbb{N}$.
Since $f_z(F_w) = \delta_{zw}$,

$$\begin{aligned} &f_{lk}^z + f_{kl}^z = 0,\\ &f_{lk}^z - f_{kl}^z = 0\,, \end{aligned} \tag{2.40}$$

for $(l,\ k) \neq (i,\ j)$. Hence

$$f_{lk}^z = 0, \tag{2.41}$$

for $(l,\ k) \neq (i,\ j)$.
Additionally for $i,\ j$ we have

$$\begin{aligned} &f_{ij}^z + f_{ji}^z = 0,\\ &f_{ij}^z - f_{ji}^z = 1\,. \end{aligned} \tag{2.42}$$

Consequently, $f_{ij}^z = \frac{1}{2}$, $f_{ji}^z = -\frac{1}{2}$, therefore $F_z = B_z$. $\square$

We now return to the proof of unique minimality of our averaging projection. For the purpose of the proof, let us denote by

$$A_{ij}(\theta)(e_l) = \begin{cases} \sin\theta e_i - \cos\theta e_j & \text{if} \quad l = i\\ \cos\theta e_i + \sin\theta e_j & \text{if} \quad l = j\\ e_l & \text{if} \quad l \in \{1,\ \ldots,\ j\} \setminus \{i,\ j\}\\ 0 & \text{if} \quad l \neq i,\ l \neq j. \end{cases} \tag{2.43}$$

for fixed $(i,\ j) \in M,\ \ \theta \in \mathbb{R}$.
It is easy to show that each operator $A_{ij}(\theta)$ is a compact and bounded on the space H.
Now we will state and prove the principal result of this paper:

Theorem 2.13 *In separable real Hilbert space the averaging projection P_a is the only norm-one projection in $K(H)$.*

PROOF. Fix $Q \in \mathrm{P}(K(H), Y)$ and $\|Q\| = 1$. By Theorem 2.11 we obtain

$$Q(\cdot) = Id - \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} f_{ij}(\cdot) F_{ij}, \tag{2.44}$$

where $f_w(F_z) = \delta_{wz}$, for $w,\ z \in M$.
By Theorem 2.12 it is enough to show that the matrix of the operator $F_z = \{f^z_{lk}\}_{(l,\ k)\in\mathbb{N}}$ has the form

$$f^z_{lk} = \begin{cases} \frac{1}{2} & \text{if} \quad (l,\ k) = (i,\ j) \\ \frac{1}{2} & \text{if} \quad (l,\ k) = (j,\ i) \\ 0 & \text{if} \quad (l,\ k) \neq (i,\ j). \end{cases} \tag{2.45}$$

To do this, fix $z = (i,\ j) \in M$. For each $\theta \in \mathbb{R}$

$$\begin{aligned} Q(A_{ij}(\theta)) = Id(A_{ij}(\theta)) - \sum_{l=1}^{\infty} \sum_{k=l+1}^{\infty} f_{lk}(A_{ij}(\theta)) F_{lk} = \\ = A_{ij}(\theta) - 2\cos(\theta) F_{(i,\ j)}. \end{aligned} \tag{2.46}$$

Since Q is a norm-one projection

$$\|A_{ij}(\theta) - 2\cos(\theta) F_{(i,\ j)}\| = \sup_{\|x\|_2 = 1} \|(A_{ij}(\theta) - 2\cos(\theta) F_{(i,\ j)})x\| \leq 1, \tag{2.47}$$

for each $\theta \in \mathbb{R}$.
Let us fix $l \in \{1, \ldots,\ j\} \setminus \{i,\ j\}$. We will show that $f^z_{ll} = 0$.
By equation (2.47) we get

$$\|(A_{ij}(\theta) - 2\cos(\theta) F_{(i,\ j)}) e_l\| \leq 1, \tag{2.48}$$

for each $\theta \in \mathbb{R}$. In particular,

$$|1 - 2\cos(\theta) f^z_{ll}| \leq 1, \tag{2.49}$$

for any $\theta \in \mathbb{R}$. Hence $f^z_{ll} = 0$ and

$$f^z_{lk} = 0, \tag{2.50}$$
$$f^z_{kl} = 0\,, \tag{2.51}$$

for any $k \in \mathbb{N}$.
We will show that $l > j$ $f^z_{ll} = 0$.

Let us define

$$A^l_{ij}(\theta)(e_k) = \begin{cases} \sin\theta e_i - \cos\theta e_j & \text{if} \qquad k = i \\ \cos\theta e_i + \sin\theta e_j & \text{if} \qquad k = j \\ e_k & \text{if} \qquad k \in \{1,\ \dots,\ j,\ l\} \setminus \{i,\ j\} \\ 0 & \text{for remaining } k. \end{cases} \tag{2.52}$$

Then

$$\begin{aligned} Q(A^l_{ij}(\theta)) = Id(A^l_{ij}(\theta)) - \sum_{l=1}^{\infty}\sum_{k=l+1}^{\infty} f_{lk}(A^l_{ij}(\theta))F_{lk} = \\ = A^l_{ij}(\theta) - 2\cos(\theta)F_{(i,\ j)}\,. \end{aligned} \tag{2.53}$$

Hence

$$\|(A^l_{ij}(\theta) - 2\cos(\theta)F_{(i,\ j)})e_l\|_2 \leq 1, \tag{2.54}$$

for each $\theta \in \mathbb{R}$.
Therefore

$$|1 - 2\cos(\theta)f^z_{ll}| \leq 1, \tag{2.55}$$

for each $\theta \in \mathbb{R}$. Hence $f^z_{ll} = 0$ and

$$\begin{aligned} f^z_{lk} = 0, \\ f^z_{kl} = 0\,, \end{aligned} \tag{2.56}$$

for each $k \in \mathbb{N}$.
Now we show that $f^z_{ii} = f^z_{jj} = 0$.
By (2.47),

$$\|(A_{ij}(\theta) - 2\cos(\theta)F_{(i,\ j)})\sin(\theta)e_i\| \leq 1. \tag{2.57}$$

Hence

$$-1 \leq \sin^2\theta - 2\sin\theta\cos\theta f^z_{ii} \leq 1, \tag{2.58}$$

for each $\theta \in \mathbb{R}$.
After simple calculations we get

$$\begin{aligned} \frac{\sin^2\theta - 1}{\sin 2\theta} \leq f^z_{ii} \leq \frac{\sin^2\theta + 1}{\sin 2\theta} \text{ for } \theta \in \left(0,\ \frac{\pi}{2}\right), \\ \frac{\sin^2\theta + 1}{\sin 2\theta} \leq f^z_{ii} \leq \frac{\sin^2\theta - 1}{\sin 2\theta} \text{ for } \theta \in \left(-\frac{\pi}{2},\ 0\right). \end{aligned} \tag{2.59}$$

Hence

$$\lim_{\theta\to\frac{\pi}{2}^-}\frac{\sin^2\theta-1}{\sin 2\theta}\le f^z_{ii},$$
$$f^z_{ii}\le\lim_{\theta\to-\frac{\pi}{2}^+}\frac{\sin^2\theta-1}{\sin 2\theta}. \tag{2.60}$$

Therefore $f^z_{ii}=0$. Proceeding analogously, we obtain that$f^z_{jj}=0$.
In order to end the proof, it is necessary to show that $f^z_{ij}=\frac{1}{2}$, $f^z_{ji}=-\frac{1}{2}$.
Set $a:=1-2f^z_{ij}$. We will show that

$$\sin\theta+|\cos\theta||a|, \tag{2.61}$$

is an eigenvalue of the operator $A_{ij}(\theta)-2\cos(\theta)F_{(i,\,j)}$.
Set

$$\delta=\mathrm{sgn}(a\cos\theta). \tag{2.62}$$

Note that

$$\begin{aligned}
(A_{ij}(\theta)-2\cos(\theta)F_{(i,\,j)})(e_i+\delta e_j)=\\
A_{ij}(\theta)(e_i+\delta e_j)-2\cos(\theta)F_{(i,\,j)})(e_i+\delta e_j)=\\
A_{ij}(\theta)e_i+\delta A_{ij}(\theta)e_j-2\cos(\theta)F_{ij}e_i-2\delta\cos(\theta)F_{ij}e_j=\\
(\sin(\theta)e_i-\cos(\theta)e_j)+\delta(\sin(\theta)e_j+\cos(\theta)e_i)=\\
-2\cos(\theta)f^z_{ji}e_j-2\delta\cos(\theta)f^z_{ij}e_i=\\
(\sin(\theta)+\delta\cos(\theta)-2\delta\cos(\theta)f^z_{ij})e_i=\\
+(-\cos(\theta)+\delta\sin(\theta)-2\cos(\theta)(f^z_{ij}-1))e_j=\\
(\sin(\theta)+\delta\cos(\theta)a)e_i+(\delta\sin(\theta)+\cos(\theta)a)e_j=\\
(\sin(\theta)+|\cos(\theta)a|)e_i+(\sin(\theta)+|\cos(\theta)a|)\delta e_j=\\
(\sin(\theta)+|\cos(\theta)a|)(e_i+\delta e_j)\,.
\end{aligned} \tag{2.63}$$

In particular for each $\theta\in\mathbb{R}$

$$\sin\theta+|\cos\theta||a|\le 1. \tag{2.64}$$

Hence

$$|a|\le\frac{1-\sin\theta}{\cos\theta}, \tag{2.65}$$

for $\theta\in\left[0,\,\frac{\pi}{2}\right)$, which gives

$$|a|\le\lim_{\theta\to\frac{\pi}{2}^-}\frac{1-\sin\theta}{\cos\theta}=0. \tag{2.66}$$

Therefore $|a|=0$, consequently $f^z_{ij}=\frac{1}{2}$. The proof is complete. □

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