Generalized Friedland's theorem for C_0-semigroups

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Abstract. Friedland's characterization of bounded normal operators is shown to hold for infinitesimal generators of C_0-semigroups. New criteria for normality of bounded operators are furnished in terms of Hamburger moment problem. All this is achieved with the help of the celebrated Ando's theorem on paranormal operators.

1. Introduction

Throughout what follows, $\mathcal{H}$ stands for a complex Hilbert space. By an operator *in* $\mathcal{H}$ we mean a linear mapping $A\colon \mathcal{D}(A) \to \mathcal{H}$ defined on a linear subspace $\mathcal{D}(A)$ of $\mathcal{H}$, called the *domain* of A. Set $\mathcal{D}^\infty(A) = \bigcap_{n=1}^\infty \mathcal{D}(A^n)$, and denote by $\mathcal{N}(A)$ and A^* the kernel and the adjoint of A respectively. A densely defined operator A in $\mathcal{H}$ is said to be *normal* if A is closed and $A^*A = AA^*$. We refer the reader to the monographs [**5**] and [**19**] for the theory of unbounded normal operators. The C^*-algebra of all bounded linear operators in $\mathcal{H}$ with domain $\mathcal{H}$ is denoted by $\boldsymbol{B}(\mathcal{H})$. An operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *paranormal* if

$$\|Ah\|^2 \leqslant \|A^2h\|\|h\|, \quad h \in \mathcal{H}.$$

The notion of a paranormal operator first appeared in [**11**] under the name of class (N). Its present name was introduced in [**8**]. It is known that bounded normal

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operators are always paranormal but not conversely (cf. [**9**]). Nevertheless, the celebrated theorem of Ando enables us to verify normality with the help of paranormality as follows: an operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and both the operators A and A^* are paranormal (cf. [**1**, Theorem 5]).

Friedland characterized in [**7**] a bounded normal operator with the help of convexity properties of the associated exponential group. More precisely, he proved that an operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if the functions $t \mapsto \log \| \mathrm{e}^{tA} h\|$ and $t \mapsto \log \| \mathrm{e}^{tA^*} h\|$ are convex on the real line $\mathbb{R}$ for every nonzero vector $h \in \mathcal{H}$. In the present paper we generalize Friedland's theorem to the case of (unbounded) infinitesimal generators of C_0-semigroups (cf. Theorem 3). We propose a completely new proof based upon Ando's theorem on paranormal operators. It is worth noticing that in the bounded operator case our proof becomes essentially shorter than that of Friedland. We conclude the paper with some other criteria for normality of bounded operators written in terms of Hamburger moment problem (cf. Proposition 7). As shown in Propositions 6 and 8 both continuous and discrete cases can be essentially simplified in the context of compact operators.

2. Preliminaries

Before formulating the main result of the paper it will be convenient to prove, for easy reference, some indispensable facts concerning convex functions. For our purpose, we extend in a natural way the notion of convexity to real variable functions which take values in $\mathbb{R} \cup \{-\infty\}$; it has to be accompanied by the convention that $0 \cdot (-\infty) = 0$. It is then clear that if f is a convex function defined on an open interval $J \subseteq \mathbb{R}$, then either $f(J) \subseteq \mathbb{R}$ or $f \equiv -\infty$.

Lemma 1. *Let $a, b \in \mathbb{R}$ be such that $a < b$ and let $f\colon [a, b) \to \mathbb{R}$.*

(i) *If f is differentiable and for every $t \in [a, b)$ there exists $\eta_t \in (t, b)$ such that f is convex on $[t, \eta_t)$, then f is a convex function of class C^1.*
(ii) *If f is convex, then the limit $\lim\limits_{t\to b-} f(t)$ exists in $(-\infty, \infty]$.*

Proof. (i) The proof of the following fact is left to the reader (the case $\gamma = \alpha$ does not require the assumptions on the behaviour of g to the left of γ).

(∗) Let α, β, γ be real numbers such that $\alpha \leqslant \gamma < \beta$ and let $g\colon [\alpha, \beta) \to \mathbb{R}$ be a function which has the Darboux property on the segments $[\alpha', \gamma]$ and $[\gamma, \beta']$ for all $\alpha' \in [\alpha, \gamma)$ and $\beta' \in (\gamma, \beta)$. If the limits $\lim\limits_{t\to\gamma-} g(t)$ and $\lim\limits_{t\to\gamma+} g(t)$ exist (finite or not), then g is continuous at γ.

Set $\mathcal{M} \overset{\mathsf{def}}{=} \{c \in (a, b]\colon f'$ is monotonically increasing on $[a, c)\}$. Owing to a well known characterization of convexity, the derivative f' is monotonically increasing on $[t, \eta_t)$ for every $t \in [a, b)$. Hence $\mathcal{M}$ is nonempty. We show that

$$\eta_c \in \mathcal{M} \text{ for every } c \in \mathcal{M} \cap (a, b). \tag{1}$$

Take $c \in \mathcal{M} \cap (a, b)$. Since the derivative f' has the Darboux property on each closed segment contained in $[a, b)$, we deduce from (∗) that f' is continuous at c. Hence f' is monotonically increasing on $[a, \eta_c)$, which proves (1). Since $\sup \mathcal{M} = \max \mathcal{M}$, we infer from (1) that $\max \mathcal{M} = b$. Thus f' is monotonically increasing on $[a, b)$, and so f is convex. Using (∗) again, we conclude that f' is continuous.

(ii) Suppose that, on the contrary, $\alpha \stackrel{\mathsf{def}}{=} \liminf_{t\to b-} f(t) < \beta \stackrel{\mathsf{def}}{=} \limsup_{t\to b-} f(t)$. Choose $\gamma \in (\alpha, \beta)$. Then there exist $t_1, t_2, t_3 \in (a, b)$ such that $t_1 < t_2 < t_3$ and

$$\max\{f(t_1), f(t_3)\} \leqslant \gamma < f(t_2). \tag{2}$$

Since $t_2 = \delta t_1 + (1-\delta)t_3$ for some $\delta \in (0,1)$, condition (2) contradicts the convexity of f. Hence the limit $\lim_{t\to b-} f(t)$ exists in $[-\infty, \infty]$. In turn, the hypothesis $\lim_{t\to b-} f(t) = -\infty$ and the automatic continuity of f on (a, b) lead to

$$f\Big(\frac{1}{2}(a+b)\Big) = \lim_{t\to b-} f\Big(\frac{1}{2}(a+t)\Big) \leqslant \frac{1}{2}\big(f(a) + \lim_{t\to b-} f(t)\big) = -\infty,$$

which contradicts $f(\frac{1}{2}(a+b)) \in \mathbb{R}$. This completes the proof. □

Remark 2. It is worth noting that part (i) of Lemma 1 is no longer true if differentiability of f is replaced by absolute continuity. Indeed, the function $f\colon [-1,1) \to \mathbb{R}$ defined by

$$f(t) = 1 - |t|, \quad t \in [-1,1),$$

is not convex, though it is absolutely continuous and for every $t \in [-1,1)$ there exists $\eta_t \in (t,1)$ such that f is convex on $[t, \eta_t)$. On the other hand, the proof of part (i) of Lemma 1 simplifies essentially for functions f of class C^1. Finally, if f has the second derivative, then the proof of (i) simplifies drastically because by a well known characterization of convexity we have $f''(c) = f''_+(c) \geqslant 0$ for all $c \in [a,b)$, which implies the desired convexity of f (here $f''_+(c)$ stands for the right-hand second derivative of f at c).

3. Generalized Friedland's theorem

The main result of this section, Theorem 3, is a generalization of Friedland's theorem (cf. [**7**, Theorem 2]) to the case of infinitesimal generators of C_0-semigroups. For fundamentals concerning C_0-semigroups we recommend the monographs [**10**], [**6**] and [**14**]. Below we adhere to the convention that $\log 0 = -\infty$.

THEOREM 3. *Suppose that A is the infinitesimal generator of a C_0-semigroup $\{S(t)\}_{t\in[0,\infty)} \subseteq \boldsymbol{B}(\mathcal{H})$. Then the following conditions are equivalent*:

(i) *A is normal,*
(ii) *for every $h \in \mathcal{H}$ the functions $t \mapsto \log\|S(t)h\|$ and $t \mapsto \log\|S(t)^*h\|$ are convex on $[0,\infty)$,*
(iii) *for every $h \in \mathcal{H}$ there exists $\varepsilon_h \in (0,\infty)$ such that the functions $t \mapsto \log\|S(t)h\|$ and $t \mapsto \log\|S(t)^*h\|$ are convex on $[0,\varepsilon_h)$.*

Moreover, if A is normal, then $\mathcal{N}(S(t)) = \mathcal{N}(S(t)^*) = \{0\}$ for all $t \in [0,\infty)$.

PROOF. (iii)⇒(ii) For $h \in \mathcal{H}$, we define the function $\varphi_h : [0,\infty) \to \mathbb{R}\cup\{-\infty\}$ by

$$\varphi_h(t) = \log\|S(t)h\|, \quad t \in [0,\infty). \tag{3}$$

Take $h \in \mathcal{D}(A)$. We claim that φ_h is convex. There are two cases to consider:

1° $S(t_0)h = 0$ for some $t_0 \in [0,\infty)$,
2° $S(t)h \neq 0$ for all $t \in [0,\infty)$.

Suppose 1° holds. Without loss of generality we may assume that

$$t_0 = \min\{t \in [0,\infty) \colon S(t)h = 0\}. \tag{4}$$

Consider first the case $t_0 > 0$. As $h \in \mathcal{D}(A)$, we infer from (4) and part c) of [**14**, Theorem 1.2.4] that $\varphi_h \in \mathcal{C}^1([0,t_0))$. Since $\varphi_h(t+s) = \varphi_{S(t)h}(s)$ for all $s,t \in [0,\infty)$ and $\varphi_{S(t)h}$ is convex on $[0,\varepsilon_{S(t)h})$ for all $t \in [0,t_0)$, we deduce from part (i) of Lemma 1 that φ_h is a real-valued convex function on $[0,t_0)$ such that $\lim_{s\to t_0-} \varphi_h(s) = -\infty$, the latter being a consequence of continuity of $S(\cdot)h$. This contradicts part (ii) of Lemma 1. Hence $t_0 = 0$, which implies $\varphi_h \equiv -\infty$. Repeating the above argument, we see that 2° implies the convexity of φ_h as well.

By [**14**, Corollary 1.2.5], the space $\mathcal{D}(A)$ is dense in $\mathcal{H}$. Hence, for every $h \in \mathcal{H}$ there exists a sequence $\{h_n\}_{n=0}^\infty \subseteq \mathcal{D}(A)$ which converges to h. This and the previous paragraph imply that φ_h is the pointwise limit of the sequence $\{\varphi_{h_n}\}_{n=0}^\infty$ of convex functions. Hence φ_h is convex itself.

Since $\{S(t)^*\}_{t\in[0,\infty)}$ is a C_0-semigroup (cf. [**14**, Corollary 1.10.6]), we can establish the convexity of the functions $t \mapsto \log\|S(t)^*h\|$, $h \in \mathcal{H}$, on the interval $[0,\infty)$ applying the above reasoning to $\{S(t)^*\}_{t\in[0,\infty)}$ instead of $\{S(t)\}_{t\in[0,\infty)}$.

(ii)⇒(i) First, notice that $\mathcal{N}(S(t)) = \{0\}$ for all $t \in [0,\infty)$. Indeed, otherwise there exists $h \in \mathcal{H} \setminus \{0\}$ and $t_0 \in [0,\infty)$ such that $S(t_0)h = 0$. Without loss of generality we can assume that (4) holds. Clearly $t_0 > 0$ and φ_h is a real-valued convex function on $[0,t_0)$ such that $\lim_{s\to t_0-} \varphi_h(s) = -\infty$, which by part (ii) of Lemma 1 leads to a contradiction. Applying this to the adjoint C_0-semigroup, we obtain $\mathcal{N}(S(t)^*) = \{0\}$ for all $t \in [0,\infty)$. Hence the "moreover" part of the conclusion is proved.

Take real $t \geqslant 0$. Employing the definition of convexity, we get

$$\log\|S(t)h\| \leqslant \frac{1}{2}\big(\log\|S(0)h\| + \log\|S(2t)h\|\big), \quad h \in \mathcal{H},$$

which implies that the operator $S(t)$ is paranormal. The convexity of $\log\|S(\cdot)^*h\|$ gives the paranormality of $S(t)^*$. Since the kernels of $S(t)$ and $S(t)^*$ coincide, the Ando theorem (cf. [**1**, Theorem 5]) guarantees the normality of $S(t)$. By the Stone theorem (cf. [**15**, Theorem 13.37]), the operator A is normal.

(i)⇒(ii) It follows from the spectral theorem that

$$S(t) = \int_{\mathbb{C}} \mathrm{e}^{t\lambda} E(\mathrm{d}\lambda), \quad t \in [0,\infty), \tag{5}$$

where E is the spectral measure of A. Applying the Hölder inequality, we obtain

$$\begin{aligned} \|S(\alpha t_1 + (1-\alpha)t_2)h\|^2 &= \int_{\mathbb{C}} |\,\mathrm{e}^{\alpha t_1\lambda}|^2 \cdot |\,\mathrm{e}^{(1-\alpha)t_2\lambda}|^2\, \mu_h(\mathrm{d}\lambda) \\ &\leqslant \Big(\int_{\mathbb{C}} (|\,\mathrm{e}^{\alpha t_1\lambda}|^2)^{\frac{1}{\alpha}}\, \mu_h(\mathrm{d}\lambda)\Big)^{\alpha} \Big(\int_{\mathbb{C}} (|\,\mathrm{e}^{(1-\alpha)t_2\lambda}|^2)^{\frac{1}{1-\alpha}}\, \mu_h(\mathrm{d}\lambda)\Big)^{1-\alpha} \\ &= \|S(t_1)h\|^{2\alpha}\|S(t_2)h\|^{2(1-\alpha)}, \quad h \in \mathcal{H},\ \alpha \in (0,1),\ t_1,t_2 \in [0,\infty), \end{aligned}$$

where $\mu_h(\cdot) = \langle E(\cdot)h, h\rangle$. This and $\|S(\cdot)h\| = \|S(\cdot)^*h\|$ yield the convexity of $\log\|S(\cdot)h\|$ and $\log\|S(\cdot)^*h\|$ for all $h \in \mathcal{H}$.

Since (ii) manifestly implies (iii), the equivalence of (i), (ii) and (iii) is established. This completes the proof. □

Remark 4. A close inspection of the proof of Theorem 3 reveals that condition (iii) is equivalent to

(iii*) for any two vectors $h \in \mathcal{X}$ and $g \in \mathcal{X}_*$, there exists real $\varepsilon > 0$ such that the functions $t \mapsto \log \|S(t)h\|$ and $t \mapsto \log \|S(t)^* g\|$ are convex on $[0, \varepsilon)$,

where $\mathcal{X}$ and $\mathcal{X}_*$ are fixed dense subsets of $\mathcal{D}^\infty(A)$ and $\mathcal{D}^\infty(A^*)$ respectively. The only thing which needs an explanation is the density of $\mathcal{X}$ and $\mathcal{X}_*$ in $\mathcal{H}$. This, however, is a consequence of the density[1] of $\mathcal{D}^\infty(A)$ and $\mathcal{D}^\infty(A^*)$ in $\mathcal{H}$ (cf. [**14**, Theorem 1.2.7 and Corollary 1.10.6]). Notice that if $h \in \mathcal{D}^\infty(A) \setminus \{0\}$, then the case 1° is excluded and consequently the function φ_h defined by (3) is of class C^∞.

Let us mention that in the case of a C_0-semigroup $\{S(t)\}_{t \in [0,\infty)}$ of normal operators the convexity of functions $\log \|S(\cdot)h\|$ and $\log \|S(\cdot)^* h\|$ can be proved without recourse to the spectral theorem. To see this, notice first that each $S(t)$ being normal is paranormal and then apply the following lemma.

LEMMA 5. *Let $\{S(t)\}_{t \in [0,\infty)} \subseteq \boldsymbol{B}(\mathcal{H})$ be a C_0-semigroup. Then $\log \|S(\cdot)h\|$ is convex on $[0, \infty)$ for all $h \in \mathcal{H}$ if and only if all the operators $S(t)$, $t \in [0, \infty)$, are paranormal. Moreover, if this is the case, then $\mathcal{N}(S(t)) = \{0\}$ for all $t \in [0, \infty)$.*

PROOF. Suppose that all the operator $S(t)$, $t \in [0, \infty)$, are paranormal. Then

$$\varphi_h(t) \leqslant \frac{1}{2}(\varphi_h(0) + \varphi_h(2t)), \quad h \in \mathcal{H},\, t \in [0, \infty),$$

where $\varphi_h(t) \stackrel{\text{def}}{=} \log \|S(t)h\|$. Replacing h by $S(u)h$ we get

$$\varphi_h(t+u) \leqslant \frac{1}{2}(\varphi_h(u) + \varphi_h(2t+u)), \quad u, t \in [0, \infty).$$

Letting $u, t \in [0, \infty)$ vary, we see that

$$\varphi_h\Big(\frac{1}{2}(s+t)\Big) \leqslant \frac{1}{2}(\varphi_h(s) + \varphi_h(t)), \quad s, t \in [0, \infty). \tag{6}$$

If $\varphi_h([0, \infty)) \subseteq \mathbb{R}$, then the continuity of φ_h and (6) imply the convexity of φ_h. If $\varphi_h([0, \infty)) \nsubseteq \mathbb{R}$, then we deduce from (6) that $\varphi_h(t) = -\infty$ for all $t \in (0, \infty)$. By reversing the steps above, we get the paranormality of the operators $S(t)$.

The "moreover" part of the conclusion is established in the proof of implication (ii)⇒(i) of Theorem 3. □

In [**7**, Lemma 1] the convexity of $\log \|S(\cdot)h\|$ on $\mathbb{R}$ has been established for C_0-groups with bounded hyponormal infinitesimal generators. Recall that hyponormal operators are always paranormal but not conversely (cf. [**8**, Theorem 2]). Notice that for C_0-semigroups of injective operators having dense range (e.g. for C_0-groups) the proof of Theorem 3 becomes essentially shorter because the case 1° disappears. It is worth noting that there are plenty of C_0-semigroups of normal operators which do not extend to C_0-groups.

We now prove an analogue of Friedland's theorem for compact C_0-semigroups. In particular, this covers the case of finite dimensional Hilbert spaces (cf. [**7**, Theorem 1], see also [**2, 3**]). Recall that a C_0-semigroup $\{S(t)\}_{t \in [0,\infty)} \subseteq \boldsymbol{B}(\mathcal{H})$ is said to be *compact* (cf. [**14**, page 48]) if the operator $S(t)$ is compact for every $t \in (0, \infty)$.

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[1] The desired density can also be deduced from the following much more general fact: if T is a closed densely defined operator in $\mathcal{H}$ with a nonempty resolvent set, then $\mathcal{D}^\infty(T)$ is dense in $\mathcal{H}$.

Proposition 6. *The infinitesimal generator of a compact C_0-semigroup $\{S(t)\}_{t\in[0,\infty)} \subseteq \boldsymbol{B}(\mathcal{H})$ is normal if and only if the function $t \mapsto \log\|S(t)h\|$ is convex on $[0,\infty)$ for every $h \in \mathcal{H}$.*

Proof. Apply Lemma 5, the automatic normality of compact paranormals (cf. [**11**, Theorem 2]) and the Stone theorem. □

One can show that in separable infinite dimensional Hilbert spaces compact C_0-semigroups of normal operator are exactly those which are unitarily equivalent to C_0-semigroups $\{S(t)\}_{t\in[0,\infty)}$ of bounded operators on ℓ^2 given by

$$S(t) = \begin{bmatrix} \mathrm{e}^{\lambda_1 t} & 0 & 0 & \\ 0 & \mathrm{e}^{\lambda_2 t} & 0 & \ddots \\ 0 & 0 & \mathrm{e}^{\lambda_3 t} & \ddots \\ & \ddots & \ddots & \ddots \end{bmatrix}, \quad t \in [0,\infty),$$

where $\{\lambda_n\}_{n=1}^\infty$ is a sequence of complex numbers such that $\mathfrak{Re}\,\lambda_n \to -\infty$ as $n \to \infty$. This can be done with the help of (5).

4. Normality via moment sequences

It was proved in [**16**, Proposition 6.2] that an algebraic operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if for some integer $j \geqslant 1$ (equivalently: for all integers $j \geqslant 1$) the sequence $\{\|A^n h\|^{2j}\}_{n=0}^\infty$ is a Hamburger moment sequence for every $h \in \mathcal{H}$; recall that a sequence $\{a_n\}_{n=0}^\infty$ of real numbers is said to be a *Hamburger moment sequence* if there exists a positive Borel measure μ on $\mathbb{R}$ such that $a_n = \int_{\mathbb{R}} t^n \mu(\mathrm{d}t)$ for all integers $n \geqslant 0$ (cf. [**4**, Chapter 6.2]). As shown in [**12, 13**] (see also [**17**]), if $A \in \boldsymbol{B}(\mathcal{H})$ is an *arbitrary* operator, then the requirement that $\{\|A^n h\|^2\}_{n=0}^\infty$ is a Hamburger moment sequence for every $h \in \mathcal{H}$ is equivalent to the subnormality of A. The question of whether the assumption that $\{\|A^n h\|^{2j}\}_{n=0}^\infty$ is a Hamburger moment sequence for every $h \in \mathcal{H}$ implies subnormality of $A \in \boldsymbol{B}(\mathcal{H})$ is still open for every integer $j \geqslant 2$ (the reverse implication is always true). The following criterions for normality in terms of moment sequences are obtained with the help of paranormality.

Proposition 7. *An operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if $\mathcal{N}(A) = \mathcal{N}(A^*)$ and for some integers $j,k \geqslant 1$ (equivalently: for all integers $j,k \geqslant 1$) the sequences $\{\|A^n h\|^{2j}\}_{n=0}^\infty$ and $\{\|A^{*n} h\|^{2k}\}_{n=0}^\infty$ are Hamburger moment sequences for every $h \in \mathcal{H}$.*

Proof. We begin with the proof of the “if” part. For every $h \in \mathcal{H}$ there exists a positive Borel measure μ_h on $\mathbb{R}$ such that $\|A^n h\|^{2j} = \int_{\mathbb{R}} t^n \mu_h(\mathrm{d}t)$ for all integers $n \geqslant 0$. Then by the Schwarz inequality we have

$$(\|Ah\|^{2j})^2 = \Big(\int_{\mathbb{R}} t\mu_h(\mathrm{d}t)\Big)^2 \leqslant \int_{\mathbb{R}} t^2\mu_h(\mathrm{d}t)\int_{\mathbb{R}} t^0\mu_h(\mathrm{d}t) = \|A^2h\|^{2j}\|h\|^{2j} \tag{7}$$

for all $h \in \mathcal{H}$. Hence A is paranormal. Similarly, we show that A^* is paranormal. Applying Ando’s theorem (cf. [**1**, Thorem 5]) we get the normality of A.

To complete the proof notice that the normality of A implies that for every $h \in \mathcal{H}$, $\{\|A^n h\|^2\}_{n=0}^\infty$ is a Hamburger moment sequence and consequently (cf. [**4**, Chapter 6.1]) all sequences $\{\|A^n h\|^{2j}\}_{n=0}^\infty$, $j \geqslant 1$, are Hamburger moment sequences. This completes the proof. □

Repeating argument from the above proof and employing the fact that compact paranormal operators are normal (cf. [**11**, Theorem 2]) we get the following.

Proposition 8. *A compact operator $A \in \boldsymbol{B}(\mathcal{H})$ is normal if and only if for some integer $j \geqslant 1$ (equivalently: for all integers $j \geqslant 1$) the sequence $\{\|A^n h\|^{2j}\}_{n=0}^{\infty}$ is a Hamburger moment sequence for every $h \in \mathcal{H}$.*

Remark 9. Propositions 7 remains true if the requirement "the sequences $\{\|A^n h\|^{2j}\}_{n=0}^{\infty}$ and $\{\|A^{*n} h\|^{2k}\}_{n=0}^{\infty}$ are Hamburger moment sequences" is replaced by "the sequences $\{\|A^n h\|\}_{n=0}^{\infty}$ and $\{\|A^{*n} h\|\}_{n=0}^{\infty}$ are logarithmically convex". Similar replacement can be done in the case of Proposition 8. For more details concerning the role played by logarithmic convexity and concavity in operator theory we refer the reader to [**18**].

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