Approximation of sets defined by polynomials with holomorphic coefficients

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Abstract

Let X be an analytic set defined by polynomials whose coefficients $a_1, \dots, a_s$ are holomorphic functions. We formulate conditions such that for all sequences $\{a_{1,\nu}\}, \dots, \{a_{s,\nu}\}$ of holomorphic functions converging locally uniformly to $a_1, \dots, a_s$ respectively the following holds true. If $a_{1,\nu}, \dots, a_{s,\nu}$ satisfy the conditions then the sequence of the sets $\{X_\nu\}$ obtained by replacing a_j's by $a_{j,\nu}$'s in the polynomials, converge to X.

Keywords Analytic set, Nash set, approximation
MSC (2000): 32C25

1 Introduction and main results

The problem of approximating analytic objects by simpler algebraic ones with similar properties appears in many contexts of complex geometry and has attracted the attention of several mathematicians (see [6], [7], [10], [11], [12], [13], [18], [19], [20]). In the present paper we concern the problem in the case where the approximated objects are complex analytic sets whereas the approximating ones are complex Nash sets (see Section 2.1). The approximation is expressed by means of the convergence of holomorphic chains (for the definition see Section 2.2).

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For sets with proper projection the existence of such approximation was discussed in [3], [4]. In a subsequent paper [5] it was proved that the order of tangency of a limit set and the approximating sets can be arbitrarily high. The first results on approximation of analytic sets by higher order tangent algebraic varieties are due to R. W. Braun, R. Meise and B. A. Taylor [7] with applications in [8].

Both in [4] and in [5] analytic sets are represented as mappings defined on an open subset of $\mathbf{C}^n$ with values in an appropriate symmetric power of $\mathbf{C}^m$. However, in many cases such sets are defined by systems of equations which in general carry more information than the sets themselves. Therefore it is natural to look for approximations of the functions appearing in the equations. Throughout this paper we restrict our attention to the case where the description is given by a system of polynomials with holomorphic coefficients whereas the approximated set is with proper projection onto an appropriate affine space. Our aim is to show how to approximate the coefficients of the polynomials to obtain Nash approximations of the set.

If the number of the functions describing the analytic set X is equal to the codimension of X then it is sufficient to take generic approximations of the coefficients in order to get local uniform approximation of X. Such approach clearly does not work in the case of a non-complete intersection as it leads to sets of dimensions strictly smaller than the dimension of X. Yet, it is natural to expect that there are algebraic relations satisfied by the coefficients such that if the approximating coefficients also satisfy the relations then the original polynomials with these new coefficients define appropriate approximations.

Before stating the main result let us recall that for any analytic set Y by $Y_{(n)}$ we denote the union of all n-dimensional irreducible components of Y.

Let $U \subset \mathbf{C}^n$ be a domain. Abbreviate $v = (v_1, \ldots, v_p)$, $z = (z_1, \ldots, z_m)$. Assuming the notation of Section 2 and treating analytic sets as holomorphic chains with components of multiplicity one we prove

Theorem 1.1 *Let $q_1, \ldots, q_s \in \mathbf{C}[v, z]$, for some $s \in \mathbf{N}$, and let $H : U \to \mathbf{C}^p$ be a holomorphic mapping. Assume that*

$$X = \{(x, z) \in U \times \mathbf{C}^m : q_i(H(x), z) = 0, i = 1, \ldots, s\}$$

is an analytic set of pure dimension n with proper projection onto U. Then there is an algebraic subvariety F of $\mathbf{C}^p$ with $H(U) \subset F$ such that for every sequence $\{H_\nu : U \to F\}$ of holomorphic mappings converging locally uniformly to H the following holds. The sequence $\{X_\nu\}$, where

$$X_\nu = \{(x, z) \in U \times \mathbf{C}^m : q_i(H_\nu(x), z) = 0, i = 1, \ldots, s\},$$

converges to X locally uniformly and the sequence $\{(X_\nu)_{(n)}\}$ converges to X in the sense of holomorphic chains.

The following example shows that the sets from $\{X_\nu\}$ are in general not purely dimensional:

Example 1.2 *Define* $X = \{(x,z) \in \mathbf{C}^2 : zxe^x = 0, z^2 - zx = 0\}$. *Then* $X = \{(x,z) \in \mathbf{C}^2 : z = 0\}$, *therefore it is purely* 1*-dimensional. On the other hand,* $\mathbf{C}^2 \times \{1\}$ *is the smallest algebraic set in* $\mathbf{C}^3$ *containing the image of the mapping* $x \mapsto (-x, xe^x, 1)$. *By approximating this mapping by* $x \mapsto (-x, (x - \frac{1}{\nu})e^x, 1)$ *one obtains* $X_\nu = \{(x,z) \in \mathbf{C}^2 : z(x - \frac{1}{\nu})e^x = 0, z^2 - zx = 0\}$ *containing an isolated point* $(\frac{1}{\nu}, \frac{1}{\nu})$.

Let U be a connected Runge domain in $\mathbf{C}^n$, let X be a purely n-dimensional analytic subset of $U \times \mathbf{C}^m$ with proper projection onto U and let $Q_1, \ldots, Q_s \in \mathcal{O}(U)[z]$, for some $s \in \mathbf{N}$, satisfy

$$X = \{(x,z) \in U \times \mathbf{C}^m : Q_1(x,z) = \ldots = Q_s(x,z) = 0\}.$$

(An example of such $Q_1, \ldots, Q_s$ are the canonical defining functions for X see [23], [9]).)

We check that combining Theorem 1.1 with one of results of L. Lempert (Theorem 3.2 from [13], see Theorem 2.3 below) one obtains Nash approximations of X by approximating its holomorphic description by a Nash description. (Let us mention that the proof of Theorem 2.3 is based on the affirmative solution to the Artin's conjecture first presented in [15], [16], see also [1], [14], [17].)

Let $H = (H_1, \ldots, H_s)$ denote the holomorphic mapping defined on U where, for every $j \in \{1, \ldots, s\}$, H_j is the mapping whose components are all the nonzero coefficients of the polynomial Q_j; by n_j denote the number of these coefficients. More precisely, the components of H_j are indexed by m-tuples from some finite set $S_j \subset \mathbf{N}^m$ in such a way that the component indexed by a fixed $(\alpha_1, \ldots, \alpha_m)$ is the coefficient standing at the monomial $z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$ in Q_j.

Let F be the intersection of all algebraic subvarieties of $\mathbf{C}^{(\sum_j n_j)}$ containing $H(U)$ and let $\tilde{U}$ be any open relatively compact subset of U. Then $\tilde{U}$ is contained in a polynomially convex compact subset of U hence by Theorem 2.3 there exists a sequence $\{H_\nu : \tilde{U} \to F\}$ of Nash mappings, $H_\nu = (H_{1,\nu}, \ldots, H_{s,\nu})$, such that $\{H_{j,\nu}\}$ converges uniformly to $H_j|_{\tilde{U}}$, for every $j = 1, \ldots, s$. Now let

$$X_\nu = \{(x,z) \in \tilde{U} \times \mathbf{C}^m : Q_{1,\nu}(x,z) = \ldots = Q_{s,\nu}(x,z) = 0\},$$

where $Q_{j,\nu} \in \mathcal{O}(\tilde{U})[z]$, for $j = 1, \ldots, s$, is defined as follows. The coefficient of $Q_{j,\nu}$ standing at the monomial $z_1^{\alpha_1} \cdot \ldots \cdot z_m^{\alpha_m}$ is the component of $H_{j,\nu}$ indexed by $(\alpha_1, \ldots, \alpha_m)$ (if $(\alpha_1, \ldots, \alpha_m) \notin S_j$ then the coefficient equals zero).

Finally, let $q_1, \ldots, q_s$ be the polynomials obtained from $Q_1, \ldots, Q_s$ by replacing the holomorphic coefficients of the latter polynomials by independent new variables. It is easy to see that $q_1, \ldots, q_s$ together with the mapping H satisfy the hypotheses of Theorem 1.1. Hence the sequence of Nash sets $\{(X_\nu)_{(n)}\}$, where X_ν defined in the previous paragraph, converges to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of holomorphic chains. Thus we recover the main result of [4]:

Corollary 1.3 *Let* X *be a purely* n*-dimensional analytic subset of* $U \times \mathbf{C}^m$ *with proper projection onto* U. *Then for every open set* $\tilde{U} \subset\subset U$ *there is a* *sequence $\{X_\nu\}$ of purely n-dimensional Nash subsets of $\tilde{U} \times \mathbf{C}^m$ converging to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of chains.*

Let us point out that starting from Theorem 1.1 one can obtain another proof of the existence of higher order approximations of X by Nash sets in a neighborhood of any $a \in X$ (for the original proof see [5]).

Note that the convergence of positive chains appearing in the paper is equivalent to the convergence of currents of integration over the considered sets (see [9], pp. 141, 206-207).

The organization of this paper is as follows. In Section 2 preliminary material is presented whereas Section 3 contains the proof of Theorem 1.1.

2 Preliminaries

2.1 Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \ldots, f_s$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \ldots = f_s(x) = 0\}.$$

The fact from [21] stated below explains the relation between Nash and algebraic sets.

Theorem 2.1 *Let X be an irreducible Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$. Conversely, every analytic irreducible component of $Y \cap \Omega$ is an irreducible Nash subset of Ω.*

2.2 Convergence of closed sets and holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean the formal sum $A = \sum_{j \in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j \in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [22], cp. also [2], [9]). The set $\bigcup_{j \in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an $n-$chain.

Below we introduce the convergence of holomorphic chains in U. To do this we first need the notion of the local uniform convergence of closed sets. Let Y, Y_ν be closed subsets of U for $\nu \in \mathbf{N}$. We say that $\{Y_\nu\}$ converges to Y locally uniformly if:

(1l) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^m$,

(2l) for every compact subset K of U such that $K \cap Y = \emptyset$ it holds $K \cap Y_\nu = \emptyset$ for almost all ν.

Then we write $Y_\nu \to Y$. For details concerning the topology of local uniform convergence see [22].

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1c) $|Z_\nu| \to |Z|$,

(2c) for each regular point a of $|Z|$ and each submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact and $|Z| \cap \overline{T} = \{a\}$, we have $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \rightarrowtail Z$. (By $Z \cdot T$ we denote the intersection product of Z and T (cf. [22]). Observe that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^{d} \alpha_j \{a_j\}$, $deg(A) = \sum_{j=1}^{d} \alpha_j$).

The following lemma from [22] will be useful to us.

Lemma 2.2 *Let $n \in \mathbf{N}$ and Z, Z_ν, for $\nu \in \mathbf{N}$, be positive n-chains. If $|Z_\nu| \to |Z|$ then the following conditions are equivalent:*
(1) $Z_\nu \rightarrowtail Z$,
(2) for each point a from a given dense subset of $Reg(|Z|)$ there exists a submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact, $|Z| \cap \overline{T} = \{a\}$ and $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

2.3 Approximation of holomorphic mappings

In the proofs of Corollary 1.3 we use the following theorem which is due to L. Lempert (see [13], Theorem 3.2).

Theorem 2.3 *Let K be a holomorphically convex compact subset of $\mathbf{C}^n$ and $f : K \to \mathbf{C}^k$ a holomorphic mapping that satisfies a system of equations $Q(z, f(z)) = 0$ for $z \in K$. Here Q is a Nash mapping from a neighborhood $U \subset \mathbf{C}^n \times \mathbf{C}^k$ of the graph of f into some $\mathbf{C}^q$. Then f can be uniformly approximated by a Nash mapping $F : K \to \mathbf{C}^k$ satisfying $Q(z, F(z)) = 0$.*

3 Proof of Theorem 1.1

Denote $B_m(r) = \{z \in \mathbf{C}^m : ||z||_{\mathbf{C}^m} < r\}$ and recall $v = (v_1, \ldots, v_p)$. Let U be a domain in $\mathbf{C}^n$. We prove the following

Proposition 3.1 *Let $q_1, \ldots, q_s \in \mathbf{C}[v,z]$, for some $s \in \mathbf{N}$, and let $H : U \to \mathbf{C}^p$ be a holomorphic mapping. Assume that*

$$X = \{(x,z) \in U \times \mathbf{C}^m : q_i(H(x), z) = 0, i = 1, \ldots, s\}$$

is an analytic set of pure dimension n with proper projection onto U. Then there is an algebraic subvariety F of $\mathbf{C}^p$ with $H(U) \subset F$ such that for every domain $\tilde{U} \subset\subset U$ and every sequence $\{H_\nu : \tilde{U} \to F\}$ of holomorphic mappings converging uniformly to H on $\tilde{U}$ the following holds. There is $r_0 > 0$ such that for every $r > r_0$ the sequence $\{X_\nu\}$, where

$$X_\nu = \{(x,z) \in \tilde{U} \times B_m(r) : q_i(H_\nu(x), z) = 0, i = 1, \ldots, s\},$$

satisfies:

(1) X_ν is n-dimensional with proper projection onto $\tilde{U}$ for almost all ν,
(2) $\max\{\sharp(X \cap (\{x\} \times \mathbf{C}^m)) : x \in U\} =$
$\max\{\sharp((X_\nu)_{(n)} \cap (\{x\} \times \mathbf{C}^m)) : x \in \tilde{U}\}$ for almost all ν,
(3) $\{X_\nu\}$, $\{(X_\nu)_{(n)}\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly.

Proof of Proposition 3.1. Define the algebraic set

$$V = \{(v,z) \in \mathbf{C}^p \times \mathbf{C}^m : q_i(v,z) = 0, i = 1, \ldots, s\}.$$

Next, by F denote the intersection of all algebraic subsets of $\mathbf{C}^p$ containing the image of H. Clearly, F is irreducible (because U is connected) hence of pure dimension, say $\bar{n}$. Fix an open connected subset $\tilde{U} \subset\subset U$. In the following lemma F is endowed with the topology induced by the standard topology of $\mathbf{C}^p$.

Lemma 3.2 *Let $r > 0$ be such that $(\tilde{U} \times B_m(r)) \cap X \neq \emptyset$ and $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X = \emptyset$. Then there is an open neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional with proper projection onto C. Moreover, for every $(a,z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ it holds $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$.*

Proof of Lemma 3.2. First we check that there is an open neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(\overline{C} \times \partial B_m(r)) \cap V = \emptyset$, which implies the properness of the projection of $(C \times B_m(r)) \cap V$ onto C.

It is sufficient to show that for every $a \in \overline{H(\tilde{U})}$ there is an open neighborhood C_a in F such that $(C_a \times \partial B_m(r)) \cap V = \emptyset$. Fix $a \in \overline{H(\tilde{U})}$. Now, if for every open neighborhood C_a of a we had $(C_a \times \partial B_m(r)) \cap V \neq \emptyset$ then there would be $(\{a\} \times \partial B_m(r)) \cap V \neq \emptyset$. But then $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X \neq \emptyset$ as $a \in \overline{H(\tilde{U})} \subset H(\overline{\tilde{U}})$, a contradiction.

Let us show that $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$ for every $(a,z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$. First observe that $dim((C \times B_m(r)) \cap V)$ cannot exceed the dimension of C because $(C \times B_m(r)) \cap V$ is with proper projection onto C. Next suppose that there is $(a,z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ such that $dim_{(a,z)}((C \times B_m(r)) \cap V) < \bar{n}$.

Let V_1 be the irreducible analytic component of $(C \times B_m(r)) \cap V$ containing (a, z) and let $\pi : \mathbf{C}^p \times \mathbf{C}^m \to \mathbf{C}^p$ denote the natural projection. It is easy to see that $H^{-1}(\pi(V_1))$ is a non-empty nowhere dense analytic subset of $H^{-1}(C)$ (nowhere-density because otherwise $H(U)$ would be contained in an algebraic set of dimension smaller than $\bar{n}$). Let P be a neighborhood of (a, z) in $C \times B_m(r)$ such that $P \cap V = P \cap V_1 \neq \emptyset$. Now consider the set

$$E = \{(w, y) \in (U \times B_m(r)) \cap X : (H(w), y) \in P \cap V\}.$$

One observes that $E \neq \emptyset$, because $H^{-1}(\{a\}) \times \{z\} \subset E$, and that E is an open subset of X, and moreover, the projection of E onto U is contained in $H^{-1}(\pi(V_1))$. This contradicts the fact that X is purely n-dimensional.

Since $(\tilde{U} \times B_m(r)) \cap X \neq \emptyset$ then $(H(\tilde{U}) \times B_m(r)) \cap V \neq \emptyset$ so by what we have proved so far $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional.■

Proof of Proposition 3.1 (continuation). Let $r_0 > 0$ be such that $(\tilde{U} \times B_m(r_0)) \cap X = (\tilde{U} \times \mathbf{C}^m) \cap X$ and let $r > r_0$. Then $(\overline{\tilde{U}} \times \partial B_m(r)) \cap X = \emptyset$ and by Lemma 3.2, there is a neighborhood C of $\overline{H(\tilde{U})}$ in F such that $(C \times B_m(r)) \cap V$ is $\bar{n}$-dimensional with proper projection onto C. Moreover, for every $(a, z) \in (\overline{H(\tilde{U})} \times B_m(r)) \cap V$ it holds $dim_{(a,z)}((C \times B_m(r)) \cap V) = \bar{n}$. Let $\{H_\nu : \tilde{U} \to F\}$ be a sequence of holomorphic mappings converging uniformly to H on $\tilde{U}$. Define the sequence $\{X_\nu\}$ as in the statement of Proposition 3.1.

First we show (1): X_ν is n-dimensional and with proper projection onto $\tilde{U}$ for almost all ν. To do this observe that for sufficiently large ν it holds $H_\nu(\tilde{U}) \subset C$ and then

$$X_\nu = \{(x, z) \in \tilde{U} \times B_m(r) : (H_\nu(x), z) \in (C \times B_m(r)) \cap V\}.$$

Thus the properness of the projection of X_ν onto $\tilde{U}$ is obvious by the choice of C in Lemma 3.2.

Now we check the following claim: for sufficiently large ν every fiber in X_ν over $\tilde{U}$ is not empty. Indeed, let C_0 denote the irreducible Nash component of C containing $H(\tilde{U})$. Then the projection of $(C_0 \times B_m(r)) \cap V$ onto C_0 is surjective which follows by Lemma 3.2. On the other hand, for sufficiently large ν, $H_\nu(\tilde{U}) \subset C_0$ which clearly implies the claim. Consequently, X_ν is n-dimensional for almost all ν.

Let us turn to (2). Since C_0 is an irreducible Nash set then $Reg(C_0)$ is connected. There is a nowhere dense Nash subset C' of C_0 such that the function $\rho : Reg(C_0) \setminus C' \to \mathbf{N}$ given by

$$\rho(v) = \sharp((\{v\} \times B_m(r)) \cap V)$$

is constant. By $\tilde{m}$ we denote the only value of ρ.

Neither $H(\tilde{U})$ nor $H_\nu(\tilde{U})$ (for large ν) can be contained in $Sing(C_0) \cup C'$ so $(H^{-1}(Sing(C_0) \cup C') \cup H_\nu^{-1}(Sing(C_0) \cup C')) \cap \tilde{U}$ is a nowhere dense analytic subset of $\tilde{U}$. This means that for the generic $x \in \tilde{U}$ the fibers in X and in X_ν over x have $\tilde{m}$ elements which completes the proof of (2).

Finally, let us prove (3). To check the condition (2l) of the definition of local uniform convergence it is sufficient to show that for every $(x_0, z_0) \in (\tilde{U} \times \mathbf{C}^m) \setminus X$ there is a neighborhood D of (x_0, z_0) in $\tilde{U} \times \mathbf{C}^m$ such that $D \cap X_\nu = \emptyset$ for almost all ν. This is obvious as there is $i \in \{1, \ldots, s\}$ such that $q_i(H(x_0), z_0) \neq 0$. Then $q_i(H_\nu(x_0), z_0) \neq 0$ for almost all ν in some neighborhood of (x_0, z_0).

As for the condition (1l), it suffices to show that for a fixed $x_0 \in \tilde{U} \setminus H^{-1}(Sing(C))$ the sequence $\{(\{x_0\} \times \mathbf{C}^m) \cap (X_\nu)_{(n)}\}$ converges to $(\{x_0\} \times \mathbf{C}^m) \cap X$ locally uniformly. Take $(x_0, z_0) \in X \cap (\tilde{U} \times \mathbf{C}^m) = X \cap (\tilde{U} \times B_m(r))$. Then by Lemma 3.2 it holds $dim_{(H(x_0), z_0)}(C \times B_m(r)) \cap V = dim(C)$. Consequently, (since $H(x_0) \in Reg(C)$ and $(C \times B_m(r)) \cap V$ is with proper projection onto C) there is a sequence $\{z_\nu\}$ converging to z_0 such that $dim_{(H_\nu(x_0), z_\nu)}(C \times B_m(r)) \cap V = dim(C)$ for almost all ν. This implies that for sufficiently large ν, the image of the projection of every open neighborhood of (x_0, z_ν) in X_ν onto $\tilde{U}$ contains a neighborhood of x_0 in $\tilde{U}$. Thus $(x_0, z_\nu) \in (X_\nu)_{(n)}$ for almost all ν and the proof is complete.$\blacksquare$

Proof of Theorem 1.1 (the end). Let F denote the intersection of all algebraic subvarieties of $\mathbf{C}^p$ containing $H(U)$ and let $\{H_\nu : U \to F\}$ be a sequence of holomorphic mappings converging locally uniformly to H. Define X_ν as in the statement of Theorem 1.1.

It is sufficient to show that for every relatively compact subset $\tilde{U}$ of U the sequences $\{X_\nu \cap (\tilde{U} \times \mathbf{C}^m)\}$ and $\{(X_\nu)_{(n)} \cap (\tilde{U} \times \mathbf{C}^m)\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly and in the sense of holomorphic chains respectively. Fix $\tilde{U} \subset\subset U$. Then by Proposition 3.1 there is r_0 such that for every $r > r_0$ the following hold. $\{X_\nu \cap (\tilde{U} \times B_m(r))\}$ and $\{(X_\nu)_{(n)} \cap (\tilde{U} \times B_m(r))\}$ converge to $X \cap (\tilde{U} \times \mathbf{C}^m)$ locally uniformly. Moreover, for almost all ν, $X_\nu \cap (\tilde{U} \times B_m(r))$ is n-dimensional with proper projection onto $\tilde{U}$ and $\max\{\sharp(X \cap (\{x\} \times \mathbf{C}^m)) : x \in \tilde{U}\} = \max\{\sharp((X_\nu)_{(n)} \cap (\{x\} \times B_m(r))) : x \in \tilde{U}\}$. Thus by Lemma 2.2 we have: $\{(X_\nu)_{(n)} \cap (\tilde{U} \times B_m(r))\}$ converges to $X \cap (\tilde{U} \times \mathbf{C}^m)$ in the sense of holomorphic chains. Since r can be taken arbitrarily large we get our claim.$\blacksquare$

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