QUATERNIONIC-VALUED ORDINARY DIFFERENTIAL EQUATIONS. THE RICCATI EQUATION

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IMUJ PREPRINT 2007/14

Abstract. We give some sufficient conditions for the existence of at least two periodic solutions of the quaternionic Riccati equation. In some cases we are able to give a full description of dynamics and detect solutions heteroclinic to the periodic ones.

1. Introduction

Campos and Mawhin [2] have initiated a study of the T-periodic solutions of quaternionic-valued first order differential equations

$$\dot{q} = F(t, q)$$

where $F : [0, T] \times \mathbb{H} \longrightarrow \mathbb{H}$ is continuous and $\mathbb{H}$ denotes the set of quaternions (see Subsection 2.3). They have considered e.g. linear monomial equation and the quaternionic Riccati equation

$$\dot{q} = q^2 + f(t)$$

where f is a real-valued T-periodic forcing term. The aim of this paper is to continue this study. We present results for the Riccati equation. The special

Date: September 14, 2007.
2000 *Mathematics Subject Classification.* Primary 34C25; Secondary 11R52.
Key words and phrases. periodic solutions, isolating segments, quaternions.

interest in this equation comes from the fact that it appears in the Euler vorticity dynamics (see [11]).

We present partial extensions to the quaternionic settings of the main results of [17, 16]. We use the notion of isolating segments introduced by Srzednicki [15]. Our segments are the simplest ones so we can apply the Brouwer fixed point theorem to the Poincaré map. In the complex case if the vector field is holomorphic then the Poincaré map is also holomorphic thus the Brouwer fixed point theorem can be strengthen to the Wolff–Denjoy theorem and the uniqueness of periodic solution inside isolating segments can be obtained (cf. [16]). This combined with the fact that the Poincaré map is a Möbius transformation (cf. [1]) allows one to give a full description of dynamics. But the quaternionic case seems much more complicated. While the concept of construction of isolating segments in the complex case can be carried over into the quaternionic case there is no quaternionic version of the Wolff–Denjoy theorem. In fact there are some theories of "regular" quaternionic-valued functions imitating the theory of holomorphic functions. The best known is the one due to Fueter [6]. But even the identity $f(q) = q$ and other polynomials are not regular in this sense. So we expect that the Poincaré map of the Riccati (and in general polynomial) equation is not regular either. Another theory was given by Leutwiler [12]. It deals with the class of functions of the reduced quaternionic variable $x_0 + ix_1 + jx_2$. But the Poincaré map is a function of full quaternionic variable $x_0 + ix_1 + jx_2 + kx_3$ so we can not use this approach. The most interesting theory was given by Cullen [4]. Polynomials and even power series of the form $\sum_{n=0}^{\infty} q^n a_n$ are regular in this sense. The theory is still being developed (cf. [9, 8]) but the quaternionic version of the Wolff–Denjoy theorem has not been obtained yet. We hope it will happen soon. But even this theory does not contain all functions we are interested in.

Presented paper is devoted to the Riccati equation of the form

$$\dot{q} = v(t, q) = qa(t)q + b(t)q + qd(t) + c(t) \tag{1.1}$$

where $a, b, c, d : \mathbb{R} \longrightarrow \mathbb{H}$. The vector field v is not regular in the sense of Cullen unless $a, b : \mathbb{R} \longrightarrow \mathbb{R}$. We deal with the Riccati equation in the form (1.1) because it can be extended to the point ∞ and in fact to the whole sphere $\mathcal{S}^4$ (cf. Subsection 2.4). Moreover, it has some geometrical properties which suits the technique we use. Nonetheless, some partial results can be obtained also for the equations

$$\dot{q} = q^2 a(t) + qd(t) + c(t),$$
$$\dot{q} = a(t)q^2 + b(t)q + c(t)$$

but we do not provide any. In some cases we are able to prove the uniqueness of periodic solutions inside isolating segments. It is possible due to the fact that Poincaré map is contracting in some sets (cf. Lemma 2.1). This idea comes from the geometric properties of the polynomial vector field and was developed in the complex case and carried over into the quaternionic one.

The paper is organised as follows. In Section 2 we collect some basics facts concerning processes, notion of isolating segments and quaternions. We also state a crucial lemma which is used instead of the Wolff–Denjoy theorem. In the last section we present main results. First of all we deal with the equation where the sets $\overline{a}(\mathbb{R})$ and $c(\mathbb{R})$ can be separated by a linear hyperplane. Later we present some improvement of the previous approach, namely by imposing some additional restrictions on $\overline{a}(\mathbb{R})$ and $c(\mathbb{R})$ we are able to present a full description of dynamics in the whole $\mathrm{cl}\,\mathbb{H} \simeq \mathcal{S}^4$ and $\mathbb{H}$. Next we consider the equation with the right hand side which has two continuous branches of simple zeros. We prove the existence of at least one periodic solution close to the branch. We present also some improvements of this result where the periodic solutions are asymptotically stable or asymptotically unstable. At the end we prove the existence of at least two periodic solutions close to the branch of double zeros of the vector field. We also detect infinitely many solutions heteroclinic to the periodic ones.

2. Basic facts

2.1. **Processes.** Let X be a topological space and $\Omega \subset \mathbb{R} \times X \times \mathbb{R}$ be an open set.

By a *local process* on X we mean a continuous map $\varphi : \Omega \longrightarrow X$, such that three conditions are satisfied:

i) for σ and x, $\{t \in \mathbb{R} : (\sigma, x, t) \in \Omega\}$ is an open interval containing 0,
ii) for σ, $\varphi(\sigma, \cdot, 0) = \mathrm{id}_X$,
iii) for x, σ, s and t, $\varphi(\sigma, x, s+t) = \varphi(\sigma + s, \varphi(\sigma, x, s), t)$.

For abbreviation, we write $\varphi_{(\sigma,t)}(x)$ instead of $\varphi(\sigma, x, t)$.

Let M be a smooth manifold and let $v : \mathbb{R} \times M \longrightarrow TM$ be a time-dependent vector field. We assume that v is so regular that for every $(t_0, x_0) \in \mathbb{R} \times M$ the Cauchy problem

$$\dot{x} = v(t, x), \tag{2.1}$$

$$x(t_0) = x_0 \tag{2.2}$$

has unique solution. Then the equation (2.1) generates a local process φ on X by $\varphi_{(t_0,t)}(x_0) = x(t_0, x_0, t)$, where $x(t_0, x_0, \cdot)$ is the solution of the Cauchy problem (2.1), (2.2).

Let T be a positive number. In the sequel T denotes the period. We assume that v is T-periodic in t. It follows that the local process φ is T-periodic, i.e.,

$$\varphi_{(\sigma+T,t)} = \varphi_{(\sigma,t)} \text{ for all } \sigma, t \in \mathbb{R},$$

hence there is a one-to-one correspondence between T-periodic solutions of (2.1) and fixed points of the Poincaré map $\varphi_{(0,T)}$.

Let $g : M \longrightarrow M$ and $n \in \mathbb{N}$. Then g^n denotes the n-tn iterate of g and g^{-n} denotes the n-th iterate of g^{-1}.

We call a point z_0 *attracting (repelling) in* $W \subset M$ if for every point $w \in W$ the equality $\lim_{n\to\infty} g^n(w) = z_0$ ($\lim_{n\to\infty} g^{-n}(w) = z_0$) holds.

We call a T-periodic solution of (2.1) *attracting (repelling) in* $W \subset M$ if the corresponding fixed point of the Poincaré map is attracting (repelling) in W.

2.2. **Periodic isolating segments.** Let X be a topological space. We assume that φ is a T-periodic local process on X.

For any set $Z \subset \mathbb{R} \times X$ and $t \in \mathbb{R}$ we define

$$Z_t = \{x \in X : (t, x) \in Z\}$$

and the *exit set* and *entrance set* of Z by

$$Z^- = \{(t, x) \in Z : \varphi(\{t\} \times \{x\} \times [0, \tau)) \not\subset Z \text{ for all } \tau > 0\},$$
$$Z^+ = \{(t, x) \in Z : \varphi(\{t\} \times \{x\} \times (\tau, 0]) \not\subset Z \text{ for all } \tau < 0\},$$

respectively.

Let $\pi_1 : \mathbb{R} \times X \longrightarrow \mathbb{R}$ and $\pi_2 : \mathbb{R} \times X \longrightarrow X$ be projections on, respectively, time and space variable.

A compact set $W \subset [a,b] \times X$ is called an *isolating segment over* $[a,b]$ for φ if it is ENR (Euclidean neighborhood retract - cf. [5]) and there are $W^{--}, W^{++} \subset W$ compact ENR's (called, respectively, the *proper exit set* and the *proper entrance set*) such that

(1) $\partial W = W^- \cup W^+$,
(2) $W^- = W^{--} \cup (\{b\} \times W_b)$, $W^+ = W^{++} \cup (\{a\} \times W_a)$,
(3) there exists homeomorphism $h : [a,b] \times W_a \longrightarrow W$ such that $\pi_1 \circ h = \pi_1$ and $h([a,b] \times W_a^{--}) = W^{--}$, $h([a,b] \times W_a^{++}) = W^{++}$.

An isolating segment W over $[a,b]$ is said to be $(b-a)$*-periodic* (or simply *periodic*) if $W_a = W_b$, $W_a^{--} = W_b^{--}$ and $W_a^{++} = W_b^{++}$.

2.3. **Quaternions.** We follow [2] and use the letters q, p to denote (real) quaternions. By

$$q = (q_0, q_1, q_2, q_3) \in \mathbb{R}^4 \tag{2.3}$$

we mean a quaternion

$$q = q_0 + q_1 i + q_2 j + q_3 k, \tag{2.4}$$

where the symbols i, j, k satisfies the following rules of multiplication

$$i^2 = j^2 = k^2 = ijk = -1,$$
$$ij = -ji = k,\ ki = -ik = j,\ jk = -kj = i.$$

We denote the set of quaternions by $\mathbb{H}$.

For a quaternion q we define the *scalar part*

$$\mathfrak{s}_q = q_0 \in \mathbb{R}$$

and *vectorial part* by

$$\mathfrak{v}_q = (q_1, q_2, q_3) \in \mathbb{R}^3.$$

Thus one can write

$$q = (\mathfrak{s}_q, \mathfrak{v}_q) \in \mathbb{R} \times \mathbb{R}^3 \tag{2.5}$$

and the multiplication of two quaternions q, p has the form

$$qp = (\mathfrak{s}_q, \mathfrak{v}_q)(\mathfrak{s}_p, \mathfrak{v}_p) = (\mathfrak{s}_q\mathfrak{s}_p - \mathfrak{v}_q \cdot \mathfrak{v}_p, \mathfrak{s}_q\mathfrak{v}_p + \mathfrak{s}_p\mathfrak{v}_q + \mathfrak{v}_q \times \mathfrak{v}_p),$$

where $\cdot$ and $\times$ denote the inner product and cross product in $\mathbb{R}^3$, respectively.

In the sequel we use all the notations (2.3), (2.4) and (2.5) so one can think of $\mathbb{H}$ as $\mathbb{R}^4$ or $\mathbb{R} \times \mathbb{R}^3$.

We introduce the *inner product* and *modulus* of the quaternions $q, p = p_0 + p_1 i + p_2 j + p_3 k$ by

$$\langle q, p \rangle = q_0 p_0 + q_1 p_1 + q_2 p_2 + q_3 p_3 = \mathfrak{s}_q \mathfrak{s}_p + \mathfrak{v}_q \cdot \mathfrak{v}_p,$$
$$|q| = \sqrt{\langle q, q \rangle} = \sqrt{\mathfrak{s}_q^2 + |\mathfrak{v}_q|^2},$$

where $|\ |$ denotes also the norm in $\mathbb{R}^n$ (and of course a standard modulus in $\mathbb{C}$). Thus $\mathbb{H}$ is a Hilbert space.

For a quaternion q we introduce the *real part* operator

$$\mathfrak{Re}(q) = q_0$$

and *imaginary part* by

$$\mathfrak{Im}(q) = q_1 i + q_2 j + q_3 k.$$

Thus $q = \mathfrak{Re}(q) + \mathfrak{Im}(q)$ and its *conjugate* has the form

$$\overline{q} = q_0 - q_1 i - q_2 j - q_3 k = (\mathfrak{s}_q, -\mathfrak{v}_q) = \mathfrak{Re}(q) - \mathfrak{Im}(q).$$

Moreover

$$[\mathfrak{Im}(q)]^2 = -|\mathfrak{Im}(q)|^2. \tag{2.6}$$

In general, the multiplication of two quaternions q, p is not commutative but its projection on the scalar part has this property, i.e.

$$\mathfrak{Re}(qp) = \mathfrak{Re}(pq).$$

However, it is not true in the case of three or more quaternions e.g. $\mathfrak{Re}[ijk] = -1 \neq 1 = \mathfrak{Re}[jik]$ but for $r \in \mathbb{H}$ we can write

$$\mathfrak{Re}[pqr] = \mathfrak{Re}[rpq] = \mathfrak{Re}[qrp] \tag{2.7}$$

i.e. only cyclic permutations are allowed. Moreover, quaternion q commutes with all other quaternions if and only if $q \in \mathbb{R}$.

Now we list some useful formulae

$$\langle q, p\rangle = \mathfrak{Re}(q\overline{p}) = \mathfrak{Re}(\overline{q}p),$$
$$\overline{(qp)} = \overline{p}\,\overline{q},\ q\overline{q} = \overline{q}q = |q|^2,\ |qp| = |pq| = |p||q|,$$
$$\frac{1}{q} = \frac{\overline{q}}{|q|^2}.$$

$\mathbb{H}$ is a noncommutative field and it is evident that it contains $\mathbb{R}$ as a subfield. It also contains $\mathbb{C}$ as a subfield.

For a nonzero quaternion q we introduce the *argument* by

$$\mathrm{Ark}(q) = |\operatorname{Arg}(\mathfrak{s}_q + |\mathfrak{v}_q| i)|.$$

It is easy to see that $\mathrm{Ark}(q) \in [0, \pi]$ and $\mathrm{Ark}(0)$ is not defined. We define the sector

$$\mathcal{S}(\alpha) = \{q \in \mathbb{H} : \mathrm{Ark}(q) < \alpha\},$$

where $0 < \alpha \leq \pi$.

Let $q, p \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ then $qp \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ and

$$(qp)'(t) = q(t)p'(t) + q'(t)p(t).$$

Let α, β be quaternions such that $|\alpha| = |\beta| = 1$. Then the map $g : \mathbb{H} \ni q \longrightarrow \alpha q \beta \in \mathbb{H}$ is an orthogonal rotation. Moreover, every orthogonal rotation in $\mathbb{H}$ has this form (cf. [14, Chapter 10]). When $\alpha = \overline{\beta}$ then the rotation $\alpha q \beta$ affects only the vectorial part of q.

For $q \in \mathbb{H}$ we define the *exponential* of q by

$$e^q = \exp(q) = \sum_{n=0}^{\infty} \frac{q^n}{n!},$$

where the series converges absolutely and uniformly on compacts subsets of $\mathbb{H}$. If $p \in \mathbb{H}$ and $pq = qp$ then $e^p e^q = e^q e^p = e^{p+q}$. Let $\alpha \in \mathbb{R}$ and $I \in \mathbb{H}$ be such that $I^2 = -1$. Then $e^{\alpha I} = \cos(\alpha) + \sin(\alpha) I$. It is easy to see that $I^2 = -1$ if and only if

$$I \in \{q \in \mathbb{H} : |I| = 1, \mathfrak{Re}[I] = 0\}.$$

2.4. **Basic notions.** We make the general assumptions about the equation (1.1) that its coefficients $a, b, c, d \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ are T-periodic. By the change of variables

$$p = q^{-1} \tag{2.8}$$

we get

$$\begin{aligned} p' = \left(\frac{\overline{q}}{|q|^2}\right)' &= \overline{q}' \frac{1}{|q|^2} - 2\overline{q}\frac{\mathfrak{Re}[q\overline{q}']}{|q|^4} = \frac{1}{|q|^4}\left(\overline{q}q\overline{q}' - 2\overline{q}\mathfrak{Re}[q\overline{q}']\right) \\ &= \frac{\overline{q}}{|q|^4}\left(-\mathfrak{Re}[q\overline{q}'] + \mathfrak{Im}[q\overline{q}']\right) = -\frac{\overline{q}}{|q|^4}\overline{q\overline{q}'} = -\frac{\overline{q}}{|q|^4}q'\overline{q} \\ &= -pq'p \end{aligned}$$

thus the equation (1.1) has the form

$$\dot{p} = -a(t) - pb(t) - d(t)p - pc(t)p.$$

Thus it is well defined in the point ∞ and in fact in the whole sphere $\operatorname{cl}\mathbb{H} \simeq \mathcal{S}^4$. By compactness of $\operatorname{cl}\mathbb{H}$ it follows that for all $\sigma, t \in \mathbb{R}$ the Poincaré map $\varphi_{(\sigma,t)} : \operatorname{cl}\mathbb{H} \longrightarrow \operatorname{cl}\mathbb{H}$ is well defined.

It is worth mentioning that the change of variables (2.8) in the equation

$$\dot{q} = q^2 a(t)$$

gives

$$\dot{p} = u(t,p) = -\frac{1}{p}a(t)p$$

where u is not continuous in the point $p = 0$ unless $a \in \mathcal{C}(\mathbb{R}, \mathbb{R})$. To see this we fix $t \in \mathbb{R}$ and set $a(t) = a_0 + a_1 i + a_2 j + a_3 k$ where $a_0, a_1, a_2, a_3 \in \mathbb{R}$. Without losing of generality we can assume that $a_1 \neq 0$. Let $\lambda \in (0, \infty)$. For $p = \lambda$ we get $u(t,p) = -a(t)$. Let now $p = \lambda j$. Then $\frac{1}{p} = -\frac{1}{\lambda}j$ and

$$\begin{aligned} u(t,p) &= \frac{1}{\lambda}j(a_0 + a_1 i + a_2 j + a_3 k)\lambda j = -a_0 + a_1 jij - a_2 j + a_3 jkj \\ &= -a_0 + a_1 i - a_2 j + a_3 k. \end{aligned}$$

Thus $u(t,p) \neq -a(t)$.

We call a solution $s : \mathbb{R} \longrightarrow \operatorname{cl}\mathbb{H}$ of the equation (1.1) *singular* if there exists $t_0 \in \mathbb{R}$ such that $s(t_0) = \infty$. In the other case it is called *regular.* Singular solution, when considered in $\mathbb{H}$, is one that blows up. Let $-\infty \leq \alpha < \omega \leq \infty$ and $s : (\alpha, \omega) \longrightarrow \mathbb{H}$ be the full solution of (1.1). We call s *forward blowing up* (shortly *f.b.*) or *backward blowing up* (*b.b.*) if $\omega < \infty$ or $\alpha > -\infty$, respectively. If $-\infty < \alpha < \omega < \infty$ then s is called *backward and forward blowing up* (*b.f.b.*).

The family $\{A_\iota\}$ is called a *decomposition* of a set X when

- $\emptyset \neq A_\iota \subset X$,
- $\bigcup\{A_\iota\} = X$,
- $A_\iota \cap A_\kappa = \emptyset$

holds for all ι, κ, $\iota \neq \kappa$.

2.5. **Fixed points.** In the present subsection we state the following improvement of the Brouwer fixed point theorem. We use them instead of the quaternionic version of the Wolff–Denjoy theorem.

Lemma 2.1. *Let $m \geq 1$ and X be a nonempty convex and closed subset of $\mathbb{R}^m$. Let $f \in \mathcal{C}(X, X)$ be such that for every $x, y \in X$, $x \neq y$ the inequality*

$$|f(x) - f(y)| < |x - y| \tag{2.9}$$

holds. If in addition the set $f(X)$ is bounded then there exists exactly one fixed point $x_0 \in X$ of f. Moreover, x_0 is asymptotically stable and attracting in X.

3. Main results

3.1. **Geometric approach.** The lack of the quaternionic version of Wolff–Denjoy theorem does not allow us to prove the uniquenes of the periodic solutions inside half-spaces in the following theorem.

Theorem 3.1. *Let the coefficients $a, c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$, $b, d \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be T-periodic. If*

$$ac \not\equiv 0 \tag{3.1}$$

and there exists the decomposition $\{A, l, C\}$ of $\mathbb{H}$ such that l is a linear hyperplane and A, C are the half-spaces and the following conditions

$$\overline{a}(\mathbb{R}) \subset A \cup \{0\}, \tag{3.2}$$

$$c(\mathbb{R}) \subset C \cup l \tag{3.3}$$

hold then the equation (1.1) *has at least two T-periodic solutions in $\mathbb{H}$. One of them is contained in A and the other in C. There are no periodic solutions contained in l. There are no b.f.b. solutions.*

Proof. Let quaternions α, β, $|\alpha| = |\beta| = 1$ be such that the orthogonal transformation $g(q) = \alpha q \beta$ fulfills the following conditions

$$\begin{aligned} g(l) &= \{q \in \mathbb{H} : \mathfrak{Re}(q) = 0\}, \\ g(A) &= \{q \in \mathbb{H} : \mathfrak{Re}(q) > 0\}, \\ g(C) &= \{q \in \mathbb{H} : \mathfrak{Re}(q) < 0\}. \end{aligned}$$

We make the change of variables

$$p = \alpha q \beta. \tag{3.4}$$

Then the equation (1.1) has the form

$$\dot{p} = p\overline{\alpha\overline{a}(t)\beta}p + b(t)p + pd(t) + \alpha c(t)\beta. \tag{3.5}$$

Thus we may assume that the assumptions of the theorem are satisfied with $l = \{q \in \mathbb{H} : \mathfrak{Re}(q) = 0\}$, $A = \{q \in \mathbb{H} : \mathfrak{Re}(q) > 0\}$, $C = \{q \in \mathbb{H} : \mathfrak{Re}(q) < 0\}$.

Let $\left\{\widehat{l}, \widehat{A}, \widehat{C}\right\}$ be the decomposition of the sphere $\operatorname{cl} \mathbb{H}$ corresponding to the decomposition $\{l, A, C\}$ of $\mathbb{H}$ i.e. $\widehat{l}$ corresponds to the $l \cup \{\infty\}$ and $\widehat{A}, \widehat{C}$ to the hemispheres.

(i) Let the condition

$$\overline{a}(\mathbb{R}) \subset A \text{ and } c(\mathbb{R}) \subset C \tag{3.6}$$

holds. We show that the sets $W = [0,T] \times \left(\widehat{l} \cup \widehat{C}\right) \subset [0,T] \times \operatorname{cl}\mathbb{H}$, $Z = [0,T] \times \left(\widehat{l} \cup \widehat{A}\right) \subset [0,T] \times \operatorname{cl}\mathbb{H}$ are T-periodic isolating segments for the process φ generated in $\operatorname{cl}\mathbb{H}$ by the equation (1.1).

We calculate the inner product in $\mathbb{R} \times \mathbb{H}$ of the vector field $(1, v(t,q))^T$ and an outward normal vector $n(t,q)$ to the set $\mathbb{R} \times A$ in every point of $\mathbb{R} \times l$.

It is easy to see that $n(t,q) = (0,-1)^T$. Thus for $(t,q) \in \mathbb{R} \times l$ by (2.6) and (2.7) we get

$$\begin{aligned}\left\langle n(t,q), (1, v(t,q))^T \right\rangle &= -\mathfrak{Re}[qa(t)q + b(t)q + qd(t) + c(t)] \\ &= -\mathfrak{Re}[q^2 a(t)] - b(t)\mathfrak{Re}[q] - d(t)\mathfrak{Re}[q] - \mathfrak{Re}[c(t)] \\ &= |\mathfrak{Im}[q]|^2 \mathfrak{Re}[a(t)] - \mathfrak{Re}[c(t)] \\ &\geq -\mathfrak{Re}[c(t)] \\ &> 0.\end{aligned} \tag{3.7}$$

Finally the vector field $(1,v)^T$ is transversal to $\mathbb{R} \times l$ in every point of $\mathbb{R} \times l$ and it points towards $\mathbb{R} \times C$. By the change of variables (2.8) the same is true in the point ∞. Thus the sets W, Z are isolating segments such that $W^{--} = \emptyset$, $W^{++} = [0,T] \times \widehat{l}$, $Z^{++} = \emptyset$ and $Z^{--} = [0,T] \times \widehat{l}$. It follows that

$$\varphi_{(0,T)}\left(\widehat{l} \cup \widehat{C}\right) \subsetneq \widehat{C} \text{ and } \varphi_{(0,-T)}\left(\widehat{l} \cup \widehat{A}\right) \subsetneq \widehat{A}. \tag{3.8}$$

(ii) Let now conditions (3.1), (3.2) and (3.3) hold. The sets W and Z may not be isolating segments because some trajectories during time intervals of positive but less than T length can be contained in the set $[0,T] \times \widehat{l}$. But the crucial inclusions (3.8) still hold.

By the Brouwer fixed point theorem there exist at least one T-periodic solution in both the sets A and C. By (3.8) every solution after entering the set $[0,T] \times \widehat{l}$ enters the set $\widehat{C}$ in time shorter than T. Thus there are no periodic solutions contained in l. By (3.1) and (3.2) every solution entering $\{\infty\}$ has to enter the half-space C in time shorter than T. Thus there are no b.f.b. solutions. ☐

Example 3.2. By Theorem 3.1 the equation

$$\dot{q} = q[j + 2\sin(t)]q + j - \sin(t)$$

has at least two 2π-periodic solutions. Here $l = \{q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} : q_2 = 0\}$.

Remark 3.3. The condition (3.1) can not be weaken to the form $a \not\equiv 0 \not\equiv c$ as shown in Example 3.4.

Example 3.4. The equation $\dot{q} = qa(t)q + c(t)$ where a, c are 2π-periodic and given by

$$a(t) = \begin{cases} 0, & \text{for } t \in [0,\pi], \\ -\sin(t), & \text{for } t \in [\pi, 2\pi], \end{cases} \qquad c(t) = \begin{cases} 2i\sin(2t), & \text{for } t \in [0,\pi], \\ 0, & \text{for } t \in [\pi, 2\pi] \end{cases}$$

with $l = \{q \in \mathbb{H} : \mathfrak{Re}(q) = 0\}$ fulfills all assumptions of Theorem 3.1 except (3.1). There exists 2π-periodic solution η contained in l and given by

$$\eta(t) = \begin{cases} -2i\cos(2t) + 2i, & \text{for } t \in [0,\pi], \\ 0, & \text{for } t \in [\pi, 2\pi]. \end{cases}$$

Remark 3.5. In the condition (3.2) the sum $A \cup \{0\}$ can not be replaced by $A \cup l$. The examples from [3, 7, 13] of the complex-valued equations

$$\dot{z} = z^2 + b(t)z + c(t)$$

without periodic solutions give examples of $b, c \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ which for the quaternionic-valued equation

$$\dot{q} = q^2 + b(t)q + c(t)$$

has no periodic solutions (cf. [2, Corollary 7.2]). In all the cases $l = \{q = q_0 + q_1 i + q_2 j + q_3 k \in \mathbb{H} : q_1 = 0\} \supset \mathbb{R}$.

Remark 3.6. When in assumptions of Theorem 3.1 $l = \{q \in \mathbb{H} : \mathfrak{Re}[q] = 0\}$, $A = \{q \in \mathbb{H} : \mathfrak{Re}[q] > 0\}$ and $C = \{q \in \mathbb{H} : \mathfrak{Re}[q] < 0\}$ hold then it is possible to allow $b, d \in \mathcal{C}(\mathbb{R}, \mathbb{H})$. In that case we need to assume that the following condition

$$\mathfrak{Im}[b] = -\mathfrak{Im}[d]. \tag{3.9}$$

is satisfied. Indeed, in (3.7) we get

$$-\mathfrak{Re}[b(t)q + qd(t)] = -\mathfrak{Re}[(b+d)(t)q] = -\mathfrak{Re}[(b+d)(t)]\mathfrak{Re}[q] = 0$$

which does not make the slightest difference in the final inequality.

The form of l, A and C is crucial to avoid the change of variables (3.4) because the term $b(t)q + qd(t)$ in (3.5) is equal to $\alpha b(t)\overline{\alpha}p + p\overline{\beta}d(t)\beta$ where $\mathfrak{Im}\left[\alpha b(t)\overline{\alpha}\right]$ and $\mathfrak{Im}\left[\overline{\beta}d(t)\beta\right]$ are the vectors $\mathfrak{v}_{b(t)}$ and $\mathfrak{v}_{d(t)}$, respectively, rotated in $\mathbb{R}^3$ which usually destroys the crucial equality (3.9).

Example 3.7. By Remark 3.6 and Theorem 3.1 the equation

$$\dot{q} = q\left(1 + e^{it}\right)q + j\cos(t)q - qj\cos(t) - 1$$

has at least two 2π-periodic solutions.

3.2. **Full description of dynamics.** We make a general assumption in the current subsection that $\mathrm{Ark}[0] = 0$.

Now we present an improvement of Theorem 3.1.

Theorem 3.8. *Let the coefficients $a, c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$, $b, d \in \mathcal{C}(\mathbb{R}, \mathbb{R})$ be T-periodic and* $\mathrm{Ark}(0) = 0$. *If the conditions* (3.1) *and*

$$\mathrm{Ark}[a] < \frac{\pi}{4}, \tag{3.10}$$

$$\max_{t\in\mathbb{R}} \mathrm{Ark}[a(t)] + \max_{t\in\mathbb{R}} \mathrm{Ark}[-c(t)] \leq \frac{\pi}{2}, \tag{3.11}$$

$$\mathfrak{Re}[b + d] \geq 0 \tag{3.12}$$

hold then the equation (1.1) *has exactly two T-periodic solutions ξ, η in $\mathbb{H}$. Moreover, $\mathfrak{Re}[\eta] > 0$ and η is asymptotically unstable while $\mathfrak{Re}[\xi] < 0$ and ξ is asymptotically stable and every other solution in $\mathcal{S}^4$ is heteroclinic to them. Every nonperiodic solution starting in $\mathcal{S}\left(\frac{\pi}{2}\right)$ is f.b. or enters $-\mathcal{S}\left(\frac{\pi}{2}\right)$. Every solution starting in $-\mathcal{S}\left(\frac{\pi}{2}\right)$ stays on for all positive times. There are no b.f.b. solutions.*

Proof. Let φ be the process generated by (1.1) and $\alpha_1 = \frac{\pi}{2} - \max_{t\in\mathbb{R}} \operatorname{Ark}[a(t)]$. We define $W = \operatorname{cl}\mathcal{S}(\alpha_1) \subset \operatorname{cl}\mathbb{H}$ and $Z = -W$. Obviously $\{0,\infty\} \subset W \cap Z$. Our goal is to prove that the following inclusions

$$\varphi_{(0,-T)}(W) \subset \operatorname{int} W, \tag{3.13}$$

$$\varphi_{(0,T)}(Z) \subset \operatorname{int} Z \tag{3.14}$$

hold. Moreover, there exist unique T-periodic solutions in both the sets W and Z which are asymptotically unstable or asymptotically stable, respectively. Finally we prove that every other solution is heteroclinic to the periodic ones.

(i) Let $\max_{t\in\mathbb{R}} \operatorname{Ark}[a(t)] > 0$ then it is easy to see that

$$|v_{c(t)}| \leq - \mathfrak{s}_{c(t)} \tan(\alpha_1), \tag{3.15}$$

$$|v_{a(t)}| \leq \mathfrak{s}_{a(t)} \tan\left(\frac{\pi}{2} - \alpha_1\right) \tag{3.16}$$

hold.

Let us assume that $\mathfrak{Re}(a) > 0$ and $\mathfrak{Re}(c) < 0$. We show that the vector field v points outward the set W in every point of ∂W. Let $q \in \partial W \setminus \{0,\infty\}$. An outward orthogonal vector to W in q has the form

$$n(q) = -|\mathfrak{v}_q| + \mathfrak{s}_q \frac{\mathfrak{Im}(q)}{|\mathfrak{v}_q|}. \tag{3.17}$$

Using (2.7) we estimate the inner product of $n(q)$ and the vector field by

$$\begin{aligned}
\langle v(t,q), n(q)\rangle =& \mathfrak{Re}\left\{[qa(t)q + b(t)q + qd(t) + c(t)]\overline{n}(q)\right\} \\
=& \mathfrak{Re}\left\{a(t)q\overline{n}(q)q\right\} + \mathfrak{Re}\left\{b(t)q\overline{n}(q) + \overline{n}(q)qd(t)\right\} + \mathfrak{Re}\left\{c(t)\overline{n}(q)\right\} \\
=& - \mathfrak{Re}\left\{a(t)\left[\frac{\mathfrak{s}_q^2}{|\mathfrak{v}_q|} + |\mathfrak{v}_q|\right]\mathfrak{Im}[q]q\right\} + [b(t)+d(t)]\mathfrak{Re}\{q\overline{n}(q)\} \\
& - \mathfrak{s}_{c(t)}|\mathfrak{v}_q| + \frac{\mathfrak{s}_q}{|\mathfrak{v}_q|}\mathfrak{v}_{c(t)} \cdot \mathfrak{v}_q \\
\geq& \frac{|q|^2}{|\mathfrak{v}_q|}\mathfrak{Re}\left\{a(t)\left[|\mathfrak{v}_q|^2 - \mathfrak{s}_q\mathfrak{Im}(q)\right]\right\} - \mathfrak{s}_{c(t)}|\mathfrak{v}_q| - \mathfrak{s}_q|\mathfrak{v}_{c(t)}| \qquad (3.18)\\
\geq& \frac{|q|^2}{|\mathfrak{v}_q|}\left[\mathfrak{s}_{a(t)}|\mathfrak{v}_q|^2 + \mathfrak{s}_q\mathfrak{v}_{a(t)} \cdot \mathfrak{v}_q\right] \\
& - \mathfrak{s}_{c(t)}\mathfrak{s}_q \tan(\alpha_1) + \mathfrak{s}_q\mathfrak{s}_{c(t)} \tan(\alpha_1) \\
\geq& |q|^2\left[\mathfrak{s}_{a(t)}|\mathfrak{v}_q| - \mathfrak{s}_q|\mathfrak{v}_{a(t)}|\right] \\
\geq& |q|^2\left[\mathfrak{s}_{a(t)}\mathfrak{s}_q \tan(\alpha_1) - \mathfrak{s}_q\mathfrak{s}_{a(t)} \tan\left(\frac{\pi}{2} - \alpha_1\right)\right] \\
>& 0.
\end{aligned}$$

Obviously, the vector field points outward W in the origin and, via the change of coordinates (2.8), in ∞. Thus (3.13) holds.

Let now $a(t) = 0$ and $c(t) = 0$ be possible. Thus in the direct estimation (3.18) we get only $\langle v(t,q), n(q)\rangle \geq 0$ but by (3.1) none trajectory starting in ∂W can stay there in the whole time interval $[0,T]$ thus (3.13) holds.

By the Brouwer fixed point theorem there exist a T-periodic solution η inside $\operatorname{int} W$. Now we prove that η is the unique T-periodic solution in W and it is asymptotically unstable and repelling in W. To do that we show that η is asymptotically stable and contracting in W for the equation (1.1) with reversed time

$$\dot{q} = -qa(-t)q - b(-t)q - qd(-t) - c(-t). \tag{3.19}$$

We show that Poincaré map of (3.19) is contracting in W, namely for all solutions χ, ζ such that $\chi(0), \zeta(0) \in W$ and $\chi \neq \zeta$ the inequality

$$\frac{d}{dt}|\zeta(t) - \chi(t)|^2 \leq 0$$

holds for every $t \in [0, T]$ and for some $t \in [0, T]$ is strict. It is enough to show that the inequality $\langle \zeta(t) - \chi(t), \zeta'(t) - \chi'(t) \rangle \leq 0$ holds for every $t \in [0, T]$ and is strict for some $t \in [0, T]$. Thus by (2.7)

$$\begin{aligned}
\langle \zeta - \chi, \zeta' - \chi' \rangle =& \mathfrak{Re}\left[(\overline{\zeta} - \overline{\chi}) \left(-\zeta a \zeta - b\zeta - \zeta d + \chi a \chi + b\chi + \chi d \right) \right] \\
=& \mathfrak{Re}\left[-(\overline{\zeta} - \overline{\chi}) \left(\zeta a \zeta - \chi a \chi \right) \right] - \mathfrak{Re}\left[(\overline{\zeta} - \overline{\chi}) [b(\zeta - \chi) + (\zeta - \chi) d] \right] \\
=& - \mathfrak{Re}\left[(\overline{\zeta} - \overline{\chi}) \left(\zeta a \zeta - \zeta a \chi + \zeta a \chi - \chi a \chi \right) \right] - |\zeta - \chi|^2 \mathfrak{Re}[b + d] \\
\leq& - \mathfrak{Re}\left[(\overline{\zeta} - \overline{\chi}) \left(\zeta a (\zeta - \chi) + (\zeta - \chi) a \chi \right) \right] \\
=& - |\zeta - \chi|^2 \mathfrak{Re}[a(\zeta + \chi)] \\
\leq& 0
\end{aligned} \tag{3.20}$$

holds. This follows by

$$\begin{aligned}
\mathfrak{Re}[a(\zeta + \chi)] =& \mathfrak{s}_a \mathfrak{s}_{\zeta + \chi} - \mathfrak{v}_a \cdot \mathfrak{v}_{\zeta + \chi} \\
\geq& \mathfrak{s}_a \mathfrak{s}_{\zeta + \chi} - \mathfrak{s}_a \tan\left(\frac{\pi}{2} - \alpha_1 \right) \mathfrak{s}_{\zeta + \chi} \tan(\alpha_1) \\
=& 0,
\end{aligned}$$

where $|\mathfrak{v}_{\zeta(t) + \chi(t)}| \leq \mathfrak{s}_{\zeta(t) + \chi(t)} \tan(\alpha_1)$ for all $t > 0$ but for t when $\zeta(t) + \chi(t) \in \operatorname{int} W$ and $a(t) \neq 0$ the inequalities are strict.

Now we apply Lemma 2.1 to the W and $\varphi_{(0,-T)}$ which implies the uniqueness of the periodic solution η inside W. Moreover, η is asymptotically unstable and repelling in W.

(ii) Let $\max_{t \in \mathbb{R}} \operatorname{Ark}[a(t)] = 0$. The case is much simpler then the previous one. Details are left to the reader.

By the estimation similar to (3.18) one can prove (3.14). Moreover, by calculations analogous to (3.20) one can use Lemma 2.1 and prove that there exists exactly one periodic solution ξ inside Z. It is asymptotically stable and attracting in Z.

Now we show that every solution χ such that $\eta \neq \chi \neq \xi$ is heteroclinic from η to ξ. It is enough to show that for every $\alpha \in (\alpha_1, \pi - \alpha_1)$ and $t \in [0, T]$ such that $a(t) \neq 0$ the vector field v in every point of the set $\partial \mathcal{S}(\alpha) \setminus \{0, \infty\}$ points outward $\mathcal{S}(\alpha)$.

Let us fix $\alpha \in (\alpha_1, \pi - \alpha_1)$ and $q \in \partial \mathcal{S}(\alpha) \setminus \{0, \infty\}$. An outward orthogonal vector to $\mathcal{S}(\alpha)$ in q is given by (3.17). Repeating (3.18) we get

$$\begin{aligned}
\langle v(t, q), n(q) \rangle \geq& |q|^2 \left[\mathfrak{s}_{a(t)} |\mathfrak{v}_q| - |\mathfrak{s}_q| |\mathfrak{v}_{a(t)}| \right] - \mathfrak{s}_{c(t)} |\mathfrak{v}_q| - |\mathfrak{s}_q| |\mathfrak{v}_{c(t)}| \\
=& (\star).
\end{aligned}$$

When $\alpha = \frac{\pi}{2}$ then $\mathfrak{s}_q = 0$ and $(\star) > 0$ provided $(ac)(t) \neq 0$. In the other case $|\mathfrak{v}_q| = |\mathfrak{s}_q \tan(\alpha)|$ holds. Thus by (3.15) and (3.16) we get

$$\begin{aligned}(\star) \geq& |q|^2 \left[\mathfrak{s}_{a(t)} \left|\mathfrak{s}_q \tan(\alpha)\right| - |\mathfrak{s}_q| \mathfrak{s}_{a(t)} \tan\left(\frac{\pi}{2} - \alpha_1\right)\right] \\ &- \mathfrak{s}_{c(t)} \left|\mathfrak{s}_q \tan(\alpha)\right| + |\mathfrak{s}_q| \mathfrak{s}_{c(t)} \tan(\alpha_1) \\ >& 0,\end{aligned}$$

provided $(ac)(t) \neq 0$. Finally every nonperiodic solution is heteroclinic.

When $\alpha = \frac{\pi}{2}$ then $\partial W = \widehat{l}$ where $l = \{q \in \mathbb{H} : \mathfrak{Re}[q] = 0\}$. Thus the nonexistence of b.f.b. solutions follows by the same argument as used in the proof of Theorem 3.1.

□

Example 3.9. By Theorem 3.8 the equation

$$\dot{q} = q(2 + k)q - 2 + e^{jt}$$

has exactly two 2π-periodic solutions. Here $\mathrm{Ark}[a] < \frac{\pi}{6}$ and $\mathrm{Ark}[-c] \leq \frac{\pi}{6}$.

Remark 3.10. It is possible to allow $b, d \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ in Theorem 3.8 but it needs to assume (3.9) (cf. Remark 3.6). The calculation (3.18) is valid due to the equality $q\overline{n}(q) = \overline{n}(q)q$.

Example 3.11. By Remark 3.10 and Theorem 3.8 the equation

$$\dot{q} = q^2 - e^{jt}q + qe^{jt} - 1$$

has exactly two 2π-periodic solutions.

Remark 3.12. The real axis is distinguished in formulation of Theorem 3.8. It is done to simplify the statement of the theorem. It can be omitted by an appropriate change of variables (cf. Example 3.13 and 3.14).

Example 3.13. It is impossible to apply directly Theorem 3.8 to the equation

$$\dot{q} = qjq + 2j - je^{kt}$$

because $\mathrm{Ark}[a] \equiv \frac{\pi}{2}$. But after the change of variables $p = jq$ we get

$$\dot{p} = p^2 - 2 + e^{kt}.$$

Now by Theorem 3.8 we get exactly two 2π-periodic solution of both equations.

Example 3.14. Let us consider the equation

$$\dot{q} = -q^2 + c(t)$$

which appears in the Euler vorticity dynamics (cf. [11]). By the change of variables $p = -q$ we get

$$\dot{p} = p^2 - c(t).$$

Now, by Theorem 3.8 if $c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ is T-periodic then both equations has exactly two T-periodic solutions provided $c \not\equiv 0$ and $\mathrm{Ark}[c] \leq \frac{\pi}{2}$.

3.3. **Simple zeros of the vector field.** Let us assume that the equation (1.1) has the following form

$$\dot{q} = u(t,z) = [q-\xi(t)]a(t)[q-\eta(t)] + c(t). \tag{3.21}$$

If $c \equiv 0$ then $\xi, \eta : \mathbb{R} \longrightarrow \mathbb{H}$ are the branches of simple zeros of the vector field otherwise c is treated as perturbation.

Remark 3.15. We investigate the vector field u instead of v because the quaternionic polynomial of order two can have more than two zeros (e.g. q^2+1). Thus finding a "nice" branches of simple zeros could be difficult (cf. [10]).

We state the main theorem in the present subsection.

Theorem 3.16. *Let $a, c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ and $\xi, \eta \in \mathcal{C}^1(\mathbb{R}, \mathbb{H})$ are T-periodic. If there exist constants $E, F > 0$, $\kappa, l \in \mathbb{R}$ such that the inequalities*

(3.22)
$$\mathfrak{Re}[a(t)(\eta(t)-\xi(t))] > E|a(t)||\eta(t)|^\kappa + \left(\frac{|\kappa|}{|\eta(t)|} + \frac{1}{E|\eta(t)|^\kappa}\right)|\eta'(t)| + \frac{|c(t)|}{E|\eta(t)|^\kappa},$$

(3.23)
$$\mathfrak{Re}[a(t)(\eta(t)-\xi(t))] > F|a(t)||\xi(t)|^l + \left(\frac{|l|}{|\xi(t)|} + \frac{1}{F|\xi(t)|^l}\right)|\xi'(t)| + \frac{|c(t)|}{F|\xi(t)|^l}$$

hold for every $t \in \mathbb{R}$ then the equation (3.21) *has at least two T-periodic solutions in $\mathbb{H}$. If one of the inequalities* (3.22), (3.23) *holds then the equation* (3.21) *has at least one T-periodic solution in $\mathbb{H}$.*

Proof. We write $K = \{q \in \mathbb{H} : |q| \leq 1\}$ and denote by φ the process generated by (3.21).

Let us assume that (3.22) holds. Our goal is to construct a T-periodic isolating segment $W \subset [0,T] \times \mathbb{H}$ for φ homeomorphic to the cylinder $[0,T] \times K$ such that the vector field $(1,u)^T$ points outward W in every point of its side.

We define segment

$$W = \{(t,q) \in [0,T] \times \mathbb{H} : |q - \eta(t)| \leq M(t)\}$$

and homeomorphism $s : [0,T] \times K \longrightarrow W \subset [0,T] \times \mathbb{H}$ by

$$s(t,q) = (t, \eta(t) + M(t)q)$$

where $M \in \mathcal{C}^1(\mathbb{R}, (0,\infty))$ is T-periodic. It is easy to see that $s([0,T] \times \partial K)$ is the side of W and an outward orthogonal vector to W in the point $s(t,q)$ has the form

$$n(t,q) = (-\mathfrak{Re}[\eta'(t)\overline{q} + M'(t)], q)^T.$$

We estimate the inner product of $(1,u)^T$ and n in every point of the side W by

$$\begin{aligned}
\left\langle n(t,q), (1, (u \circ s)(t,q))^T \right\rangle = &- \mathfrak{Re}[\eta'(t)\overline{q} + M'(t)] \\
&+ \mathfrak{Re}\left\{\overline{q}\,[\eta(t) - \xi(t) + M(t)q]\,a(t)M(t)q\right\} + \mathfrak{Re}\,[\overline{q}c(t)] \\
\geq &- |\eta'(t)| - |M'(t)| + M(t)\mathfrak{Re}[a(t)(\eta(t)-\xi(t))] \\
&- M^2(t)|a(t)| - |c(t)| \\
= &(\star).
\end{aligned}$$

Thus if $(\star) > 0$ then the vector field points outward W. Let $M(t) = E|\eta(t)|^\kappa$. Thus using

$$|M'| = \left|(E|\eta|^\kappa)'\right| = \left|E\kappa|\eta|^{\kappa-2}\mathfrak{Re}[\eta'\overline{\eta}]\right| \leq E|\kappa||\eta|^{\kappa-1}|\eta'|$$

the inequality $(\star) > 0$ follows by (3.22). T-periodicity of η and M implies that W is a T-periodic isolating segment. Moreover $\varphi_{(0,-T)}(W_0) \subset W_0$ thus by the Brouwer fixed point theorem there exists at least one T-periodic solution inside W.

Let now (3.23) holds. We write $M(t) = F|\eta(t)|^l$. Thus by the argument analogous to above the set

$$Z = \{(t,q) \in [0,T] \times \mathbb{H} : |q - \xi(t)| \leq M(t)\}$$

is a T-periodic isolating segment such that the vector field $(1,u)^T$ points inward Z in every point of its side. Thus by the Brouwer fixed point theorem there exists at least one T-periodic solution inside Z.

Let both (3.22) (3.23) hold. There exist periodic solutions inside W and Z. To finish the proof it is enough to show that at least two of them are different. Let $\widehat{W} = W \setminus Z$. By (3.22) we get $\xi(t) \neq \eta(t)$ for every $t \in \mathbb{R}$ thus $\widehat{W}_t \neq \emptyset$ for every $t \in \mathbb{R}$. Moreover, the vector field points outward $\widehat{W}$ in every point of it side thus it is an isolating segment and by the Brouwer fixed point theorem there exist a periodic solution α inside $\widehat{W}$. As above there exists a periodic solution β inside Z so $\alpha \neq \beta$. $\square$

We present an improvement of Theorem 3.16.

Theorem 3.17. *Let assumptions of Theorem 3.16 be fulfilled. If in addition the inequalities*

$$\mathfrak{Re}[a(t)(\eta(t) - \xi(t)] > 2E|a(t)||\eta(t)|^\kappa, \tag{3.24}$$

$$\mathfrak{Re}[a(t)(\eta(t) - \xi(t)] > 2F|a(t)||\xi(t)|^l \tag{3.25}$$

hold for every $t \in \mathbb{R}$ then the equation (3.21) *has at least two T-periodic solutions in $\mathbb{H}$. One of them is asymptotically stable and another one is asymptotically unstable.*

If inequalities (3.22) *and* (3.24) *hold thus there exists at least one T-periodic asymptotically unstable solution in $\mathbb{H}$. If* (3.23) *and* (3.25) *hold then there exists at least one T-periodic asymptotically stable solution in $\mathbb{H}$.*

Proof. Let us assume that (3.23) and (3.25) hold. Let φ be the process generated by (3.21) and Z be an isolating segment as in the proof of Theorem 3.16. Our goal is to apply Lemma 2.1 to the set Z_0 and map $\varphi_{(0,T)}|_{Z_0}$. Let χ and ζ be different solutions of (3.21) such that $\chi(0), \zeta(0) \in Z_0$. Thus as in the proof of Theorem 3.8 it is enough to show that $\langle \zeta - \chi, \zeta' - \chi' \rangle < 0$. The vector field points inward Z in every point of the side of Z so $|\zeta(t) - \xi(t)| \leq F|\xi(t)|^l$ and $|\chi(t) - \xi(t)| \leq F|\xi(t)|^l$

for every $t \in [0, T]$. Finally we get

$$\begin{aligned}\langle \zeta - \chi, \zeta' - \chi' \rangle =& \mathfrak{Re}\left\{\overline{\zeta - \chi}[(\zeta - \xi)a(\zeta - \eta) - (\chi - \xi)a(\chi - \eta)]\right\}\\ =& \mathfrak{Re}\left\{\overline{\zeta - \chi}[(\zeta - \xi)a(\zeta - \eta) - (\zeta - \xi)a(\chi - \eta)]\right\}\\ &+ \mathfrak{Re}\left\{\overline{\zeta - \chi}[(\zeta - \xi)a(\chi - \eta) - (\chi - \xi)a(\chi - \eta)]\right\}\\ =& \mathfrak{Re}\left\{\overline{\zeta - \chi}[(\zeta - \xi)a(\zeta - \chi) + (\zeta - \chi)a(\chi - \eta)]\right\}\\ =& |\zeta - \chi|^2 \mathfrak{Re}\left\{a[\zeta - \xi + \chi - \xi + \xi - \eta]\right\}\\ \le& |\zeta - \chi|^2 \left\{\mathfrak{Re}[a(\xi - \eta)] + 2F|a||\xi(t)|^l\right\}\\ <& 0\end{aligned}$$

by (3.25). Thus the T-periodic solution inside Z is asymptotically stable.

The technical details of the proof when inequalities (3.22) and (3.24) hold are similar to the above. □

Example 3.18. By Theorems 3.16 and 3.17 the equation

$$\dot{q} = \left[q - e^{jt}\right] i \left[q + 10i + e^{kt}\right] + 1$$

has at least two 2π-periodic solutions in $\mathbb{H}$. One of them is asymptotically stable and another one is asymptotically unstable. It is enough to take $E = F = 1$ and $\kappa = l = 0$.

Remark 3.19. For a locally Lipschitz mapping $f : \mathbb{R} \longrightarrow \mathbb{H}$ and an open set $U \subset \mathbb{R}$ let $L(U, f)$ be a Lipschitz constant of f in the set U. We write

$$L_f(t) = \inf\{L(U, f) : U \text{ is a neighborhood of } t\}.$$

Obviously, if $f \in \mathcal{C}^1(\mathbb{R}, \mathbb{C})$ then $L_f(t) = |f'(t)|$.

It is easy to see (cf. [17, 16]) that it is enough $\xi, \eta \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ be locally Lipschitz in Theorems 3.16 and 3.17. In this case in (3.22) and (3.23) terms $|\eta'(t)|$ and $|\xi'(t)|$ should be replaced by $L_\eta(t)$ and $L_\xi(t)$, respectively.

We carry over a concept of the critical line condition from the complex case (cf. [17, 16]). The present formulation is quite different from the complex one which is due to the special form of the equation (3.21).

Definition 3.20. We call the set $\{q \in \mathbb{H} : \mathfrak{Re}[q] = 0\}$ the critical hyperplane. The equation (3.21) fulfills the critical hyperplane condition if the inequality

$$\mathfrak{Re}[a(t)(\eta(t) - \xi(t))] > 0 \tag{3.26}$$

holds for every $t \in \mathbb{R}$.

By comparing the growth rates of the both sides of the inequalities (3.22), (3.23), (3.24) and (3.25) one can prove the following corollaries.

Corollary 3.21. *Let $a, c, \xi, \eta \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ be T-periodic and ξ, η be Lipschitz. If the equation* (3.21) *fulfills the critical hyperplane condition* (3.26) *then every equation*

$$\dot{q} = [q - \xi(t)]Ra(t)[q - \eta(t)] + c(t), \tag{3.27}$$

$$\dot{q} = [q - R\xi(t)]a(t)[q - R\eta(t)] + c(t), \tag{3.28}$$

$$\dot{q} = [q - R\xi(t)]Ra(t)[q - R\eta(t)] + c(t) \tag{3.29}$$

has at least two T-periodic solutions in $\mathbb{H}$ provided $R \in \mathbb{R}$ is big enough. One of them is asymptotically stable and another one is asymptotically unstable.

Proof. Let $B = \min\{\mathfrak{Re}[a(t)(\eta(t) - \xi(t))] : t \in \mathbb{R}\} > 0$, $A = \max\{|a(t)| : t \in \mathbb{R}\}$, $C = \max\{|c(t)| : t \in \mathbb{R}\}$, $D = \max\{|\eta'(t)| : t \in \mathbb{R}\}$. We set $\kappa = 0$.

We fix E such small that $B - 2EA > 0$. Then inequalities (3.22) and (3.24) for (3.27) are satisfied provided $R > \frac{C+D}{E(B-EA)}$.

Now we fix E such big that $B - \frac{D}{E} > 0$ Then inequalities (3.22) and (3.24) for (3.28) are satisfied provided $R > \max\left\{\frac{E^2A+C}{EB-D}, \frac{2EA}{B}\right\}$.

Finally we fix $E > 0$. Then inequalities (3.22) and (3.24) for (3.28) are satisfied provided $R > \max\left\{\frac{EA}{B} + \frac{C+D}{EB}, 1, \frac{2EA}{B}\right\}$.

The proof that inequalities (3.23) and (3.25) hold is analogous. □

Example 3.22. By Theorems 3.16 and 3.17 the equation

$$\dot{q} = \left(q - ke^{jt}\right) R \left(q - 3 + ie^{kt}\right)$$

has at least two 2π-periodic solutions (one asymptotically stable and another one asymptotically unstable) provided $R > \frac{4}{9}$. Here $\xi = ke^{jt}$, $\eta = 3 - ie^{kt}$, $|\xi| = |\xi'| = |\eta'| \equiv 1$, $|\eta| \equiv \sqrt{10}$. We take $\kappa = l = 0$ and $E = F \approx \frac{3}{2}$.

Corollary 3.23. *Let $a, \xi, \eta \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ be 1-periodic and ξ, η be Lipschitz. If the equation* (3.21) *fulfills the critical hyperplane condition* (3.26) *then the equation*

$$\dot{q} = \left[q - \xi\left(\frac{t}{T}\right)\right] a\left(\frac{t}{T}\right)\left[q - \eta\left(\frac{t}{T}\right)\right]$$

has at least two T-periodic solutions in $\mathbb{H}$ provided T is big enough. One of them is asymptotically stable and another one is asymptotically unstable.

Proof. Let A, B, D be as in the proof of Corollary 3.21. We set $\kappa = 0$ and fix E such small that $B - 2EA > 0$. Then inequalities (3.22) and (3.24) are satisfied provided $R > \frac{D}{E(B-EA)}$. The proof that inequalities (3.23) and (3.25) hold is analogous. □

Example 3.24. By Theorems 3.16 and 3.17 the equation

$$\dot{q} = \left(q - ke^{\frac{jt}{T}}\right)\left(q - 3 + ie^{\frac{kt}{T}}\right)$$

has at least two $2\pi T$-periodic solution (one asymptotically stable and another one asymptotically unstable) provided $T > \frac{4}{9}$. Here $\xi = ke^{\frac{jt}{T}}$, $\eta = 3 - ie^{\frac{kt}{T}}$, $|\xi| \equiv 1$, $|\xi'| = |\eta'| \equiv \frac{1}{T}$, $|\eta| \equiv \sqrt{10}$. We take $\kappa = l = 0$ and $E = F \approx \frac{3}{2}$.

The method presented in Theorems 3.16 and 3.17 detects the existence of periodic solutions when $|\xi$, $|\eta|$, $|a|$ and T are big enough. Unfortunatelly it can fail in the case of small ones as shown in the following examples.

Example 3.25. By Theorem 3.8 and Remark 3.10 the equation

$$\begin{aligned}\dot{q} =& R\left[q^2 + \left(3 + ke^{it}\right)q - q\left(3 + ke^{it}\right) - \left(3 + ke^{it}\right)^2\right] \\ =& \left(q + 3 + ke^{it}\right) R \left(q - 3 - ke^{it}\right)\end{aligned}$$

has exactly two 2π-periodic solutions in $\mathbb{H}$ for every $R > 0$. Here $l = \{q \in \mathbb{H} : \mathfrak{Re}(q) = 0\}$. But Theorem 3.16 implies the existence of at least two periodic solutions only for $R > R_0$ where $R_0 > 0$. Here $\eta = -\xi = 3 + ke^{it}$. By setting $\kappa = l = 0$ and $E = F = 3$ we get $R > \frac{1}{9}$.

Example 3.26. By Theorem 3.8 and Remark 3.10 the equation

$$\begin{aligned}\dot{q} =& q^2 + \left(3 + ke^{\frac{it}{T}}\right) q - q\left(3 + ke^{\frac{it}{T}}\right) - \left(3 + ke^{\frac{it}{T}}\right)^2 \\ =& \left(q + 3 + ke^{\frac{it}{T}}\right)\left(q - 3 - ke^{\frac{it}{T}}\right)\end{aligned}$$

has exactly two $2\pi T$-periodic solutions in $\mathbb{H}$ for every $R > 0$. Here $l = \{q \in \mathbb{H} : \mathfrak{Re}(q) = 0\}$. But Theorem 3.16 implies the existence of at least two periodic solutions only for $T > T_0$ where $T_0 > 0$. Here $\eta = -\xi = 3 + ke^{\frac{it}{T}}$. By setting $\kappa = l = 0$ and $E = F = 3$ we get $T > \frac{1}{9}$.

3.4. **Double zeros of the vector field.** Let us assume that the equation (1.1) has the following form

$$\dot{q} = u(t,z) = [q - f(t)]a(t)\overline{f'}(t)[q - f(t)] + c(t). \tag{3.30}$$

When $c \equiv 0$ then $f : \mathbb{R} \longrightarrow \mathbb{H}$ is the branch of double zeros of the vector field, otherwise c is treated as perturbation.

Remark 3.27. The term $\overline{f'}(t)$ comes from the complex case (cf. [17, 16]).

We state the main theorem in the present subsection.

Theorem 3.28. *Let $a \in \mathcal{C}^1(\mathbb{R}, \mathbb{H} \setminus \{0\})$, $c \in \mathcal{C}(\mathbb{R}, \mathbb{H})$ and $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{H})$ be T-periodic. If there exist numbers $\alpha, \kappa \in \mathbb{R}$ such that the inequalities*

$$\operatorname{Ark}[a(t)] < \alpha \leq \frac{\pi}{6}, \tag{3.31}$$

$$|f'(t)|\sin(\alpha) > |\kappa||a(t)|^{\kappa-1}|a'(t)| + 2|a(t)|^{\kappa}\frac{|f''(t)|}{|f'(t)|} + |c(t)|, \tag{3.32}$$

$$\cos(\operatorname{Ark}[a(t)] + 2\alpha) > [1 + \sin(\alpha)]|a(t)|^{-2\kappa-1} \tag{3.33}$$

hold for every $t \in \mathbb{R}$ than the equation (3.30) *has at least two T-periodic solutions in $\mathbb{H}$. One of them is asymptotically unstable and another one is asymptotically stable. Moreover the equation has infinitely many solutions which are heteroclinic to the periodic ones.*

Proof. Let φ denote the process generated by the equation (3.30). Our goal is to construct W, Z two T-periodic isolating segments for φ and apply Lemma 2.1.

Throughout the current proof we assume $\operatorname{Ark}[0] = 0$ and write

$$\mathcal{S}^2 = \left\{q \in \mathbb{H} : q^2 = -1\right\}.$$

We define a map $s : \mathbb{R} \times [0,1] \times [0,1] \times \mathcal{S}^2 \longrightarrow \mathbb{R} \times \mathbb{H}$ by

$$s(t,x,y,I) = \left(t, f(t) + M(t)x\frac{f'(t)}{|f'(t)|}\left[\cos(\alpha) + y\sin(\alpha)I\right]\right),$$

where $M \in \mathcal{C}^1(\mathbb{R}, (0,\infty))$ is T-periodic. We set

$$W = s\left([0,T] \times [0,1] \times [0,1] \times \mathcal{S}^2\right).$$

It is easy to see that $W_0 = W_T$ and W_t is the set $\{q \in \mathbb{H} : \operatorname{Ark}[q] \leq \alpha, \mathfrak{Re}[q] \leq M(t)\}$ rotated by $\frac{f'(t)}{|f'(t)|}$ and shifted by $f(t)$. Let

$$\begin{aligned}K =& s\left([0,T] \times [0,1] \times \{1\} \times \mathcal{S}^2\right), \\ L =& s\left([0,T] \times \{1\} \times [0,1] \times \mathcal{S}^2\right).\end{aligned}$$

We prove that W is an isolating segment such that

$$W^{++} = \emptyset, \tag{3.34}$$

$$W^{--} = K \cup L. \tag{3.35}$$

First of all $\partial(W_t) = K_t \cup L_t$ thus to prove (3.34) and (3.35) it is enough to show that in every point of $K \cup L$ the vector field $(1, u)^T$ points outward the set W.

It is easy to see that an outward orthogonal vector to W in the point $s(t, x, 1, I) \in K$ is given by

$$n(t, x, 1, I) = \left(n_1(t, x, 1, I), \frac{f'(t)}{|f'(t)|} I e^{\alpha I} \right)^T \in \mathbb{R} \times \mathbb{H} \simeq \mathbb{R} \times \mathbb{R}^4,$$

where

$$n_1(t, x, 1, I) = \mathfrak{Re} \left[I e^{-\alpha I} |f'(t)| + M'(t) x I + M(t) x I \frac{\overline{f'}(t)}{|f'(t)|} \left(\frac{f'(t)}{|f'(t)|} \right)' \right].$$

The vector field u in the set K has the form

$$(u \circ s)(t, x, 1, I) = M(t) x \frac{f'(t)}{|f'(t)|} e^{\alpha I} a(t) \overline{f'}(t) M(t) x \frac{f'(t)}{|f'(t)|} e^{\alpha I} + c(t).$$

We estimate the inner product of n and vector field $(1, u)^T$ in every point of K by

$$\begin{aligned}
\langle n, (1, u \circ s)^T \rangle =& \mathfrak{Re} \left[I e^{-\alpha I} |f'| + M' x I + M x I \frac{\overline{f'}}{|f'|} \left(\frac{f'}{|f'|} \right)' \right] \\
&+ \mathfrak{Re} \left[\overline{\frac{f'}{|f'|} I e^{\alpha I}} \left\{ M x \frac{f'}{|f'|} e^{\alpha I} a \overline{f'} M x \frac{f'}{|f'|} e^{\alpha I} + c \right\} \right] \\
\geq& |f'| \mathfrak{Re} \left[I e^{-\alpha I} \right] - |M'| - M \left| \left(\frac{f'}{|f'|} \right)' \right| \\
&- \mathfrak{Re} \left[e^{-\alpha I} I \frac{\overline{f'}}{|f'|} M x \frac{f'}{|f'|} e^{\alpha I} a \overline{f'} M x \frac{f'}{|f'|} e^{\alpha I} \right] - |c| \\
\geq& |f'| \sin(\alpha) - |M'| - 2M \frac{|f''|}{|f'|} - M^2 x^2 |f'| \mathfrak{Re} \left[I e^{\alpha I} a \right] - |c| \\
=& (\star),
\end{aligned}$$

where

$$\left| \left(\frac{f'}{|f'|} \right)' \right| = \left| \frac{f''}{|f'|} - f' \frac{\mathfrak{Re} \left[f'' \overline{f'} \right]}{|f'|^3} \right| \leq 2 \frac{|f''|}{|f'|}. \tag{3.36}$$

It is enough to show that $(\star) > 0$. By (3.31) we get

$$-\mathfrak{Re} \left[I e^{\alpha I} a \right] = \mathfrak{Re} \left[e^{\left(-\frac{\pi}{2} + \alpha \right) I} a \right] \geq \mathfrak{Re} \left[e^{\left(-\frac{\pi}{2} + \alpha - \mathrm{Ark}[a] \right) I} \right] > 0,$$

so it is enough to show that

$$|f'| \sin(\alpha) > |M'| + 2M \frac{|f''|}{|f'|} + |c| \tag{3.37}$$

holds. But taking $M(t) = |a(t)|^\kappa$ one can get $|M'| = \left| \kappa |a|^{\kappa - 2} \mathfrak{Re}[\overline{a} a'] \right| \leq |\kappa| |a|^{\kappa - 1} |a'|$ and the inequality (3.37) follows by (3.32).

Now it is easy to see that an outward orthogonal vector to W in the point $s(t,1,y,I) \in L$ is given by

$$n(t,1,y,I) = \left(n_1(t,1,y,I), \frac{f'(t)}{|f'(t)|} \right)^T \in \mathbb{R} \times \mathbb{H} \simeq \mathbb{R} \times \mathbb{R}^4,$$

where

$$\begin{aligned} n_1(t,1,y,I) = &- \mathfrak{Re}\left\{|f'(t)|\right\} \\ &- \mathfrak{Re}\left\{ \left[M'(t) + M(t)\frac{\overline{f'}(t)}{|f'(t)|} \left(\frac{f'(t)}{|f'(t)|} \right)' \right] [\cos(\alpha) + y\sin(\alpha)I] \right\}. \end{aligned}$$

The vector field u in the set L has the form

$$\begin{aligned} (u \circ s)(t,1,y,I) = &M(t)\frac{f'(t)}{|f'(t)|}[\cos(\alpha) + y\sin(\alpha)I]a(t)\overline{f'}(t) \\ &\cdot M(t)\frac{f'(t)}{|f'(t)|}[\cos(\alpha) + y\sin(\alpha)I] + c(t). \end{aligned}$$

We estimate the inner product of n and the vector field $(1,u)^T$ in every point of L using (3.36) and (3.37) by

$$\begin{aligned} \langle n, (1, u\circ s)^T \rangle = &- \mathfrak{Re}\left\{|f'|\right\} - \mathfrak{Re}\left\{M'[\cos(\alpha) + y\sin(\alpha)I]\right\} \\ &- \mathfrak{Re}\left\{ M\frac{\overline{f'}}{|f'|} \left(\frac{f'}{|f'|} \right)' [\cos(\alpha) + y\sin(\alpha)I] \right\} \\ &+ \mathfrak{Re}\left[\frac{\overline{f'}}{|f'|} M \frac{f'}{|f'|}[\cos(\alpha) + y\sin(\alpha)I]a\overline{f'}M \right. \\ &\left. \cdot \frac{f'}{|f'|}[\cos(\alpha) + y\sin(\alpha)I] \right] + \mathfrak{Re}\left[\frac{\overline{f'}}{|f'|} c \right] \\ \geq &- |f'| - |M'| - 2M\frac{|f''|}{|f'|} + M^2|f'|\mathfrak{Re}\left\{ a[\cos(\alpha) + y\sin(\alpha)I]^2 \right\} - |c| \\ > &- |f'|[1 + \sin(\alpha)] + M^2|f'|\mathfrak{Re}\left\{ ae^{2\alpha I} \right\} \\ \geq &M^2|f'||a|\cos(\mathrm{Ark}[a] + 2\alpha) - |f'|[1 + \sin(\alpha)] \\ = &(\star\star) \end{aligned}$$

But by (3.33) the inequality $(\star\star) > 0$ holds. Finally (3.34) and (3.35) hold. Thus $\varphi_{(0,-T)}(W_0) \subset W_0$ and there exists at least one periodic solution of (3.30) inside W.

Now we define a map $\widehat{s} : \mathbb{R} \times [0,1] \times [0,1] \times \mathcal{S}^2 \longrightarrow \mathbb{R} \times \mathbb{H}$ by

$$\widehat{s}(t,x,y,I) = \left(t, f(t) - M(t)x\frac{f'(t)}{|f'(t)|} [\cos(\alpha) + y\sin(\alpha)I] \right),$$

where $M \in \mathcal{C}^1(\mathbb{R}, (0,\infty))$ is T-periodic. We set

$$Z = \widehat{s}\left([0,T] \times [0,1] \times [0,1] \times \mathcal{S}^2 \right).$$

Let

$$\begin{aligned} \widehat{K} = &\widehat{s}\left([0,T] \times [0,1] \times \{1\} \times \mathcal{S}^2 \right), \\ \widehat{L} = &\widehat{s}\left([0,T] \times \{1\} \times [0,1] \times \mathcal{S}^2 \right). \end{aligned}$$

We prove that Z is an isolating segment such that

$$
\begin{aligned}
Z^{++} =& \widehat{K} \cup \widehat{L}, \\
Z^{--} =& \emptyset.
\end{aligned}
$$

We do it similarly to the case of W. The only difference is that the vector field $(1, u)^T$ points inward the set Z in every point of $\widehat{K} \cup \widehat{L}$. Thus $\varphi_{(0,T)}(Z_0) \subset Z_0$ and there exists at least one periodic solution of (3.30) inside Z.

Now our goal is to prove that there is exactly one T-periodic solution inside Z and it is asymptotically stable. Moreover, we prove that there is exactly one T-periodic solution inside W and it is asymptotically unstable.

First we deal with Z. Our goal is to use Lemma 2.1 for $\varphi_{(0,T)}|_{Z_0}$ and Z_0. It is easy to see that Z_0 is convex. Let $\widehat{\xi}$, $\widehat{\eta}$ be distinct solutions of (3.30) such that $\widehat{\xi}(0) \in Z_0$ and $\widehat{\eta}(0) \in Z_0$. Then $\widehat{\xi}([0, T]) \subset Z$ and $\widehat{\eta}([0, T]) \subset Z$. As in the proof of Theorem 3.8 it is enough to show that

$$
\left\langle \widehat{\xi}(t) - \widehat{\eta}(t), \widehat{\xi}'(t) - \widehat{\eta}'(t) \right\rangle < 0
$$

holds for every $t \in (0, \infty)$. For $t \geq 0$ we can write $\widehat{\xi}(t) = f(t)+\xi(t)$, $\widehat{\eta}(t) = f(t)+\eta(t)$ where there exist functions $x_\xi, x_\eta : (0, \infty) \longrightarrow (0, 1)$, $y_\xi, y_\eta : (0, \infty) \longrightarrow [0, 1)$, $I_\xi, I_\eta : (0, \infty) \longrightarrow \mathcal{S}^2$ such that

$$
\begin{aligned}
\xi(t) =& - M(t) x_\xi(t) \frac{f'(t)}{|f'(t)|} [\cos(\alpha) + y_\xi(t) \sin(\alpha) I_\xi(t)], \\
\eta(t) =& - M(t) x_\eta(t) \frac{f'(t)}{|f'(t)|} [\cos(\alpha) + y_\eta(t) \sin(\alpha) I_\eta(t)].
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\langle \widehat{\xi} - \widehat{\eta}, \widehat{\xi}' - \widehat{\eta}' \right\rangle =& \left\langle \xi - \eta, \widehat{\xi}' - \widehat{\eta}' \right\rangle \\
=& \mathfrak{Re} \left\{ \overline{\xi - \eta} \left[\xi a \overline{f'} \xi + c - \eta a \overline{f'} \eta - c \right] \right\} \\
=& \mathfrak{Re} \left\{ \overline{\xi - \eta} \left[\xi a \overline{f'} \xi - \xi a \overline{f'} \eta + \xi a \overline{f'} \eta - \eta a \overline{f'} \eta \right] \right\} \\
=& \mathfrak{Re} \left\{ \overline{\xi - \eta} \left[\xi a \overline{f'} (\xi - \eta) + (\xi - \eta) a \overline{f'} \eta \right] \right\} \\
=& |\xi - \eta|^2 \mathfrak{Re} \left\{ a \overline{f'} (\xi + \eta) \right\}
\end{aligned}
$$

$$
\begin{aligned}
=& - |\xi - \eta|^2 |f'| M \Big\{ x_\xi \mathfrak{Re} \left\{ a[\cos(\alpha) + y_\xi \sin(\alpha) I_\xi] \right\} \\
& + x_\eta \mathfrak{Re} \left\{ a[\cos(\alpha) + y_\eta \sin(\alpha) I_\eta] \right\} \Big\} \\
<& 0
\end{aligned}
$$

because by (3.31) one gets

$$
\begin{aligned}
\mathfrak{Re} \left\{ a[\cos(\alpha) + y_\xi \sin(\alpha) I_\xi] \right\} \geq& \mathfrak{Re} \left\{ a[\cos(\alpha) + \sin(\alpha) I_\xi] \right\} \\
\geq& |a| \cos(\mathrm{Ark}[a] + \alpha) \\
>& 0.
\end{aligned}
$$

Thus by Lemma 2.1 there exists exactly one T-periodic solution ζ inside Z. It is asymptotically stable and attracting in Z.

In similar way we apply Lemma 2.1 to W and prove that there exist exactly one T-periodic solution λ inside W. It is asymptotically unstable and repelling in W.

It is easy to see that $W \cap Z = \{(t,q) \in [0,T] \times \mathbb{H} : q = f(t)\}$. Thus every solution ξ of (3.30) such that $\xi(t) \in W \cap Z$ is heteroclinic from λ to ζ. □

Example 3.29. By Theorem 3.28 the equation

$$\dot{q} = \left[q - 4e^{kt}\right]\left(-48ke^{-kt}\right)\left[q - 4e^{kt}\right] + j$$

has at least two T-periodic solutions in $\mathbb{H}$ and infinitely many solutions which are heteroclinic to them. Here $f(t) = 4e^{kt}$, $a \equiv 16$, $\alpha = \frac{\pi}{6}$ and $\kappa = -\frac{3}{10}$.

Remark 3.30. Theorem 3.28 holds also for the equation

$$\dot{q} = [q - f(t)]\overline{f'}(t)a(t)[q - f(t)] + c(t).$$

The proof is almost the same. We define s in a bit different way, namely

$$s(t,x,y,I) = \left(t, f(t) + M(t)x\left[\cos(\alpha) + y\sin(\alpha)I\right]\frac{f'(t)}{|f'(t)|}\right).$$

There is also a different order of terms in an outward orthogonal vector n.

Remark 3.31. It is enough in Theorem 3.28 to assume that $a \in \mathcal{C}(\mathbb{R}, \mathbb{H} \setminus \{0\})$ is T-periodic and locally Lipschitz. In that case one uses $L_a(t)$ instead of $|a'(t)|$ in the inequality (3.32) (cf. Remark 3.19).

Corollary 3.32. *Let $a \in \mathcal{C}(\mathbb{R}, \mathbb{H} \setminus \{0\})$, $f \in \mathcal{C}^2(\mathbb{R}, \mathbb{H})$ be T-periodic and a be locally Lipschitz. If the inequalities $|f'(t)| > 0$ and $\operatorname{Ark}[a(t)] < \frac{\pi}{6}$ hold for every $t \in \mathbb{R}$ then every equation*

$$\begin{aligned}\dot{q} =& [q - f(t)]Ra(t)\overline{f'}(t)[q - f(t)],\\ \dot{q} =& [q - f(t)]\overline{f'}(t)Ra(t)[q - f(t)]\end{aligned}$$

has at least two T-periodic solutions in $\mathbb{H}$ provided $R \in \mathbb{R}$ is big enough. One of them is asymptotically unstable and another one is asymptotically stable. Moreover every equation has infinitely many solutions which are heteroclinic to the periodic ones.

Proof. We fix $\alpha = \frac{\pi}{6}$ and $\kappa = -\frac{1}{4}$. The left hand side of (3.32) does not depend on R and is positive while the right hand side is proportional to $R^{-\frac{1}{4}}$. Similarly, the left hand side of (3.33) does not depend on R and is positive while the right hand side is proportional to $R^{-\frac{1}{2}}$. □

Example 3.33. By Theorem 3.28 and Remark 3.30 the equation

$$\dot{q} = \left[q - e^{kt}\right]\left(-ke^{-kt}Re^{i\frac{\pi}{12}}\right)\left[q - e^{kt}\right]$$

has at least two T-periodic solutions in $\mathbb{H}$ and infinitely many solutions which are heteroclinic to them provided $R > R_0$ where $R_0 < 100$. Here $f(t) = e^{kt}$, $a(t) \equiv Re^{i\frac{\pi}{12}}$, $\alpha = \frac{\pi}{6}$ and $\kappa = -\frac{61}{200}$.

Acknowledgements

I would like to express my gratitude to G. Gentili and D. C. Struppa for their help with acquainting me with the properties of quaternionic regular functions.

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