C-holomorphic functions with algebraic graphs I

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Abstract. In this paper we are dealing with c-holomorphic functions defined on algebraic sets and having algebraic graphs. We prove a Liouville-type lemma which allows us to extend Strzeboński's results to these functions. Moreover, using a result of Tworzewski and Winiarski we provide a bound for the growth exponent in terms of the degrees of the sets involved. We prove also that algebricity of the graph is equivalent to the function being the restriction of a rational function (a Serre-type theorem). Then we turn to considering proper c-holomorphic mappings with algebraic graphs and we prove a Bezout-type theorem together with effective Nullstellensätze in the spirit of Płoski and Tworzewski. Finally, following Płoski, we show that the Łojasiewicz exponent at infinity of such a mapping is attained and rational. This completes in some sense our work concerning the local Łojasiewicz exponent of c-holomorphic mappings.

1. Introduction

For the convenience of the reader we recall the definition of a c-holomorphic mapping. Let $A \subset \Omega$ be an analytic subset of an open set $\Omega \subset \mathbb{C}^m$.

Definition 1.1. ([Ł], [Wh]) A mapping $f\colon A \to \mathbb{C}^n$ is called *c-holomorphic* if it is continuous and the restriction of f to the subset of regular points $\mathrm{Reg}A$ is holomorphic. We denote by $\mathcal{O}_c(A, \mathbb{C}^n)$ the ring of c-holomorphic mappings, and by $\mathcal{O}_c(A)$ the ring of c-holomorphic functions.

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It is a way (due to R. Remmert) of generalizing the notion of holomorphic mapping onto sets having singularities and a more convenient one than the usual notion of *weakly holomorphic functions* (i.e. functions defined and holomorphic on Reg$A$ and locally bounded on A). The following theorem is fundamental for all what we shall do (cf. [Wh] 4.5Q):

Theorem 1.2. *A mapping $f\colon A \to \mathbb{C}^n$ is c-holomorphic iff it is continuous and its graph $\Gamma_f := \{(x, f(x)) \mid x \in A\}$ is an analytic subset of $\Omega \times \mathbb{C}^n$.*

For a more detailed list of basic properties of c-holomorphic mappings see [Wh], [D1].

2. C-holomorphic functions with algebraic graphs

Let $|\cdot|$ denote any of the usual norms on $\mathbb{C}^m$ (we shall not distinguish in notation the norms for different m as long as there is no real need for such a dinstinction). We begin with the following Liouville-type lemma concerning c-holomorphic mappings whose graphs are algebraic sets (it is a consequence of the Rudin-Sadullaev criterion):

Lemma 2.1. *Let $A \subset \mathbb{C}^m$ be a purely k-dimensional analytic set and let $f \in \mathcal{O}_c(A, \mathbb{C}^n)$. Then Γ_f is algebraic if and only if A is algebraic and there are constants $M, s > 0$ such that*

$$|f(x)| \leq M(1 + |x|^s), \quad \text{for } x \in A.$$

Proof. For the 'only if' part remark that by Chevalley's theorem A is algebraic as the proper projection of the graph. Besides, by [Ł] VII.7.2 we get immediately

$$\Gamma_f \subset \{(z, w) \in \mathbb{C}^m \times \mathbb{C}^n \mid |w| \leq M(1 + |z|^s)\}$$

for some $M, s > 0$.

For simplicity sake we assume now that the considered norms are the ℓ_1 norms i.e. sum of moduli of the coordinates.

To prove the 'if' part we apply [Ł] VII.7.4 to A:

$$A \subset \{(u, v) \in \mathbb{C}^k \times \mathbb{C}^{m-k} \mid |v| \leq M'(1 + |u|)\}$$

(in well-chosen coordinates) for some $M' > 0$. Take now $(u, v, w) \in \Gamma_f$, we have then $|f(u, v)| \leq M(1 + |(u, v)|^s) = M[1 + (|u| + |v|)^s]$ and we compute:

$$\begin{aligned}
|f(u, v)| + |v| &\leq M[1 + (|u| + |v|)^s] + M'(1 + |u|) \leq \\
&\leq M\{1 + [|u| + M'(1 + |u|)]^s\} + M'(1 + |u|) \leq \\
&\leq M + M(M' + 1)^s(1 + |u|)^s + M'(1 + |u|) \leq \\
&\leq C + C(1 + |u|)^{s'} + C(1 + |u|)^{s'} \leq \\
&\leq 3C(1 + |u|)^{s'},
\end{aligned}$$

for $s' := \max\{1, s\}$ and $C := \max\{M, M(M' + 1)^s, M'\}$. Now Rudin-Sadullaev criterion yields Γ_f algebraic. $\square$

Remark 2.2. The condition 'A is algebraic' in the equivalence is not redundant since any polynomial restricted to e.g. $A = \{y = e^x\}$ satisfies the inequality but has a non algebraic graph (otherwise A would be algebraic too).

Note also that $|f(x)| \leq M(1 + |x|^s)$ on A iff $|f(x)| \leq M(1 + |x|)^s$ on A.

Hereafter we are interested in particular in c-holomorphic functions with algebraic graphs. For simplicity sake we will denote their ring by $\mathcal{O}_c^{\mathrm{a}}(A)$ when $A \subset \mathbb{C}^m$ is a fixed algebraic set. As a matter of fact, we will assume most of the time that $A \subset \mathbb{C}^m$ is a pure k-dimensional algebraic set of degree $d := \deg A$ (meaning the degree of the projective completion of A). Obviously, we shall assume also $k \geq 1$ unless something else is stated.

In connection with Strzeboński's paper [S], for any $f \in \mathcal{O}_c^{\mathrm{a}}(A)$ with any algerbaic set A, we introduce its *growth exponent*

$$\mathcal{B}(f) := \inf\{s \geq 0 \mid |f(x)| \leq C(1+|x|)^s,\ \textit{on } A \textit{ with some constant } C > 0\}$$

and the set of all possible growth exponents on A:

$$\mathcal{B}_A := \{\mathcal{B}(f) \mid f \in \mathcal{O}_c(A) \colon \Gamma_f \textit{ is algebraic}\}.$$

Observe that there is in fact

$$\mathcal{B}(f) = \inf\{s \geq 0 \mid |f(x)| \leq \mathrm{const} \cdot |x|^s,\ x \in A \colon |x| \geq M \textit{ with some } M \geq 1\}.$$

The growth exponent replaces in the c-holomorphic setting the notion of the *degree* of a polynomial (if $A = \mathbb{C}^m$, then obviously $\mathcal{O}_c^{\mathrm{a}}(A) = \mathbb{C}[x_1, \ldots, x_m]$ and so $\mathcal{B} = \deg$).

Recall the following important lemma from [S]:

Lemma 2.3 ([S] lemma 2.3). *Let $P(x,t) = t^d + a_1(x)t^{d-1} + \ldots + a_d(x)$ be a polynomial with $a_j \in \mathbb{C}[x_1, \ldots, x_k]$. Then $\delta(P) := \max_{j=1}^d(\deg a_j/j)$ is the minimal exponent $s > 0$ for which the inclusion*

$$P^{-1}(0) \subset \{(x,t) \in \mathbb{C}^k \times \mathbb{C} \mid |t| \leq C(1+|x|)^s\}$$

holds with some $C > 0$.

Note that in view of our preceding observation it is merely an avatar of the following Płoski's crucial lemma:

Lemma 2.4 ([P] lemma (2.1)). *If $P(x,t)$ is as above, then $\delta(P)$ is the minimal exponent $q > 0$ such that*

$$\{(x,t) \in \mathbb{C}^k \times \mathbb{C} \mid P(x,t) = 0, |x| \geq R\} \subset \{(x,t) \in \mathbb{C}^k \times \mathbb{C} \mid |t| \leq C|x|^q\}$$

for some $R, C > 0$.

From 2.3 we easily obtain the c-holomorphic counterpart of Strzeboński's result by taking the image Γ of the graph Γ_f by the projection $\pi \times \mathrm{id}_{\mathbb{C}}$, where $\pi \colon \mathbb{C}^m \to \mathbb{C}^k$ is a projection realizing $\deg A$. Then one can apply lemma 2.3 to Γ. It remains to observe that $\mathcal{B}(f) = \delta(P)$, where P is the minimal polynomial describing Γ.

Theorem 2.5. *For any algebraic set A and any $f \in \mathcal{O}_c^{\mathrm{a}}(A)$, the number $\mathcal{B}(f)$ is rational and is a growth exponent of f. Moreover, one has*

$$\mathbb{Z}_+ \subset \mathcal{B}_A \subset \{p/q \mid p, q \in \mathbb{N} \colon 1 \leq q \leq d,\ p, q \textit{ relatively prime}\},$$

where d is the maximum of degrees of all the irreducible components of A.

P. Tworzewski conjectures that in this case the second inclusion is in fact an equality (when A is irreducible).

In the second part of this paper we shall need some more information about $\mathcal{B}(f)$. It is easy to see from the definition that for any $h_1, h_2 \in \mathcal{O}_c^{\mathrm{a}}(A)$ there is $\mathcal{B}(h_1h_2) \leq \mathcal{B}(h_1) + \mathcal{B}(h_2)$ and $\mathcal{B}(h_1 + h_2) \leq \max\{\mathcal{B}(h_1), \mathcal{B}(h_2)\}$. But what will turn out to be most important is that for any positive integer n there is $\mathcal{B}(f^n) = n\mathcal{B}(f)$.

Using the results of [TW1] we are able to give an estimate of $\mathcal{B}(f)$. Indeed, by applying [TW1] theorem 3 we get in lemma 2.3 above the estimate $\delta(P) \leq \deg P^{-1}(0) - d + 1$. It remains to specify what actually $\deg P^{-1}(0)$ and d are. We shall need a proposition which is interesting in itself and is a simple consequence of the following general one which is also of interest:

Proposition 2.6. *Let $f\colon (A,0) \to (\mathbb{C}^k_w, 0)$ be a non-constant c-holomorphic germ on a pure k-dimensional analytic germ $A \subset \mathbb{C}^m$. Then we can choose coordinates in $\mathbb{C}^m_z = \mathbb{C}^k_x \times \mathbb{C}^{m-k}_y$ in such a way that for the projections $\pi(x,y) = x$, $\eta := (\pi \times \mathrm{id}_{\mathbb{C}^k_w})$, $p(x,y,w) = w$, $\varrho(x,w) = w$, $\zeta(x,w) = x$ and the set $\Gamma := \eta(\Gamma_f)$, we have*

(i) $\pi^{-1}(0) \cap C_0(A) = \{0\}$, *i.e.* $\mu_0(\pi|_A) = \deg_0 A$;
(ii) $\mu_0(p|_{\Gamma_f}) = \mu_0(\varrho|_\Gamma)$, *i.e.* $m_0(f) = \mu_0(\varrho|_\Gamma)$ *and so* $\mu_0(\eta|_{\Gamma_f}) = 1$;
(iii) $\mu_0(\zeta|_\Gamma) = \deg_0 A$,

and this holds true for the generic choice of coordinates.

Here $\mu_0(\pi|_A)$ denotes the *covering number* of the branched covering $\pi|_A$ with 0 as the unique point in the fibre $\pi^{-1}(0)$ (see [Ch]).

So as to prove this proposition we begin with a most easy lemma:

Lemma 2.7. *If $E \subset \mathbb{C}^m$ is such that $\#E = \mu > 0$ and $k \leq m$, then for the generic epimorphism $L \in \mathrm{L}(\mathbb{C}^m, \mathbb{C}^k)$ one has $\#L(E) = \mu$.*

Proof. It suffices to prove the assertion for $k = m-1$. The set $\{\ell \in G_1(\mathbb{C}^m) \mid \exists x,y \in E\colon x \neq y, x \in \ell + y\}$ is finite. Thus for the generic $\ell \in G_1(\mathbb{C}^m)$ the set $\bigcup_{x\in E} x + \ell$ consists of μ distinct lines. The orthogonal projection π^ℓ along ℓ is hence the sought after epimorphism. □

Proof of proposition 2.6. We know that for the generic projection π we have $\pi^{-1}(0) \cap C_0(A) = \{0\}$ (cf. [Ch]). Let us take such a projection which in addition 'separates' the points in the maximal fibre of f, i.e. for w such that $f^{-1}(w)$ consists of $m_0(f)$ points, $\pi(f^{-1}(w))$ consists also of $m_0(f)$ points (cf. the previous lemma). That means $\pi|_{f^{-1}(w)}$ is an injection.

Observe that $p = \varrho \circ \eta$ and $p|_{\Gamma_f}$, $\eta|_{\Gamma_f}$, $\varrho|_\Gamma$ are proper. Moreover $m_0(f) = \mu_0(p|_{\Gamma_f})$. Now it remains to observe that

$$\varrho^{-1}(w) \cap \Gamma = \pi(f^{-1}(w)) \times \{w\}$$

and so by the choice of π we have $\mu_0(\varrho|_\Gamma) = m_0(f)$. Thence $\mu_0(\eta|_{\Gamma_f}) = 1$, which means that $\eta\colon \Gamma_f \to \Gamma$ is one-to-one. Indeed, if $(x_0, w_0) \in \mathbb{C}^k \times \mathbb{C}^k$ is fixed,

$$\begin{aligned}\#\eta^{-1}(x_0,w_0) \cap \Gamma_f &= \#\{y \in \mathbb{C}^{m-k} \mid (x_0,y) \in A,\ f(x_0,y) = w_0\} = \\ &= \#\{z \in f^{-1}(w_0) \mid \pi(z) = x_0\} = \\ &= \#f^{-1}(w_0) \cap \pi^{-1}(x_0).\end{aligned}$$

The latter is equal to one iff $f|_{\pi^{-1}(x_0)\cap A}$ is injective which is equivalent to $\pi|_{f^{-1}(w_0)}$ being an injection. That we know to be true. Thus, in particular, for any $x, w \in \mathbb{C}^k$ there exists exactly one $y \in \mathbb{C}^{m-k}$ such that $f(x,y) = w$.

Therefore $\mu_0(\zeta|_\Gamma) = \deg_0 A =: d$. Indeed, if we take $x_0 \in \mathbb{C}^k$ near zero such that $\#\pi^{-1}(x_0) \cap A = d$, then obviously $\#f(\pi^{-1}(x_0) \cap A) \leq d$. On the other hand if there were $y \neq y'$ such that $f(x_0, y) = f(x_0, y') =: w_0$, then the set $\eta^{-1}(x_0, w_0) \cap \Gamma_f$ would include the two points $(x_0, y, w_0) \neq (x_0, y', w_0)$ and this would hold true for x_0 arbitrarily close to zero. This would contradict $\mu_0(\eta|_{\Gamma_f}) = 1$. $\square$

Note. It may be useful, in reference to [D1], to observe that in the situation from the proposition above it is easy to check that the *Łojasiewicz exponent* $\mathcal{L}(f; 0) = 1/q_0(\Gamma, \varrho)$ (with the notations from theorem (2.6) in [D1]).

Proposition 2.8. *Let $A \subset \mathbb{C}^m$ be an algebraic irreducible set of dimension k and let $f \in \mathcal{O}_c^{\mathrm{a}}(A)$. If f is non-constant, then for the generic choice of coordinates in $\mathbb{C}_z^m = \mathbb{C}_x^k \times \mathbb{C}_y^{m-k}$ the projection $\pi(x, y) = x$ restricted to A realizes* $\mathrm{deg}A$ *and $f|_{\pi^{-1}(x) \cap A}$ is injective for the generic x.*

Proof. If $k = 0, m$, there is nothing to do (if $k = m$, then $\pi = \mathrm{id}_{\mathbb{C}^m}$ and by Serre's Theorem f is a polynomial). Suppose $0 < k < m$ and assume without loss of generality that $0 \in A$ and $f(0) = 0$. By the identity principle $f^{-1}(0)$ has pure dimension $k-1$ (see [D2]) and clearly is an algebraic set. For the generic choice of coordinates the c-holomorphic mapping $g(z) := (f(z), z_1, \ldots, z_{k-1})$ has isolated fibres. By a similar argument to the one used in the proof of proposition 2.6, for the generic projection $\pi\colon \mathbb{C}^m \to \mathbb{C}^k$ onto the first k coordinates realizing $\mathrm{deg}A$, the restriction $g|_{\pi^{-1}(\pi(z)) \cap A}$ is injective for the generic z.

Choose now coordinates so that $g(z)$ and $\pi(z) = (z_1, \ldots, z_k)$ satisfy the conditions mentioned above. If $z, z' \in A$ are such that $\pi(z) = \pi(z')$ and $f(z) = f(z')$, then by construction $g(z) = g(z')$ and so $z = z'$. $\square$

Theorem 2.9. *Let $f \in \mathcal{O}_c^{\mathrm{a}}(A)$ with $A \subset \mathbb{C}^m$ of dimension $k \geq 0$. Then*

$$\mathcal{B}(f) \leq \mathrm{deg}\Gamma_f - \mathrm{deg}A + 1.$$

Proof. If f is constant or $k = 0$, then $\mathcal{B}(f) = 0$ and the estimate holds. We may assume thus f non-constant and $k > 0$.

Suppose first that A is irreducible.

Let $\pi\colon \mathbb{C}^m \to \mathbb{C}^k$ be a projection realizing the degree $\mathrm{deg}A$ and let $\Gamma := (\pi \times \mathrm{id}_{\mathbb{C}})(\Gamma_f)$. The latter is clearly an algebraic set due to Remmert-Chevalley Theorem. A straightforward application of the main result of [TW1] gives now $\mathcal{B}(f) \leq \mathrm{deg}\Gamma - d + 1$, where d is the covering number of the branched covering $\zeta\colon \mathbb{C}^k \times \mathbb{C} \to \mathbb{C}^k$ on Γ.

It is easy to see that $\mathrm{deg}\Gamma \leq \mathrm{deg}\Gamma_f$. Indeed, if $\ell \subset \mathbb{C}^{k+1}$ is an affine complex line such that $\mathrm{deg}\Gamma = \#(\ell \cap \Gamma)$, then the set $L := \{z \in \mathbb{C}^m \mid (\pi \times \mathrm{id}_{\mathbb{C}})(z) \in \ell\}$ is an affine space of dimension $m + 1 - k$ intersecting Γ_f in a zero-dimensional set. This follows from the properness of $\pi \times \mathrm{id}_{\mathbb{C}}$ on Γ_f. Therefore, we have $\#(L \cap \Gamma_f) \leq \mathrm{deg}\Gamma_f$ (cf. [Ł] VII.11). Clearly $\#(\ell \cap \Gamma) \leq \#(L \cap \Gamma_f)$.

The point is how to choose π so as to have $d = \mathrm{deg}A$. We are able to do this thanks to proposition 2.8 asserting that for the generic π we have $\#f(\pi^{-1}(x) \cap A) = \mathrm{deg}A$ for the generic $x \in \mathbb{C}^k$.

Now, if A is reducible, then we apply the preceding argument to each irreducible component $S \subset A$ and $f|_S$ getting

$$\mathcal{B}(f|_S) \leq \mathrm{deg}\Gamma_{f|_S} - \mathrm{deg}S + 1.$$

Observe that $\mathcal{B}(f) = \max\{\mathcal{B}(f|_S) \mid S \subset A \textit{ an irreducible component}\}$ (cf. [S]) and $\mathrm{deg}\Gamma_f = \sum_S \mathrm{deg}\Gamma_{f|_S}$, since $\Gamma_{f|_S}$ is irreducible iff $S \subset A$ is irreducible (thus

$\Gamma_{f|_S}$ are the irreducible components of the graph). Therefore

$$\mathcal{B}(f) \leq \max_{S \subset A}(\deg \Gamma_{f|_S} - \deg S) + 1,$$

but since for each irreducible component S' there is

$$\deg \Gamma_{f|_{S'}} - \deg S' \leq \sum_{S \subset A}(\deg \Gamma_{f|_S} - \deg S),$$

we finally obtain the required inequality. $\square$

Example 2.10. Let A be the algebraic curve $\{(x,y) \in \mathbb{C}^2 \mid x^3 = y^2\}$ and consider the c-holomorphic function

$$f(x,y) = \begin{cases} \dfrac{y}{x}, & \mathit{for}\,(x,y) \in A \setminus (0,0) \\ 0, & \mathit{for}\, x = y = 0 \end{cases}$$

It is easy to see (cf. lemma 2.1) that Γ_f is algebraic and $\mathcal{B}(f) = 1/2$. Actually the algebricity of Γ_f is not really surprising because f is the restriction to Reg$A$ of a rational function. We will show that in fact the algebricity of the graph is equivalent in this case to the fact that the function has a rational 'extension'.

3. Universal denominators and Algebraic Graph Theorem

Using Oka's theorem about universal denominators (cf. [TsY] and [Wh]) one can show that any c-holomorphic function admits locally a universal denominator. We will detail this a little more in the proof of the following theorem. For the convienience of the reader let us start with one useful construction of a universal denominator.

Proposition 3.1. *Let $A \subset U \times \mathbb{C}_t \times \mathbb{C}_y^{m-k}$ be a pure k-dimensional analytic set, where $U \subset \mathbb{C}_x^k$ is open and connected, such that $0 \in A$ and the natural projection $\pi(x,t,y) = x$ is proper on A and realizing* $\deg_0 A =: d$. *Then after a change of coordinates in $\mathbb{C} \times \mathbb{C}^{m-k}$ there exists a monic polynomial $P \in \mathcal{O}(U)[t]$ of degree d such that $Q(x,t,y) := \frac{\partial P}{\partial t}(x,t)$ is a universal denominator at each point $a \in A$.*

Proof. Let $\rho(x,t,y) = (x,t)$ and $\xi(x,t) = x$ be the natural projections. For any point $x \in U$ not critical for $\pi|_A$ we have exactly d distinct points $(t_1, y^1), \dots, (t_d, y^d)$ over it in A. If we fix x, then taking if necessary a rotation in $\mathbb{C} \times \mathbb{C}^{m-k}$, we may assume that all the points $t_1, \dots, t_d$ are distinct. Thus ξ on $\rho(A)$ has multiplicity d as a branched covering. Note that by the Remmert theorem $\rho(A) \subset U \times \mathbb{C}$ is an analytic hypersurface. Thus there exist a reduced Weierstrass polynomial $P \in \mathcal{O}[t]$ such that $P^{-1}(0) = \rho(A)$. Its degree is obviously d.

Now for fixed x in a simply connected neighbourhood V not intersecting the critical set of P we have $(t_1(x), y^1(x)), \dots, (t_d(x), y^d(x))$, exactly d distinct points. Put

$$h(x,t) := \sum_{j=1}^{d} f(x, t_j(x), y^j(x)) \prod_{\iota \neq j}(t - t_\iota(x)), \quad (x,t) \in V \times \mathbb{C}.$$

Observe that $h(x, t_j(x)) = f(x, t_j(x), y^j(x)) Q(x, t_j(x))$. Clearly the function $h(x,t)$ is locally holomorphic apart from the critical set of f (because the functions $t_j(x)$ and $y^j(x)$ are locally holomorphic) and locally bounded near the critical points of P, and so by the Riemann theorem we obtain a holomorphic function $h \in \mathcal{O}(U \times \mathbb{C} \times \mathbb{C}^{m-k})$ being an extension of Qf. $\square$

Theorem 3.2. *Let $A \subset \mathbb{C}^m$ be a purely k-dimensional algebraic set and let $f \in \mathcal{O}_c(A)$. Then Γ_f is algebraic if and only if there exists a rational function $R \in \mathbb{C}(x_1, \dots, x_m)$ equal to f on A (in particular $R|_A$ is continuous). More precisely, there exists a polynomial $Q \in \mathbb{C}[x_1, \dots, x_m]$ of degree $< \deg A$ such that $f = P/Q$ on A for some polynomial $P \in \mathbb{C}[x_1, \dots, x_m]$.*

Proof. If $m = 1$, then either A is the whole $\mathbb{C}$, and then by the identity principle Γ_f is algebraic if and only if f is a polynomial (cf. Serre's theorem on the algebraic graph), or A is a finite set and we apply a Lagrange interpolation. In both cases $Q \equiv 1$. Hence we may confine us to the case $m \geq 2$.

If $k = 0$, then $\#A < \infty$. We follow lemma 2.7. The set

$$\{\ell \in G_1(\mathbb{C}^m) \mid \exists x, y \in A,\ x \neq y \colon y \in x + \ell\}$$

is finite (even algebraic). Take thus a line $\ell \in G_1(\mathbb{C}^m)$ such that for all $x \in A$, $(x + \ell) \cap A = \{x\}$. If we denote by π^ℓ the natural projection along ℓ onto its orthogonal complement $\ell^\perp$, then $\#\pi^\ell(A) = \#A$. Continuing this *procédé* we find a one-dimensional subspace $L \subset \mathbb{C}^m$ such that $\#\pi(A) = \#A$, where π is the orthogonal projection onto L. Now the Lagrange interpolation for $\pi(A)$ and the values $f(a), \pi(a) \in \pi(A)$ yields a polynomial $P \in \mathbb{C}[t]$. Then $\widetilde{P}(x) := P(\pi(x))$ is the polynomial interpolating f on A. The 'only if' part is clear.

Assume now that $k \geq 1$. Since A is algebraic of pure dimension k, there are coordinates in $\mathbb{C}^m$ such that the projection π onto the first k coordinates is proper on A (so it is a branched covering) and it realizes $\deg A$. Now if we take $\rho(x_1, \dots, x_m) = (x_1, \dots, x_{k+1})$, then we are able to apply proposition 3.1 getting a polynomial (since by Chevalley's theorem $\rho(A)$ is an algebraic hypersurface) Q being a local universal denominator for A. Since $\mathbb{C}^m$ is a domain of holomorphy, Q is in fact a global universal denominator for A (actually this follows directly from the proof of 3.1).

That means that there exists $h \in \mathcal{O}(\mathbb{C}^m)$ such that

$$f = \frac{h}{Q} \quad \text{on } \mathrm{Reg}A.$$

Note that for points $a \in \mathrm{Sng}A$, if we take any sequence $\mathrm{Reg}A \ni a_\nu \to a$, then by continuity we obtain $f(a)Q(a) = h(a)$. Therefore either a is a point in which h/Q is well defined, or it is a point of indeterminacy of h/Q. In the latter case, the function h/Q has a finite and well defined limit along A, namely $f(a)$. Thus h/Q is continous on A.

As a matter of fact h is not uniquely determined. The proof of our theorem consists now in showing

(a) If h is a polynomial, then Γ_f is algebraic;
(b) If Γ_f is algebraic, then we may choose $h \in \mathbb{C}[x_1, \dots, x_m]$.

Ad (a): Let $X := (A \times \mathbb{C}) \cap \{(x,t) \in \mathbb{C}^m \times \mathbb{C} \mid h(x) = Q(x)t\}$. It is an algebraic set of dimension at least k. Over points $x \in A \setminus Q^{-1}(0)$ this is exactly the graph of f. Thus for each such point x and the only one t for which $(x,t) \in X$, we have $\dim_{(x,t)} X = k$. On the other hand, since Q does not vanish on any irreducible component of A, the set $A \cap Q^{-1}(0)$ has pure dimension $k - 1$ (see [D2]). For each point $x \in A \cap Q^{-1}(0)$ we have a whole line $\{x\} \times \mathbb{C} \subset X$. Thus the set X has pure dimension k.

Set $\Gamma := \Gamma_f \setminus (Q^{-1}(0) \times \mathbb{C}) = \Gamma_f \setminus [(A \cap Q^{-1}(0)) \times \mathbb{C}]$. Then we have $\Gamma \subset X$ and so for closures $\overline{\Gamma} \subset \overline{X} = X$. But by continuity $\overline{\Gamma} = \Gamma_f$, and since Γ_f has pure dimension k it must be the union of some irreducible components of X. Since X is algebraic, so is Γ_f.

Ad (b): This follows from Serre's algebraic graph theorem (for regular functions, see [Ł]). Indeed, fQ is a holomorphic function in $\mathbb{C}^m$ with algebraic graph over the algebraic set A (to see this apply lemma 2.1; one can remark by the way that $\mathcal{B}(fQ) \leq \mathcal{B}(f) + \mathcal{B}(Q|_A)$). Thus it is on A a regular function which means that it is in fact the restriction to A of a polynomial P. □

Remark 3.3. It is easy to check that in the theorem above we obtain

$$\mathcal{B}(f) \geq \mathcal{B}(P|_A) - \mathcal{B}(Q|_A).$$

4. Proper c-holomorphic mappings with algebraic graphs

C-holomorphic functions with algebraic graphs are a promising generalization of polynomials onto algebraic sets. Most of the theorems known for instance for polynomial dominating mappings should have their analogues for c-holomorphic proper mappings with algebraic graphs. Note, however, that in this setting we are naturally obliged to make do more with the geometric structure than the algebraic one (that is a hindrance when trying to extend the results of [D2] to the c-holomorphic algebraic case).

We consider now the following situation:
Let $A \subset \mathbb{C}^m$ be algebraic of pure dimension $k > 0$ and suppose $f \in \mathcal{O}_c(A, \mathbb{C}^k)$ is a proper mapping with algebraic graph. It is clear then that for each component f_j of f has an algebraic graph.

Since Γ_f is algebraic with proper projection onto $\mathbb{C}^k$, then $\#f^{-1}(w)$ is constant for the generic $w \in \mathbb{C}^k$. We call this number, denoted by $\mathrm{d}(f)$, the *geometric degree* of f just as in the polynomial case. We call *critical* for f any point $w \in \mathbb{C}^k$ for which $\#f^{-1}(w) \neq \mathrm{d}(f)$. In that case one has actually $\#f^{-1}(w) < \mathrm{d}(f)$ (cf. e.g. [Ch], the projection onto $\mathbb{C}^k$ restricted to Γ_f is a $\mathrm{d}(f)$-sheeted branched covering). Obviously $\mathrm{d}(f) \leq \deg\Gamma_f$ (cf. [Ł]).

Similarly to the polynomial case, we have the following

Proposition 4.1. *Let $f\colon A \to \mathbb{C}^k$ be a c-holomorphic proper mapping with algebraic graph. Then*

$$\mathrm{d}(f) \leq \deg A \prod_{j=1}^{k} \mathcal{B}(f_j).$$

Before we begin the proof recall (see [Ł] VII.§7 and [Ch]) that if $\Gamma \subset \mathbb{C}^n$ is algebraic of pure dimension k, then $\deg\Gamma = \#(L \cap \Gamma)$ for any $L \subset \mathbb{C}^n$ affine subspace of dimension $n-k$ transversal to A and such that $L_\infty \cap \overline{\Gamma} = \varnothing$, where $\overline{\Gamma}$ is the projective closure and L_∞ denotes the points of L at infinity (i.e. the intersection of $\overline{L}$ with the hyperplane at infinity in $\mathbb{P}_n$). The point is that the condition $L_\infty \cap \overline{\Gamma} = \varnothing$ is equivalent to the inclusion

$$\Gamma \subset \{u + v \in L' + L \mid |v| \leq \mathrm{const} \cdot (1 + |u|)\}$$

where L' is any k-dimensional affine subspace such that $L' + L = \mathbb{C}^m$. Moreover, for any $n-k$-dimensional affine subspace L cutting A in a zero-dimensional set (with no additional hypotheses) there is $\#(L \cap \Gamma) \leq \deg\Gamma$.

Proof of proposition 4.1. Let q_j be any positive integers such that $q_j\mathcal{B}(f_j) \in \mathbb{N}$ for $j = 1,\dots,k$. Then set $F := (f_1^{q_1},\dots,f_k^{q_k})$. We still have $F \in \mathcal{O}_c^{\mathrm{a}}(A)$ and F is proper with $\mathrm{d}(F) = \mathrm{d}(f)\prod_j q_j$. Besides, $\mathcal{B}(F_j) = q_j\mathcal{B}(f_j)$.

The idea now is to follow the idea used in the proof of proposition (4.6) from [D1] inspired by the methods of Płoski and Tworzewski. To that aim consider the algebraic set

$$\Gamma := \{(z,w) \in A \times \mathbb{C}^k \mid w_j^{\mathcal{B}(F_j)} = F_j(z),\ j = 1,\dots,k\}.$$

Clearly, for any $a \in \Gamma$, there is $\dim_a\Gamma \geq k$ and since Γ has proper projection $p(z,w) = z$ onto A, the converse inequality holds too and so Γ is pure k-dimensional.

Take now any affine subspace $\ell \subset \mathbb{C}^m$ of dimension k such that $\#(\ell\cap A) = \mathrm{deg}A$ and

$$A \subset \{x+y \in \ell^{\perp} + \ell \mid |y| \leq C(1+|x|)\},$$

where $\ell^{\perp}$ is an orthogonal complementary to ℓ, $x+y = z$ and $C > 0$ a constant. Then by construction $L := \ell + \mathbb{C}^k$ (seen in $\mathbb{C}^{m+k}$) is transversal to Γ and we have $\#(L\cap\Gamma) = \mathrm{deg}A\prod_j \mathcal{B}(F_j)$. We may assume that the norm in consideration is the sum of moduli. Now observe that for $(z,w) \in \Gamma$,

$$|w_j|^{\mathcal{B}(F_j)} = |F_j(z)| \leq c_j|z|^{\mathcal{B}(F_j)} \text{ when } |z| \geq R_j,$$

for some $c_j, R_j > 0$. Then $|w| \leq (\max_j c_j)|z|$ when $|z| \geq \max_j R_j$. Therefore, there exists a constant $K > 0$ such that

$$\Gamma \subset \{(x,y,w) \in \ell^{\perp} + \ell + \mathbb{C}^k \mid |y| + |w| \leq K(1+|x|)\}$$

and so $\mathrm{deg}\Gamma = \mathrm{deg}A\prod_j \mathcal{B}(F_j)$.

Finally, it suffices to remark that one has $\mathrm{d}(F) \leq \mathrm{deg}\Gamma$ since we have $\mathrm{d}(F) = \#((\mathbb{C}^m \times \{0\}^k)\cap\Gamma)$. $\square$

Example 4.2. Let $A := \{y^2 = x^3\} \subset \mathbb{C}^2$ and $f(x,y) = y/x$ when $x \neq 0$, $f(0,0) = 0$. One has $f \in \mathcal{O}_c^{\mathrm{a}}(A)$. Since f is injective, $\mathrm{d}(f) = 1$. Clearly $\mathrm{deg}A = 3$ and it is easy to check that $\mathcal{B}(f) = 1/3$. Thus $\mathrm{d}(f) = \mathrm{deg}A \cdot \mathcal{B}(f)$.

This example hints at a more general observation:

Proposition 4.3. *Let $\Gamma = \gamma(\mathbb{C}) \subset \mathbb{C}^m$ be an algebraic curve such that $\mathrm{deg}\Gamma = \max_j \mathrm{deg}\gamma_j$, where $\gamma = (\gamma_1,\dots,\gamma_m)\colon \mathbb{C} \to \mathbb{C}^m$ is an injective polynomial mapping. Let $f \in \mathcal{O}_c^{\mathrm{a}}(\Gamma)$ be non-constant. Then*

$$\mathrm{d}(f) = \mathrm{deg}\Gamma \cdot \mathcal{B}(f)$$

Proof. The idea of the proof is similar to that of theorem (3.2) from [D1]. We may assume that $|\cdot|$ is the maximum norm. Let $d := \mathrm{deg}\Gamma$ and observe that since

$$\lim_{|t|\to+\infty} \frac{|\gamma(t)|}{|t|^d} = \text{const.} > 0,$$

the inequality $|f(x)| \leq \text{const}\cdot|x|^{\mathcal{B}(f)}$ for $x \in A$ with $|x| \gg 1$ is equivalent to

$$|f(\gamma(t))| \leq \text{const}\cdot|t|^{d\mathcal{B}(f)}, \quad |t| \gg 1.$$

Observe now that $f\circ\gamma$ is a polynomial by Serre's Graph Theorem and so there are two positive constants c_1, c_2 such that

$$c_1|t|^{\deg(f\circ\gamma)} \leq |f(\gamma(t))| \leq c_2\cdot|t|^{\deg(f\circ\gamma)}, \quad |t| \gg 1.$$

But $\deg(f \circ \gamma) = \mathrm{d}(f)$ because γ being injective, we have $\#(f \circ \gamma)^{-1}(w) = \# f^{-1}(w)$ for the generic $w \in \mathbb{C}$. We can now find two positive constants c'_1, c'_2 so that

$$\lim_{|t| \to +\infty} |t|^{d\mathcal{B}(f) - \mathrm{d}(f)} \geq c'_1 \quad \text{and} \quad \lim_{|t| \to +\infty} |t|^{\mathrm{d}(f) - d\mathcal{B}(f)} \leq c'_2.$$

Therefore $\mathrm{d}(f) = d\mathcal{B}(f)$. □

In the sequel we shall use intensively the notion of characteristic polynomial relative to f.

For any $g \in \mathcal{O}^{\mathrm{a}}_c(A)$ let us introduce the *characteristic polynomial* of g relative to f: for any $w \in \mathbb{C}^k$ not critical for f we put

$$P_g(w,t) := \prod_{x \in f^{-1}(w)} (t - g(w)) = t^{\mathrm{d}(f)} + a_1(w) t^{\mathrm{d}(f)-1} + \ldots + a_{\mathrm{d}(f)}(w)$$

extending the coefficients through the critical locus of f thanks to the Riemann Extension Theorem (they are continuous; see below their form). Therefore $P_g \in \mathcal{O}(\mathbb{C}^k)[t]$.

Proposition 4.4. *In the introduced setting, P_g is a pure-bred polynomial, i.e. $P_g \in \mathbb{C}[w_1, \ldots, w_k, t]$.*

Proof. This follows from the expressions for the coefficients:

$$a_j(w) = (-1)^j \sum_{1 \leq \iota_1 < \ldots < \iota_j \leq \mathrm{d}(f)} g(x^{(\iota_1)}) \cdot \ldots \cdot g(x^{(\iota_j)}),$$

where $f^{-1}(w) = \{x^{(1)}, \ldots, x^{(\mathrm{d}(f))}\}$ consists of $\mathrm{d}(f)$ points.

Since $g \in \mathcal{O}^{\mathrm{a}}_c(A)$, there is $|g(x)| \leq C_1(1 + |x|^r)$ for $x \in A$ with some constants $C_1, r > 0$ (cf. lemma 2.1). By assumption, Γ_f has proper projection onto $\mathbb{C}^k$ and so by [Ł],

$$\Gamma_f \subset \{(z,w) \in \mathbb{C}^m \times \mathbb{C}^k \mid |z| \leq C_2(1 + |w|)^s\}$$

for some constants $C_2, s > 0$. Therefore, for any $x \in A$, $|x| \leq C_2(1 + |f(x)|)^s$. We obtain thus

$$|g(x)| \leq C_1(1 + C_2(1 + |f(x)|)^{rs}) \leq C_1 2 \max\{1, C_2\}(1 + |f(x)|)^{rs}.$$

That means in particular that for any w not critical for f and for all j,

$$|a_j(w)| \leq \mathrm{const} \cdot (1 + |w|)^p,$$

since $w = f(x^{(j)})$. Here $p = \max\{1, rs\}$. By continuity this inequality can be extended to the whole of $\mathbb{C}^k$ and so by Liouville's Theorem $a_j \in \mathbb{C}[w_1, \ldots, w_k]$ for all j. □

5. Nullstellensätze

We shall deal first with the 0-dimensional case, i.e. we assume that $f = (f_1, \ldots, f_k)$ is a proper c-holomorphic mapping with algebraic graph over a set of pure dimension $k > 0$ as in the preceding section. It is clear that it is surjective. With all the notations introduced so far we have

Theorem 5.1. *Let $g \in \mathcal{O}_c^{\mathrm{a}}(A)$ be such that $g^{-1}(0) \supset f^{-1}(0)$. Then there are k functions $h_j \in \mathcal{O}_c^{\mathrm{a}}(A)$ such that*

$$g^{\mathrm{d}(f)} = \sum_{j=1}^{k} h_j f_j \quad \textit{on the whole of} \quad A.$$

Proof. Let P_g be the characteristic polynomial of g relative to f. From the definition we have clearly $P_g(f(x), g(x)) = 0$ for $x \in A$, which means

$$g(x)^{\mathrm{d}(f)} = -a_1(f(x))g(x)^{\mathrm{d}(f)-1} + \ldots - a_{\mathrm{d}(f)}(f(x)).$$

Now, any $a_j \in \mathbb{C}[w_1, \ldots, w_k]$ and since $g = 0$ on $f^{-1}(0)$, it follows from the expression of a_j (see the proof of proposition 4.4) that $a_j(0) = 0$ for any j. Therefore $a_j(w) = \sum_{\iota=1}^{k} a_{j,\iota}(w) w_\iota$ with $a_{j,\iota} \in \mathbb{C}[w_1, \ldots, w_k]$ and the assertion follows. $\square$

Remark 5.2. Example 3.3 from [D2] shows that the coefficients h_j may well be strictly c-holomorphic, i.e. having no holomorphic extension onto a neighbourhood of A in $\mathbb{C}^m$ (even locally).

We are able now to generalize this to the case of a set-theoretical complete intersection in connection with [D2] and [PT]. Suppose that $f\colon A \to \mathbb{C}^n$ is c-holomorphic with algebraic graph, A has pure dimension $k > 0$ and $f^{-1}(0)$ is pure $(k-n)$-dimensional. This means exactly that the intersection $\Gamma_f \cap (\mathbb{C}^m \times \{0\}^k)$ is proper (i.e. it is a set-theoretical complete intersection). In such a case we may consider the *algebraic effective cycle of zeroes* of f:

$$Z_f := \Gamma_f \cdot (\mathbb{C}^k \times \{0\}^k) = \sum_{j=1}^{r} i(\Gamma_f, \mathbb{C}^k \times \{0\}^k; V_j) V_j,$$

where $f^{-1}(0) = \bigcup_{j=1}^{r} V_j$ is the decomposition into irreducible components and $i(\Gamma_f, \mathbb{C}^k \times \{0\}^k; V_j)$ is the intersection multiplicity along V_j computed following [Dr].

Since all V_j are algebraic we may define the *degree of the cycle* Z_f to be the number

$$\mathrm{deg} Z_f = \sum_{j=1}^{r} i(\Gamma_f, \mathbb{C}^k \times \{0\}^k; V_j) \cdot \mathrm{deg} V_j.$$

Note that for $k = n$ we clearly obtain $\mathrm{deg} Z_f = \mathrm{d}(f)$.

Theorem 5.3. *In the introduced setting, for any $g \in \mathcal{O}_c^{\mathrm{a}}(A)$ such that $g^{-1}(0) \supset f^{-1}(0)$ there are n functions $h_j \in \mathcal{O}_c^{\mathrm{a}}(A)$ yielding*

$$g^{\mathrm{deg} Z_f} = \sum_{j=1}^{n} h_j f_j \quad \textit{on the whole of } A.$$

Proof. The idea of the proof is similar to that of [PT] and [D2].

We start with choosing coordinates in $\mathbb{C}^m$ in such a way that $\{0\}^{k-n} \times \mathbb{C}^{m-(k-n)}$ intersects Z_f properly with multiplicity $\mathrm{deg} Z_f$, i.e. all the components $V_j \subset f^{-1}(0)$ project properly onto the first $k - n$ coordinates with multiplicity $\mathrm{deg} V_j$. Then the mapping

$$\varphi\colon A \ni x \to (f(x), x_1, \ldots, x_{k-n}) \in \mathbb{C}^k$$

is c-holomorphic with algebraic graph and all its fibres are zero-dimensional. We will first show that $\mathrm{d}(\varphi) = \deg Z_f$.

It is quite obvious that $\mathrm{d}(\varphi)$ coincides with the multiplicity of the projection

$$\pi\colon \mathbb{C}^m \times \mathbb{C}^n \ni (x, y) \mapsto (y, x_1, \ldots, x_{k-n}) \in \mathbb{C}^k$$

restricted to Γ_f. In turn, this multiplicity is equal to $\deg(\Gamma_f \cdot \pi^{-1}(0))$. Finally observe that by [TW2] Theorem 2.2 we obtain

$$\begin{aligned}\Gamma_f \cdot \pi^{-1}(0) &= (\Gamma_f \cdot (\mathbb{C}^m \times \{0\}^n)) \cdot_{\mathbb{C}^m \times \{0\}^n} (\{0\}^{k-n} \times \mathbb{C}^{m-(k-n)}) = \\ &= Z_f \cdot (\{0\}^{k-n} \times \mathbb{C}^{m-(k-n)}).\end{aligned}$$

Take now $P_g \in \mathbb{C}[w_1, \ldots, w_k, t]$ to be the characteristic polynomial of g relative to φ. Since $g^{-1}(0) \supset f^{-1}(0)$, we have

$$P_g^{-1}(0) \cap (\{0\}^n \times \mathbb{C}^{k-n} \times \mathbb{C}) = \{0\}^n \times \mathbb{C}^{k-n} \times \{0\}.$$

Therefore all the coefficients a_j of P_g must vanish on $\{0\}^n \times \mathbb{C}^{k-n}$. Writing $w = (y, z) \in \mathbb{C}^n \times \mathbb{C}^{k-n}$ we obtain for any j, $a_j(w) = \sum_{\iota=1}^n y_\iota a_{j,\iota}(w)$ with some polynomials $a_{j,\iota}$. The result sought after follows now from $P_g(\varphi(x), g(x)) \equiv 0$. $\square$

6. The Łojasiewicz exponent at infinity

We are still dealing with f as in section 4, i.e. $f = (f_1, \ldots, f_k)$ is a c-holomorphic mapping with algebraic graph over an algebraic set of pure dimension $k > 0$. Let $g \in \mathcal{O}_c^{\mathrm{a}}(A)$. Thanks to the polynomial P_g we shall be able to prove an analogue of theorem 2.6 from [D1] for the Łojasiewicz exponent at infinity of f. This notion will be introduced after the following proposition:

Proposition 6.1. *In the introduced setting, $\delta(P_g)$ (see lemma 2.3) is the minimal exponent $q > 0$ for which*

$$(*) \qquad |g(x)| \leq C|f(x)|^q, \quad \text{when } x \in A, |x| \geq R$$

with some $C, R > 0$.

Proof. This is a consequence of lemma 2.4. Indeed, observe that by construction, $P_g(f(x), g(x)) = 0$ and so by this lemma

$$|g(x)| \leq C|f(x)|^{\delta(P_g)}, \quad \text{when } x \in A, |f(x)| \geq R$$

for some $C, R > 0$. However, the properness of f is clearly equivalent to the condition (since $k > 0$)

$$\lim_{\substack{|x| \to +\infty \\ x \in A}} |f(x)| = +\infty.$$

Therefore, we can find an $r > 0$ for which $|x| \geq r$ implies $|f(x)| \geq R$.

On the other hand, any such inequality $(*)$ with an exponent $q > 0$ implies in particular that if $w \in \mathbb{C}^k$ is not critical for f, $|w| \geq R$, and $P_g(w, t) = 0$, then $|t| \leq C|w|^q$. By continuity (since critical points form a nowheredense set) this can be extended to all $w \in \mathbb{C}^k$ such that $|w| \geq R$ and $P_g(w, t) = 0$. Then lemma 2.4 yields $q \geq \delta(P_g)$. $\square$

Thanks to that proposition, taking $g(x) = x_j$, i.e. the coordinate functions on $\mathbb{C}^k$, and the maximum norm on $\mathbb{C}^k$, we clearly see that f satisfies the following

Łojasiewicz inequality at infinity (being the c-holomorphic counterpart of the *Hörmander-Łojasiewicz inequality* for polynomials):

$$\mathrm{const}\cdot|x|^{1/\max_{j=1}^m \delta(P_{x_j})} \leq |f(x)|, \quad \text{if } x \in A, |x| \geq R$$

with some $R > 0$ for which $|f(x)| \geq 1$. Note that if such an inequality holds with an exponent $q > 0$ and $R \geq 1$, then it holds also with any exponent $q' < q$ for the same R. It is thus interesting to introduce the notion of *Łojasiewicz exponent at infinity* posing

$$\mathcal{L}_\infty(f) := \sup\{q > 0 \mid \mathrm{const}\cdot|x|^q \leq |f(x)| \textit{ for all } x \in A \textit{ big enough}\}.$$

We have just seen that $\mathcal{L}_\infty(f) \geq (1/\max_{j=1}^m \delta(P_{x_j}))$. Actually, we have the following theorem being a c-holomorphic counterpart of the Gorin and Płoski result ([P] proposition (1.6) and corollary (2.6)):

Theorem 6.2. *In the introduced setting,* $\mathcal{L}_\infty(f) = (1/\max_{j=1}^m \delta(P_{x_j}))$ *and so* $\mathcal{L}_\infty(f)$ *is attained and is a rational number. Moreover,* $\mathcal{L}_\infty(f) = p/q$*, with integers* $p, q \geq 1$ *such that* $p \leq \mathrm{d}(f)$*.*

Proof. Take an exponent $q > 0$ for which $C|x|^q \leq |f(x)|$ when $x \in A$ and $|x| \geq R$, with $C, R > 0$. We may assume, without loss of generality, that the norm in consideration is the maximum norm. Then, clearly, for each $j = 1, \ldots, k$ there is $C|x_j|^q \leq |f(x)|$ whenever $x \in A$ satisfies $|x| \geq R$. By proposition 6.1 we must have then $\delta(P_{x_j}) \leq 1/q$ for any j. That proves the assertion. The form p/q as well as the bound on p follow from the definition of $\delta(P_{x_j})$. □

We will give a bound on $\mathcal{L}_\infty(f)$ in terms of the growth exponents $\mathcal{B}(f_j)$ and the degree $\deg A$ in a forthcoming paper.

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