Approximation of analytic sets along Nash subvarieties

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Abstract

Let X be an analytic subset of pure dimension n of an open set $U \subset \mathbf{C}^m$ and let E be a Nash subset of U such that $E \subset X$. Then for every $a \in E$ there is an open neighborhood V of a in U and a sequence $\{X_\nu\}$ of complex Nash subsets of V of pure dimension n converging to $X \cap V$ in the sense of holomorphic chains such that the following hold for every $\nu \in \mathbf{N}: E \cap V \subset X_\nu$ and the multiplicity of X_ν at x equals the multiplicity of X at x for every x in a dense open subset of $E \cap V$.

Keywords analytic set germ, Nash set germ, approximation
MSC (2000): 32B10, 32C25

1 Introduction and main results

A natural question in analysis and geometry is whether analytic objects can be approximated by simpler algebraic ones with similar properties. Besides the fact that the question presents an independent interest, it is strongly motivated by applications. In particular, algebraic approximation is one of the central techniques used in numerical computations. This paper addresses the question in the case where the approximated objects are (germs of) complex analytic sets whereas the approximating ones are (germs of) complex Nash sets (i.e. the unions of irreducible analytic components of algebraic sets intersected with an open subset of a complex vector space, see Section 2.1). The approximation is expressed by means of the convergence of holomorphic chains (for a definition see Section 2.3).

Since the sixties of the last century, there has been interest in the problem of transforming the germ of an analytic set in $\mathbf{K}^m$ onto an algebraic germ in $\mathbf{K}^m$, where $\mathbf{K} = \mathbf{C}$ or $\mathbf{R}$ (see the articles by M. Artin [2], J. Bochnak [8], J. Bochnak and W. Kucharz [9], M. A. Buchner and W. Kucharz [13], T. Mostowski [22], A. Nobile [23], J. Cl. Tougeron [31], H. Whitney [35]). This problem is related to what is discussed in the present paper in the following way. Let $\phi : V \to W$ be a biholomorphism where V and W are complex algebraic (or Nash) and

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complex analytic subsets of an open ball $B \subset \mathbf{C}^m$ respectively. Then for every $a \in W$ there exist local approximations of W in a neighborhood of a by Nash sets $\phi_\nu(V \cap \tilde{B})$ where $\{\phi_\nu : \tilde{B} \to B\}$ is a sequence of polynomial injections converging uniformly to ϕ in some neighborhood $\tilde{B} \subset B$ of $\phi^{-1}(a)$.

Unfortunately, not every germ of an analytic set is biholomorphically equivalent to an algebraic germ as was first observed by H. Whitney [35]. Then such an analytic germ is not equivalent to a Nash germ either (a consequence of the equivalence of every Nash germ to an algebraic one, proved by J. Bochnak and W. Kucharz [9]). Nevertheless, local approximations by Nash sets exist for every analytic set, as shown in [5] and [6] (see also [4]). In particular, in [6] it is proved that in a neighborhood of a fixed point the order of tangency of approximating Nash sets and the limit set can be arbitrarily high. (Let us mention that the first results on approximation of analytic sets by higher order algebraic varieties are due to R. W. Braun, R. Meise and B. A. Taylor (see [11]).)

The simplest biholomorphic invariant of an analytic set X is its multiplicity $\mu_x(X)$ at a given point x. Thus a natural question arises whether X can be approximated by a sequence $\{X_\nu\}$ of Nash sets in such a way that $\mu_x(X) = \mu_x(X_\nu)$ for every ν and every x in a fixed set $E \subset X \cap \bigcap_{\nu=1}^{\infty} X_\nu$. The affirmative answer to the question in the case where E is an isolated point is given in [6]. It follows from the above mentioned result of that paper. In the present article we show that such approximation is possible in a certain neighborhood of every fixed point along any Nash subvariety with a removed nowhere dense analytic subset. This requires a different approach that the isolated point case since we cannot hope to obtain an arbitrarily high order of approximation along an arbitrary Nash subvariety of X.

Assuming the notation of Section 2, and treating analytic sets as holomorphic chains with components of multiplicity one, we prove the following

Theorem 1.1 *Let X be an analytic subset of pure dimension n of an open set Ω in $\mathbf{C}^m$ and let E be a Nash subset of Ω such that $E \subset X$. Then for every $x_0 \in E$ there is an open neighborhood V of x_0 in Ω and a sequence $\{X_\nu\}$ of complex Nash subsets of V of pure dimension n converging to $X \cap V$ in the sense of holomorphic chains such that for every $\nu \in \mathbf{N}$ the following hold:*
(1) $E \cap V \subset X_\nu$,
(2) $\mu_x(X_\nu) = \mu_x(X)$ for every $x \in (E \cap V) \setminus F_\nu$ where F_ν is a nowhere dense analytic subset of $E \cap V$.

In general in the assertion of Theorem 1.1 we cannot drop the assumption that the multiplicities coincide outside a thin subset of E as the following example shows.

Example. Define

$$X = \{(x, y, t, z) \in \mathbf{C}^4 : z^2(z - y + e^t) + x^4 = 0\},$$
$$E = \{(x, y, t, z) \in \mathbf{C}^4 : x = z = 0\}.$$

Then for every $(0, y_0, t_0, 0) \in E$ the cone tangent to X at $(0, y_0, t_0, 0)$ intersects the space $\{0_3\} \times \mathbf{C}$ at the isolated point $\{0_4\}$. Hence, by Proposition 2.3,

$\mu_{(0,y_0,t_0,0)}(X) = \mu_{(0,y_0,t_0,0)}(\rho|_X)$, where $\rho : \mathbf{C}^3_{x,y,t} \times \mathbf{C}_z \to \mathbf{C}^3_{x,y,t}$ is a natural projection. This implies that $\mu_{(0,y_0,t_0,0)}(X) = 2$ for every $(0, y_0, t_0, 0) \in E \setminus F$ where

$$F = \{(0, y, t, 0) \in \mathbf{C}^4 : y = e^t\}$$

and $\mu_{(0,y_0,t_0,0)}(X) = 3$ for $(0, y_0, t_0, 0) \in F$. Now, it is easy to see that the subset of E of points at which the multiplicity of X_ν equals 3 cannot be a transcendental curve for any Nash set X_ν.

On the other hand, if E is an analytic curve then for every $x_0 \in E$ there is a biholomorphic deformation of a neighborhood of x_0 in $\mathbf{C}^m$ after which we are able to avoid removing subsets of E in Theorem 1.1. The following proposition is a consequence of Theorem 1.1, the fact that every analytic curve is locally biholomorphically equivalent to an algebraic curve and the fact that proper analytic subsets of an irreducible analytic curve are isolated points.

Proposition 1.2 *Let X be an analytic subset of pure dimension n of an open set Ω in $\mathbf{C}^m$ and let $E \subset X$ be an analytic curve. Then for every $x_0 \in E$ there is an open neighborhood V of x_0 in Ω a biholomorphism $\phi : V \to W \subset \mathbf{C}^m$ and a sequence $\{X_\nu\}$ of complex Nash subsets of W of pure dimension n converging to $\phi(X \cap V)$ in the sense of holomorphic chains such that for every $\nu \in \mathbf{N}$ the following hold:*
(1) $\phi(E \cap V) \subset X_\nu$,
(2) $\mu_x(X_\nu) = \mu_x(\phi(X \cap V))$ for every $x \in \phi(E \cap V)$.

The basic obstacle one comes across in the proof of the presented results is the fact that in the case of set-theoretic non-complete intersections, generic approximations of describing functions yield sets whose dimension is strictly smaller than the dimension of the given set. To overcome this difficulty we use L. Lempert's theorem on approximation of holomorphic mappings with values in singular varieties (see Theorem 3.6). This theorem is formulated in terms of basic notions of complex analysis, however, its original proof involves a powerful machinery from commutative algebra (cf. [20]). More precisely, it relies on the affirmative solution to the M. Artin's conjecture for which the reader is referred to [1], [24], [25], [26], [28]. Let us mention that the local version of Theorem 3.6 (sufficient to obtain Theorem 1.1) can be derived from M. Artin's results of [2]. Recently an elementary proof of the local version (using methods different from those of [2]) has been given in [7]. In the present paper we treat Theorem 3.6 as a black box: most reasonings are based on a detailed study of the local properties of analytic varieties. For other results on algebraic approximation of analytic mappings between complex spaces see [10], [16], [17], [18], [29], [30].

The notion of the multiplicity of an analytic set at some point is central for intersection theory (applications to which partially motivate our interest in Theorem 1.1). We plan to use the techniques developed in this paper applying the methods of algebraic intersection theory in the analytic setting in a subsequent publication.

Finally, let us mention that the convergence of positive chains appearing in the paper is equivalent to the convergence of currents of integration over the considered sets (see [19], [15]; the equivalence in the considered context is discussed in [14], pp. 141, 206-207).

The organization of this paper is as follows. In Section 2 we present preliminaries about Nash sets, multiplicities of analytic sets, holomorphic chains and symmetric powers. Section 3 contains proofs of our main results.

2 Preliminaries

2.1 Nash sets

Let Ω be an open subset of $\mathbf{C}^n$ and let f be a holomorphic function on Ω. We say that f is a Nash function at $x_0 \in \Omega$ if there exist an open neighborhood U of x_0 and a polynomial $P : \mathbf{C}^n \times \mathbf{C} \to \mathbf{C}$, $P \neq 0$, such that $P(x, f(x)) = 0$ for $x \in U$. A holomorphic function defined on Ω is said to be a Nash function if it is a Nash function at every point of Ω. A holomorphic mapping defined on Ω with values in $\mathbf{C}^N$ is said to be a Nash mapping if each of its components is a Nash function.

A subset Y of an open set $\Omega \subset \mathbf{C}^n$ is said to be a Nash subset of Ω if and only if for every $y_0 \in \Omega$ there exists a neighborhood U of y_0 in Ω and there exist Nash functions $f_1, \ldots, f_s$ on U such that

$$Y \cap U = \{x \in U : f_1(x) = \ldots = f_s(x) = 0\}.$$

We will use the following fact from [32], p. 239. Let $\pi : \Omega \times \mathbf{C}^k \to \Omega$ denote a natural projection.

Theorem 2.1 *Let X be a Nash subset of $\Omega \times \mathbf{C}^k$ such that $\pi|_X : X \to \Omega$ is a proper mapping. Then $\pi(X)$ is a Nash subset of Ω and $dim(X) = dim(\pi(X))$.*

The fact from [32] stated below explains the relation between Nash and algebraic sets.

Theorem 2.2 *Let X be an irreducible Nash subset of an open set $\Omega \subset \mathbf{C}^n$. Then there exists an algebraic subset Y of $\mathbf{C}^n$ such that X is an analytic irreducible component of $Y \cap \Omega$. Conversely, every analytic irreducible component of $Y \cap \Omega$ is an irreducible Nash subset of Ω.*

2.2 Multiplicities of analytic sets

Let A be a purely n-dimensional locally analytic subset of $\mathbf{C}^m$ and let L be an $m - n$ dimensional affine subspace of $\mathbf{C}^m$ such that a is an isolated point of $L \cap A$. Then there is a domain $U \subset \mathbf{C}^m$ such that $U \cap A \cap L = \{a\}$ and the projection $\pi_L : U \cap A \to \pi_L(U) \subset L^{\perp}$ along L is a k-sheeted analytic cover. The number k will be called the multiplicity of π_L at a and denoted by $\mu_a(\pi_L|_A)$ (see [14] p. 102). Now put

$$\mu_a(A) = \min\{\mu_a(\pi_L|_A) : a \text{ is an isolated point of } A \cap L\}.$$

The number $\mu_a(A)$ will be called the multiplicity of A at a. (For the properties of this notion see [14] p.120.)

Let us recall that the tangent cone $C(Y, \mathbf{0})$ for an analytic subset Y of an open neighborhood of $\mathbf{0} \in \mathbf{C}^m$ is the set of all vectors $v \in \mathbf{C}^m$ for which there are a sequence $\{p_\nu\} \subset Y$ and a sequence $\{c_\nu\} \subset \mathbf{C}$ such that $\{p_\nu\}$ converges to $\mathbf{0}$ and $\{c_\nu p_\nu\}$ converges to v. The following proposition and lemma from [14], pp. 122, 102, will be useful to us.

Proposition 2.3 *Let Y be an n-dimensional analytic subset of some neighborhood of $\mathbf{0} \in \mathbf{C}^m$ such that $\mathbf{0} \in Y$ and let L be an $(m-n)$-dimensional linear subspace of $\mathbf{C}^m$ such that $L \cap Y = \{\mathbf{0}\}$. Then $\mu_{\mathbf{0}}(\pi_L|_Y) = \mu_{\mathbf{0}}(Y)$ if and only if $L \cap C(Y, \mathbf{0}) = \{\mathbf{0}\}$.*

Lemma 2.4 *Let A be a pure n-dimensional analytic subset of a domain $U = U' \times U''$ in $\mathbf{C}^m$ such that the projection $\pi : A \to U'$ is an analytic cover. Then for each natural number p the set $\{z \in A : \mu_z(\pi|_A) \geq p\}$ is analytic.*

2.3 Holomorphic chains

Let U be an open subset in $\mathbf{C}^m$. By a holomorphic chain in U we mean the formal sum $A = \sum_{j \in J} \alpha_j C_j$, where $\alpha_j \neq 0$ for $j \in J$ are integers and $\{C_j\}_{j \in J}$ is a locally finite family of pairwise distinct irreducible analytic subsets of U (see [33], cp. also [3], [14]). The set $\bigcup_{j \in J} C_j$ is called the support of A and is denoted by $|A|$ whereas the sets C_j are called the components of A with multiplicities α_j. The chain A is called positive if $\alpha_j > 0$ for all $j \in J$. If all the components of A have the same dimension n then A will be called an n-chain.

Below we introduce the convergence of holomorphic chains in U. To do this we first need the notion of the local uniform convergence of closed sets. Let Y, Y_ν be closed subsets of U for $\nu \in \mathbf{N}$. We say that $\{Y_\nu\}$ converges to Y locally uniformly if:

(1l) for every $a \in Y$ there exists a sequence $\{a_\nu\}$ such that $a_\nu \in Y_\nu$ and $a_\nu \to a$ in the standard topology of $\mathbf{C}^m$,

(2l) for every compact subset K of U such that $K \cap Y = \emptyset$ it holds $K \cap Y_\nu = \emptyset$ for almost all ν.

Then we write $Y_\nu \to Y$. For details concerning the topology of local uniform convergence see [33]. Let us mention that for a compact subset R of U, if $R \cap Y_\nu \to R \cap Y$ then $dist(R \cap Y_\nu, R \cap Y)$ converges to zero, where $dist$ is the Hausdorff distance.

We say that a sequence $\{Z_\nu\}$ of positive n-chains converges to a positive n-chain Z if:

(1c) $|Z_\nu| \to |Z|$,
(2c) for each regular point a of $|Z|$ and each submanifold T of U of dimension $m-n$ transversal to $|Z|$ at a such that $\overline{T}$ is compact and $|Z| \cap \overline{T} = \{a\}$, we have $deg(Z_\nu \cdot T) = deg(Z \cdot T)$ for almost all ν.

Then we write $Z_\nu \rightarrowtail Z$. By $Z \cdot T$ we denote the intersection product of Z and T (cf. [33]). Observe that the chains $Z_\nu \cdot T$ and $Z \cdot T$ for sufficiently large ν have finite supports and the degrees are well defined. Recall that for a chain $A = \sum_{j=1}^{d} \alpha_j \{a_j\}$, $deg(A) = \sum_{j=1}^{d} \alpha_j$.

2.4 Symmetric powers

Let $(\mathbf{C}^k)^d_{sym}$ and $\langle x_1, \dots, x_d \rangle$ denote $(\mathbf{C}^k)^d/_\sim$ and the equivalence class of $(x_1, \dots, x_d) \in (\mathbf{C}^k)^d$ respectively, where $(x'_1, \dots, x'_d) \sim (x_1, \dots, x_d)$ if and only if $(x'_1, \dots, x'_d) = (x_{p(1)}, \dots, x_{p(d)})$, for some permutation p. We endow $(\mathbf{C}^k)^d_{sym}$ with a metric ρ_k given by

$$\rho_k(\langle x_1, \dots, x_d \rangle, \langle y_1, \dots y_d \rangle) = \inf_p \sup_i ||x_i - y_{p(i)}||_{\mathbf{C}^k},$$

where $||(z_1, \dots, z_k)||_{\mathbf{C}^k} = \max_{i=1,\dots,k} |z_i|$, whereas p is any permutation of $(1, \dots, d)$ (the subscript k in ρ_k will be often omitted).

Then there exist an integer N and a mapping $\phi : (\mathbf{C}^k)^d_{sym} \to \mathbf{C}^N$ with the following properties (cf. [34] pp. 366-368, 152-154):

(a) ϕ is injective and ϕ, ϕ^{-1} are continuous and proper,
(b) $\phi \circ \pi_{sym} : (\mathbf{C}^k)^d \to \mathbf{C}^N$ is a polynomial mapping, where $\pi_{sym}(x_1, \dots, x_d) = \langle x_1, \dots, x_d \rangle$,
(c) $\phi((\mathbf{C}^k)^d_{sym})$ is an algebraic subset of $\mathbf{C}^N$.

(As for (c), in [34] the analyticity of $\phi((\mathbf{C}^k)^d_{sym})$ is proved. It is done by observing that this set is the image of a complex vector space by a proper polynomial mapping. Then, by Theorem 2.1, $\phi((\mathbf{C}^k)^d_{sym})$ is a Nash subset of $\mathbf{C}^N$. Hence irreducibility of $\phi((\mathbf{C}^k)^d_{sym})$ implies that it is an algebraic subset of $\mathbf{C}^N$ (cf. [32] p. 237).)

Let us mention that ϕ can be obtained by taking $\phi \circ \pi_{sym}$ equal to the collection of elementary symmetric functions

$$(\mathbf{C}^k)^d \to \bigoplus_{1 \le p \le d} S^p(\mathbf{C}^k),\ (x_1, \dots, x_d) \mapsto \bigoplus_{1 \le p \le d} \sum_{j_1 < \dots < j_p} x_{j_1} \cdots x_{j_p}$$

into the symmetric algebra of $\mathbf{C}^k$ (identifying the vector space $\bigoplus_{1 \le p \le d} S^p(\mathbf{C}^k)$ with $\mathbf{C}^N$ for some N).

3 Proofs

To prove Theorem 1.1 we construct a system (S) of Nash equations and a system of holomorphic functions describing X, in a neighborhood of the fixed x, satisfying the equations from (S). These objects will have the following property. For any sequence of systems of Nash functions satisfying (S) converging locally uniformly to the originally constructed holomorphic solution, the sets described by these Nash systems will satisfy the assertion of Theorem 1.1.

Let us turn to the construction. First, some of the equations from (S) will be responsible for the fact that higher order derivatives of certain functions (zero-sets of which contain X in a neighborhood of x) vanish on E. Here we shall need the following lemma which is a sort of higher order Nullstellensatz.

Let E_a, f_a denote the germs at a of the set E and the function f respectively and let $I(E_a)$ be the ideal of the germs of holomorphic functions vanishing on E in some neighborhood of a.

Lemma 3.1 *Let Y be an analytic subset of an open set $U \subset \mathbf{C}^n$, irreducible at $y \in U$ and let k_0 be a fixed integer. Then there are a neighborhood V of y in U and a holomorphic function $\beta : V \to \mathbf{C}$ such that $\beta_y \notin I(Y_y)$ and the following is satisfied. For every holomorphic function $f : U \to \mathbf{C}$ with*

$$\frac{\partial^k f}{\partial x_1^{t_1} \ldots \partial x_n^{t_n}}(x) = 0, \text{ for every } x \in Y, 0 \leq t_1 + \ldots + t_n = k \leq k_0$$

it holds $\beta f_y \in (I(Y_y))^{k_0+1}$.

Proof. Without loss of generality we assume $y = 0 \in \mathbf{C}^n$. Functions and their germs at zero will be denoted by the same letters (subscript omitted). Let $\mathcal{O}_m$ denote the ring of the germs of functions holomorphic in some neighborhood of $0 \in \mathbf{C}^m$, for $0 \leq m \leq n$.

Let d denote the dimension of Y at 0. By the Rückert's Parametrization (see [27], p. 28) we may assume that there exist polynomials $P, Q_j \in \mathcal{O}_d[x_{d+1}]$, $j = d+2, \ldots, n$ such that P is unitary and irreducible (i.e. its discriminant $\delta \in \mathcal{O}_d \setminus (0)$) and the following holds. There is an integer $q \geq 1$ such that

$$\delta^q I(Y_0) \subset I = \{P, \delta x_{d+2} - Q_{d+2}, \ldots, \delta x_n - Q_n\}\mathcal{O}_n \subset I(Y_0).$$

We show that there is m such that $\delta^m f \in I^{k_0+1}$ which completes the proof. Obviously, $\delta^q f \in I$, so assume that $\delta^{m_0} f \in I^s$ for some m_0 and $1 \leq s \leq k_0$. It is sufficient to check that, for some m_1, $\delta^{m_1} f \in I^{s+1}$. It holds

$$\delta^{m_0} f = \sum_{t_{d+1}+\ldots+t_n=s} h_{t_{d+1},\ldots,t_n} P^{t_{d+1}} (\delta x_{d+2} - Q_{d+2})^{t_{d+2}} \ldots (\delta x_n - Q_n)^{t_n},$$

where $h_{t_{d+1},\ldots,t_n}$ are germs of holomorphic functions at $0 \in \mathbf{C}^n$. Fix any point $(a, b_{d+1}, \ldots, b_n)$ in a neighborhood of 0 in Y such that $\delta(a) \neq 0$. To complete the proof it is sufficient to show that $h_{t_{d+1},\ldots,t_n}(a,b_{d+1},\ldots,b_n)=0$ for every $t_{d+1}+\ldots+t_n=s$.

Since $\delta(a)\neq 0$, we may assume that, after a biholomorphic change of coordinates in the neighborhood of $(a,b_{d+1},\ldots,b_n)$, it holds $b_j=0$, for $j=d{+}1,\ldots,n$, and

$$\delta^{m_0}f=\sum_{t_{d+1}+\ldots+t_n=s} h_{t_{d+1},\ldots,t_n}x_{d+1}^{t_{d+1}}x_{d+2}^{t_{d+2}}\ldots x_n^{t_n}.$$

Suppose that $h_{t_{d+1},\ldots,t_n}(a,0,\ldots,0)\neq 0$ for some $t_{d+1},\ldots,t_n$. Then there exist $v_{d+1},\ldots,v_n$ such that

$$\sum_{t_{d+1}+\ldots+t_n=s} h_{t_{d+1},\ldots,t_n}(a,0,\ldots,0)v_{d+1}^{t_{d+1}}v_{d+2}^{t_{d+2}}\ldots v_n^{t_n}\neq 0.$$

This implies that the s-th derivative of the function

$$F(t)=\delta^{m_0}f(a,tv_{d+1},\ldots,tv_n)$$

at zero is different from zero. A contradiction with the hypothesis.■

In the sequel we treat every purely n-dimensional analytic set as a holomorphic n-chain such that each of its components appears with multiplicity one. Then the notion of the convergence in the sense of chains (denoted by "$\rightarrowtail$", see Section 2.3) is well defined in this context.

The proof of Theorem 1.1 involves the fact that if an affine space $L\subset\mathbf{C}^m$ of dimension $m-n$ intersects X at x transversally then $\mu_x(\pi_L|_X)=\mu_x(X)$, where π_L denotes the projection of $\mathbf{C}^m$ onto the orthogonal complement of L (see Proposition 2.3). Thus in order to obtain $\mu_x(X)=\mu_x(X_\nu)$ for x from a fixed set, where X_ν are approximating varieties, it is sufficient to make sure that the transversality condition holds also for X_ν and that $\mu_x(\pi_L|_X)=\mu_x(\pi_L|_{X_\nu})$. These conditions can be equivalently expressed by the fact that certain Nash equations are satisfied by suitably chosen descriptions of X,X_ν. The aim of the following lemma is to formulate such Nash equations in the case where X and X_ν are hypersurfaces.

Let $U=U_1\times U_2$ be an open subset of $\mathbf{C}^n_{x_1,\ldots,x_n}=\mathbf{C}^d_{x_1,\ldots,x_d}\times\mathbf{C}^{n-d}_{x_{d+1},\ldots,x_n}$ such that $0_n\in U$ and let $\pi:\mathbf{C}^n\times\mathbf{C}\to\mathbf{C}^n,\rho:\mathbf{C}^n_{x_1,\ldots,x_n}\times\mathbf{C}\to\mathbf{C}^{d+1}_{x_1,\ldots,x_{d+1}}$ be natural projections.

Lemma 3.2 *Let $\tilde p(x,z)=z^r+a_1(x)z^{r-1}+\ldots+a_r(x)\in\mathcal{O}(U)[z]$ be a polynomial with non-zero discriminant, $a_j(0_n)=0$ for $j=1,\ldots,r$. Next, let $S\subset Y=\{(x,z)\in U\times\mathbf{C}:\tilde p(x,z)=0\}$ be a purely d-dimensional Nash subset of $U\times\mathbf{C}$ irreducible at $0_{n+1}\in\mathbf{C}^n\times\mathbf{C}$, $d<n$, such that:*

(1) S is with proper projection onto U_1, $\rho|_{S\setminus F}$ is injective,
(2) $C(Y,a)\cap(\{0_n\}\times\mathbf{C})=\{0_{n+1}\}$, for every $a\in S\setminus F$,

where F is a nowhere dense analytic subset of S. Then there are an open neighborhood $\tilde U$ of 0_n in U, holomorphic functions $g_1,\ldots,g_t:\tilde U\to\mathbf{C}$ and Nash *functions* $F_i(x, u_1, \ldots, u_r, v_1, \ldots, v_t)$, $i = 1, \ldots, s$ *with the following property. For all sequences* $\{a_{1,\nu}\}, \ldots, \{a_{r,\nu}\}, \{g_{1,\nu}\}, \ldots, \{g_{t,\nu}\}$ *of holomorphic functions converging to* $a_1, \ldots, a_r, g_1, \ldots, g_t$ *respectively uniformly on* $\tilde{U}$ *such that for every* $i = 1, \ldots, s$ *and* $x \in \tilde{U}$, $\nu \in \mathbf{N}$,

$$F_i(x, a_{1,\nu}(x), \ldots, a_{r,\nu}(x), g_{1,\nu}(x), \ldots, g_{t,\nu}(x)) = 0,$$

the following hold:

(3) $Y_\nu = \{(x, z) \in \tilde{U} \times \mathbf{C} : z^r + a_{1,\nu}(x) z^{r-1} + \ldots + a_{r,\nu}(x) = 0\} \rightarrowtail Y \cap (\tilde{U} \times \mathbf{C})$,
(4) $\mu_a(\pi|_Y) = \mu_a(\pi|_{Y_\nu})$ *for every* $a \in (S \cap (\tilde{U} \times \mathbf{C})) \setminus S_\nu$ *for almost all* ν,
(5) $C(Y_\nu, a) \cap (\{0_n\} \times \mathbf{C}) = \{0_{n+1}\}$ *for every* $a \in (S \cap (\tilde{U} \times \mathbf{C})) \setminus S_\nu$ *for almost all* ν,

where S_ν *is a nowhere dense analytic subset of* $S \cap (\tilde{U} \times \mathbf{C})$.

Proof. To explain the idea of the proof suppose for a moment that $a_j(x) = 0$ for $x \in \pi(S)$, $j = 1, \ldots, r$ (which implies that $S \subset U \times \{0\}$ and $\mu_b(\pi|_Y) = r$ for $b \in S$). Then, having in mind that the cone tangent to Y at $b \in S$ is defined by the initial polynomial of the Taylor expansion of $\tilde{p}$ at b, (2) is equivalent to the fact that partial derivatives of a_j up to sufficiently large order vanish on $\pi(S)$. Now using Lemma 3.1 it is easy to formulate Nash equations satisfied by a_j (together with some other holomorphic functions) such that (5) holds provided these equations are satisfied by $a_{j,\nu}$ (and the other functions converging to the original solution). This will be done in detail, in the general situation, below. (Observe that $a_{j,\nu}(x) = 0$ for $x \in \pi(S)$, $j = 1, \ldots, r$ automatically imply (4).)

To apply this sketch in general, we should know that some neighborhood of 0_{n+1} in S can be embedded in an n dimensional manifold which may not hold true. However, such embedding is possible if we replace Y by its image by a certain Nash mapping as specified below. (Of course it may be $\mu_b(\pi|_Y) < r$ for b from a dense open subset of S. This does not lead to any difficulties as we shall see in the sequel.)

Let $\delta : U_1 \to \mathbf{C}$ be the discriminant of the unitary reduced polynomial from $\mathcal{O}(U_1)[x_{d+1}]$ describing $\rho(S)$. (Note that δ is a non-zero Nash function.) It is well known (see e.g. [21], ch. VI.2) that, by (1), there is a Nash function $\alpha : (U_1 \setminus \{\delta = 0\}) \times U_2 \to \mathbf{C}$ such that $S \cap ((U_1 \setminus \{\delta = 0\}) \times U_2 \times \mathbf{C}) \subset graph(\alpha)$ and $\delta \cdot \alpha$ can be extended over all $U_1 \times U_2$. This implies that for the mapping

$$\hat{\gamma} : U \times \mathbf{C} \to U \times \mathbf{C},\ \hat{\gamma}(x_1, \ldots, x_n, z) = (x_1, \ldots, x_n, z \cdot \delta(x_1, \ldots, x_d))$$

the set $\hat{\gamma}(S) \subset graph(\phi)$, where $\phi : U \to \mathbf{C}$, $\phi = \delta \cdot \alpha$ is a Nash function. Moreover, $\hat{\gamma}$ is a biholomorphism on $(U_1 \setminus \{\delta = 0\}) \times U_2 \times \mathbf{C}$ (recall that $S \setminus \{\delta = 0\}$ is an open dense subset of S). Taking $\gamma = \Psi \circ \hat{\gamma}$, where $\Psi : U \times \mathbf{C} \to U \times \mathbf{C}$ is given by $\Psi(x, z) = (x, z - \phi(x))$, we obtain $\gamma(S) \subset U \times \{0\}$.

Observe that the condition (2) is satisfied with Y, S, F replaced by $\gamma(Y), \gamma(S)$, $\tilde{F} = \gamma(F) \cup \{\delta = 0\}$ respectively. Moreover,

$$\mu_b(\pi|_Y) = \mu_{\gamma(b)}(\pi|_{\gamma(Y)})$$

for $b \in S \setminus \{\delta = 0\}$. Therefore to complete the proof of the lemma it is sufficient to construct holomorphic functions $g_1, \ldots, g_t : \tilde{U} \to \mathbf{C}$ and Nash functions $F_i(x, u_1, \ldots, u_r, v_1, \ldots, v_t)$, $i = 1, \ldots, s$, such that the following is satisfied. For all sequences $\{a_{1,\nu}\}, \ldots, \{a_{r,\nu}\}, \{g_{1,\nu}\}, \ldots, \{g_{t,\nu}\}$ converging to $a_1, \ldots, a_r, g_1, \ldots, g_t$ uniformly on $\tilde{U}$ such that for every $i = 1, \ldots, s$ and $x \in \tilde{U}$

$$F_i(x, a_{1,\nu}(x), \ldots, a_{r,\nu}(x), g_{1,\nu}(x), \ldots, g_{t,\nu}(x)) = 0,$$

the following hold for $\nu \in \mathbf{N}, a \in (\gamma(S) \cap (\tilde{U} \times \mathbf{C})) \setminus G_\nu$:

(a) $C(\gamma(Y_\nu), a) \cap (\{0_n\} \times \mathbf{C}) = \{0_{n+1}\}$,
(b) $\mu_a(\pi|_{\gamma(Y)}) = \mu_a(\pi|_{\gamma(Y_\nu)})$,

where G_ν is an analytic nowhere dense subset of $\gamma(S) \cap (\tilde{U} \times \mathbf{C})$. (Note that (3) will follow automatically by the fact that $\tilde{p}$ has non-zero discriminant.)

To do this denote $\hat{a} = (a_1, \ldots, a_r)$ and observe that

$$\gamma(Y) = \{(x, z) \in U \times \mathbf{C} : P(x, z) = 0\},$$

for

$$P(x, z) = z^r + z^{r-1} b_1(\hat{a}(x), \phi(x), \delta(x)) + \ldots + b_r(\hat{a}(x), \phi(x), \delta(x)),$$

where $b_1, \ldots, b_r \in \mathbf{C}[u_1, \ldots, u_r, v, w]$ are polynomials independent of $\hat{a}, \delta, \phi$. Indeed, the fact that $\gamma(Y)$ is described by P outside $\{\delta = 0\}$ is obvious (direct calculations). On the other hand, $\{\delta = 0\}$ is nowhere dense in U and $\gamma(Y)$ is bounded over every compact subset of U so $\gamma(Y)$ is an analytic subset of $U \times \mathbf{C}$. Then there is the unique unitary polynomial in z of degree r (with non-zero discriminant) describing $\gamma(Y)$. The uniqueness immediately implies that it must be P (because the coefficients of the polynomials are equal outside $\{\delta = 0\}$).

The facts that S is irreducible at 0_{n+1}, $a_j(0_n) = 0$ for $j = 1, \ldots, r$ imply that $\gamma(S)$ is irreducible at 0_{n+1}. This, in view of (2) gives, by Lemma 2.4, that there are $\mu \in \mathbf{N}$ and a nowhere dense analytic subset $\hat{F}$ of $\gamma(S)$ such that

$$C(\gamma(Y), a) \cap (\{0_n\} \times \mathbf{C}) = \{0_{n+1}\} \text{ and } \mu_a(\pi|_{\gamma(Y)}) = \mu$$

for every $a \in \gamma(S) \setminus \hat{F}$ (shrinking U if necessary; then $\mu = \min\{\mu_a(\pi|_{\gamma(Y)}) : a \in \gamma(S)\}$). Then, since $\gamma(S) \subset U \times \{0\}$, (taking into account that the tangent cone is described by the initial homogenous polynomial of the Taylor expansion of P at the given point) we obtain

$$\frac{\partial^i b_j(\hat{a}, \phi, \delta)}{\partial x_1^{\alpha_{1,i}} \ldots \partial x_n^{\alpha_{n,i}}}(x) = 0$$

for every $x \in \pi(\gamma(S))$, $\alpha_{1,i} + \ldots + \alpha_{n,i} = i \leq \mu + j - r - 1$, $j = r - \mu + 1, \ldots, r$. This in turn, by Lemma 3.1, implies that there are Nash functions $h_1, \ldots, h_{\tilde{t}}$ describing the set $\pi(\gamma(S))$ (in some neighborhood of zero in $\mathbf{C}^n$) such that for every $j \in \{r-\mu+1,\dots,r\}$ there are holomorphic functions $\theta_j, \eta_{j,\kappa_{1,j},\dots,\kappa_{\tilde{t},j}}$, where θ_j is a non-zero function on $\pi(\gamma(S))$ such that

$$\text{(c)} \qquad \theta_j b_j(\hat{a},\delta,\phi) - \textstyle\sum_{\kappa_{1,j}+\dots+\kappa_{\tilde{t},j}=\mu+j-r} h_1^{\kappa_{1,j}} \dots h_{\tilde{t}}^{\kappa_{\tilde{t},j}} \eta_{j,\kappa_{1,j},\dots,\kappa_{\tilde{t},j}} = 0$$

in some neighborhood of $0_n \in \mathbf{C}^n$.

We shall show that the required functions F_i, $i=1,\dots,s$, can be obtained by taking the left-hand sides of the equations from the system (c) in which $\theta_j, a_1,\dots,a_r, \eta_{j,\kappa_{1,j},\dots,\kappa_{t,j}}$ are replaced by independent new variables. Then the replaced functions, apart from $a_1,\dots,a_r$, will be the looked for $g_1,\dots,g_t$.

To this end, suppose that sequences $\{a_{1,\nu}\},\dots,\{a_{r,\nu}\}$, $\{g_{1,\nu}\},\dots,\{g_{t,\nu}\}$ are converging uniformly to $a_1,\dots,a_r,g_1,\dots,g_t$ in some open neighborhood of zero, where $g_1,\dots,g_t$ are as above. Moreover, assume that these sequences satisfy the equations specified in the previous paragraph. Put $\hat{a}_\nu = (a_{1,\nu},\dots,a_{r,\nu})$ and observe that by (c) for some neighborhood $\tilde{U}$ of zero

$$\frac{\partial^i b_j(\hat{a}_\nu,\phi,\delta)}{\partial x_1^{\alpha_{1,i}}\dots\partial x_n^{\alpha_{n,i}}}(x) = 0$$

for $x \in \pi(\gamma(S)) \cap \tilde{U}$, $\alpha_{1,i}+\dots+\alpha_{n,i} = i \le \mu+j-r-1$, $j = r-\mu+1,\dots,r$. Next put

$$P_\nu(x,z) = z^r + z^{r-1} b_1(\hat{a}_\nu(x),\phi(x),\delta(x)) + \dots + b_r(\hat{a}_\nu(x),\phi(x),\delta(x))$$

and observe that

$$\gamma(Y_\nu) = \{(x,z) \in \tilde{U}\times\mathbf{C} : P_\nu(x,z) = 0\}.$$

In view of the vanishing of certain $b_j(\hat{a}_\nu,\phi,\delta)$ and by $\gamma(S) \subset U \times \{0\}$ we have

$$P_\nu(a) = \frac{\partial P_\nu}{\partial z}(a) = \dots = \frac{\partial^{\mu-1}P_\nu}{\partial z^{\mu-1}}(a) = 0,$$

for every $a \in \gamma(S) \cap (\tilde{U}\times\mathbf{C})$. It is easy to see that (b) holds for all $a \in \gamma(S)\cap(\tilde{U}\times\mathbf{C})$ such that

$$\frac{\partial^\mu P_\nu}{\partial z^\mu}(a) \neq 0 \neq \frac{\partial^\mu P}{\partial z^\mu}(a)$$

i.e. outside a nowhere dense analytic subset of $\gamma(S)\cap(\tilde{U}\times\mathbf{C})$. Then the fact that partial derivatives of $b_j(\hat{a}_\nu,\phi,\delta)$ of order smaller than or equal to $\mu+j-r-1$ vanish on $\pi(\gamma(S))\cap\tilde{U}$ implies that (a) holds for every a from $\gamma(S)\cap(\tilde{U}\times\mathbf{C})$ with a removed nowhere dense analytic subset.∎

The following lemma shows how to reduce the problem of the equity of the multiplicities at a fixed point of sets of arbitrary codimension to the case of hypersurfaces which was discussed in Lemma 3.2.

Let $\tilde{\pi} : \mathbf{C}^n\times\mathbf{C}^k \to \mathbf{C}^n$, $\pi : \mathbf{C}^n\times\mathbf{C} \to \mathbf{C}^n$ be natural projections. Let $L_i : \mathbf{C}^k \to \mathbf{C}$, for $i=1,\dots,p$, be $\mathbf{C}$-linear forms such that for every $(\tilde{m}+1)$-element subset A of $\mathbf{C}^k$ there is $i \in \{1,\dots,p\}$ such that $L_i|_A$ is injective, where

$\tilde{m}$ is a fixed integer. (The existence of such forms follows for example by the proof of Lemma 1 of [6].) Let U be an open connected subset of $\mathbf{C}^n$. For any $X \subset U \times \mathbf{C}^k$ put $X_i := \Phi_{L_i}(X)$, where $\Phi_{L_i} : \mathbf{C}^n \times \mathbf{C}^k \to \mathbf{C}^n \times \mathbf{C}$ is given by the formula $\Phi_{L_i}(x, v) = (x, L_i(v))$.

Lemma 3.3 *Let $X \subset U \times \mathbf{C}^k$ be an analytic subset of pure dimension n with proper projection onto U and let E be an analytic subset of $U \times \mathbf{C}^k$, $E \subset X$. For $i = 1, \dots, p$ assume:*

(0) $\max\{\sharp X \cap (\{x\} \times \mathbf{C}^k) : x \in U\} = \max\{\sharp X_i \cap (\{x\} \times \mathbf{C}) : x \in U\} = \tilde{m}$,
(1) $C(X, a) \cap (\{0_n\} \times \mathbf{C}^k) = \{0_{n+k}\}$,
(2) $\mu_{\Phi_{L_i}(a)}(\pi|_{X_i}) = \mu_a(\tilde{\pi}|_X)$,

for every $a \in E \setminus F$ where F is an analytic subset of E. Next, let $\{X^\nu\}$ be a sequence of analytic subsets of $U \times \mathbf{C}^k$ of pure dimension n, each of which contains E, such that $X^\nu \rightarrowtail X$ and such that for every $\nu \in \mathbf{N}$ the following hold:

(3) $C(X_i^\nu, b) \cap (\{0_n\} \times \mathbf{C}) = \{0_{n+1}\}$,
(4) $\mu_b(\pi|_{X_i^\nu}) = \mu_b(\pi|_{X_i})$,

for every $b \in \Phi_{L_i}(E \setminus F^\nu)$, $i = 1, \dots, p$, where F^ν is an analytic subset of E. Then

$$\mu_a(X) = \mu_a(X^\nu)$$

for every $a \in E \setminus E^\nu$ and almost all $\nu \in \mathbf{N}$, where $E^\nu = F \cup F^\nu$.

Proof. The definition of the tangent cone and (3) immediately imply that

$$C(X^\nu, a) \cap (\{0_n\} \times \mathbf{C}^k) = \{0_{n+k}\}$$

for every $a \in E \setminus F^\nu$. Indeed, suppose that $(0_n, h) \in C(X^\nu, a) \subset \mathbf{C}^n \times \mathbf{C}^k$, $h \neq 0_k$ for some $a \in E \setminus F^\nu$. Then there is $j \in \{1, \dots, p\}$ such that $L_j(h) \neq 0$. Moreover, there are $\{(x_l, y_l)\} \subset X^\nu$, $\lambda_l \in \mathbf{C}$ with

$$(x_l, y_l) \to a = (x, y)$$

as l tends to infinity such that

$$\lambda_l(x - x_l, y - y_l) \to (0_n, h).$$

Consequently,

$$\lambda_l(\Phi_{L_j}(x, y) - \Phi_{L_j}(x_l, y_l)) \to (0_n, L_j(h)),$$

which implies that $(0_n, L_j(h)) \in C(X_j^\nu, \Phi_{L_j}(a))$, a contradiction with (3).

Hence in view of Proposition 2.3 and (1) it is sufficient to prove that

$$\mu_a(\tilde{\pi}|_{X^\nu}) = \mu_a(\tilde{\pi}|_X)$$

for every $a \in E \setminus (F \cup F^\nu)$ and almost all $\nu \in \mathbf{N}$. To do this we need the following simple

Remark 3.4 *Let Y be an analytic subset of $U \times \mathbf{C}^k$ of pure dimension n with proper projection onto U and let $L : \mathbf{C}^k \to \mathbf{C}$ be a $\mathbf{C}$-linear form. Assume that*

$$\max\{\sharp Y \cap (\{x\} \times \mathbf{C}^k) : x \in U\} = \max\{\sharp \Phi_L(Y) \cap (\{x\} \times \mathbf{C}) : x \in U\}.$$

Then for every $a \in Y$ such that $\Phi_L|_{(\{\tilde{\pi}(a)\}\times\mathbf{C}^k)\cap Y}$ is injective it holds $\mu_a(\tilde{\pi}|_Y) = \mu_{\Phi_L(a)}(\pi|_{\Phi_L(Y)})$.

Let us finish the proof of Lemma 3.3. Since $X^\nu \rightarrowtail X$, (0) holds for X^ν for almost all ν. Let ν be so large that X^ν satisfies (0). Pick any $a \in E \setminus (F \cup F^\nu)$ and a form L_i, $i \in \{1, \ldots, p\}$, such that $\Phi_{L_i}|_{(\{\tilde{\pi}(a)\}\times\mathbf{C}^k)\cap X^\nu}$ is injective. By Remark 3.4, (0), (2) and (4) we obtain

$$\mu_a(\tilde{\pi}|_X) = \mu_{\Phi_{L_i}(a)}(\pi|_{X_i}) = \mu_{\Phi_{L_i}(a)}(\pi|_{X_i^\nu}) = \mu_a(\tilde{\pi}|_{X^\nu})$$

and the proof is complete.■

Proof of Theorem 1.1. *Step 1.* Let us start with some preparations. We may restrict our attention to the case where E is of dimension d strictly smaller than n. This is due to the fact that n-dimensional irreducible components of E are the components of X so the assertion follows immediately by Theorem 1.1 of [5] (cp. Step 2 below). We additionally assume that E is irreducible at x_0 (from the proof it will be clear that by this assumption we do not restrict generality). Next, passing to the image of a neighborhood of x_0 by a linear isomorphism, if necessary, we may assume that $x_0 = 0_m$ and X is an analytic subset of $U \times \mathbf{C}^k$ ($k = m - n$) with proper projection onto U, where $U = U_1 \times U_2 \subset \mathbf{C}^d \times \mathbf{C}^{n-d}$ is an open connected neighborhood of $0_n \in \mathbf{C}^n$ and E is with proper projection onto U_1.

Put $\tilde{m} = \max\{\sharp(X \cap (\{x\} \times \mathbf{C}^k)) : x \in U\}$. Next, without loss of generality, further assumptions can be made: there are non-zero $\mathbf{C}$-linear forms $L_1, \ldots, L_p : \mathbf{C}^k \to \mathbf{C}$ such that for every $(\tilde{m}+1)$-element subset A of $\mathbf{C}^k$ there is $i \in \{1, \ldots, p\}$ such that $L_i|_A$ is injective (cp. the paragraph preceding Lemma 3.3) and (after another change of the coordinates if necessary) X satisfies the hypotheses (0), (1) and (2) of Lemma 3.3 with F nowhere dense in E. Here (1) requires an explanation: denote $\mu = \min\{\mu_a(X) : a \in E\}$. There is a linear change of coordinates, arbitrarily close to the identity, after which there is $a \in E$ such that $\mu_a(\tilde{\pi}|_X) = \mu$. Then, by Lemma 2.4 we have $\{a \in E : \mu_a(\tilde{\pi}|_X) = \mu\} = E \setminus V$, where V is a nowhere dense analytic subset of E. Now, by Proposition 2.3, it holds $C(X, a) \cap (\{0_n\} \times \mathbf{C}^k) = \{0_{n+k}\}$ for every $a \in E \setminus V$. As for (2), shrinking U if necessary we may assume that $Reg(\tilde{\pi}(E))$ is connected. Take any $b_0 \in Reg(\tilde{\pi}(E))$ such that the fiber in $(Reg(\tilde{\pi}(E)) \times \mathbf{C}^k) \cap X$ over b_0 has the maximal cardinality. Next we may assume, applying an arbitrarily close to the identity change of the coordinates, that $\Phi_{L_j}|_{(X\cap(\{b_0\}\times\mathbf{C}^k))}$ is injective for every $j \in \{1, \ldots, p\}$. Then the injectivity condition holds with b_0 replaced by every $b \in \tilde{\pi}(E) \setminus \tilde{V}$ where $\tilde{V}$ is an analytic nowhere dense subset of $\tilde{\pi}(E)$. Now, in view of (0), (2) follows immediately.

We complete the first step of the proof by showing that the hypotheses of Lemma 3.2 may be assumed to be satisfied with Y, S replaced by $\Phi_{L_j}(X), \Phi_{L_j}(E)$

respectively for every $j \in \{1, \ldots, p\}$ (then $\tilde{p}$ is taken to be the polynomial with holomorphic coefficients defined on U and non-zero discriminant describing $\Phi_{L_j}(X)$). Indeed, observe that in the previous paragraph the coordinates can be changed in such a way that there is an analytic nowhere dense subset $\tilde{E}$ of E such that the projection of $E \setminus \tilde{E}$ onto $U_1 \times \mathbf{C}_{x_{d+1}}$ is injective (it is sufficient to ensure, assuming that E is purely dimensional, that the injectivity of the projection holds over $\{b_0\} \times \mathbf{C}_{x_{d+1}}$, where $b_0 \in U_1$ is such that the fiber in E over b_0 is of the maximal cardinality). Moreover, note that $ker\Phi_{L_j} \subset \{0_n\} \times \mathbf{C}^k$ hence $(ker\Phi_{L_j}) \cap C(X, a) = \{0_{n+k}\}$ for every a in E with a removed nowhere dense analytic subset. Consequently, (see [14], p. 81) $C(X_j, \Phi_{L_j}(a)) \cap (\{0_n\} \times \mathbf{C}) = \{0_{n+1}\}$ for every $j \in \{1, \ldots, p\}$ and a in E outside a nowhere dense analytic subset.

Step 2. We recall the basic construction from [5], concerning approximation of holomorphic chains by Nash chains, that will be useful in the sequel. Let $B \subset \mathbf{C}^n$ denote a polydisc centered at $0_n \in \mathbf{C}^n$ and let $B\theta = \{x \in \mathbf{C}^n : \frac{1}{\theta}x \in B\}$, where $0 < \theta \leq 1$. Let $\psi : B \to I$ be a holomorphic mapping, where $I = \phi((\mathbf{C}^k)^{\tilde{m}}_{sym})$. (For the definitions of ϕ and $(\mathbf{C}^k)^{\tilde{m}}_{sym}$ see Section 2.4.) Next put $\iota = k \cdot \tilde{m}$ and identify the space $((\mathbf{C})^k)^{\tilde{m}}$ with $\mathbf{C}^\iota$. Let $W(z_1, \ldots, z_\iota) = \phi \circ \pi_{sym}(z_1, \ldots, z_\iota)$, for $z_i \in \mathbf{C}, i = 1, \ldots, \iota$. ($\pi_{sym}$ is introduced in Section 2.4; here π_{sym} is treated, via the identification above, as a mapping defined on $\mathbf{C}^\iota$. Recall that W is a polynomial mapping.) Now define

$$X(\psi, \theta) = \{(x, z_1, \ldots, z_\iota) \in B\theta \times \mathbf{C}^\iota : \psi(x) = W(z_1, \ldots, z_\iota)\}.$$

Observe that the fact that the image of ψ is contained in I and the properties of ϕ imply that for $\theta \leq 1$ the set $X(\psi, \theta)$ is a purely n-dimensional subset of $B\theta \times \mathbf{C}^\iota$ with proper projection onto $B\theta$. The pure dimension n of $X(\psi, \theta)$ requires an explanation: in view of the properness of the projection of $X(\psi, \theta)$ onto $B\theta$, it is a consequence of the non-emptiness of all the fibers over $B\theta$ and the continuity of the mapping $\phi^{-1} \circ \psi : B\theta \to (\mathbf{C}^k)^{\tilde{m}}_{sym}$.

Now let Y be any purely n-dimensional analytic subset of $B\theta \times \mathbf{C}^k$ with proper projection onto $B\theta$ such that the generic cardinality of the fiber of Y over $B\theta$ equals $\tilde{m}$ and let $pr : \mathbf{C}^n \times \mathbf{C}^k \times \mathbf{C}^{\iota - k} \to \mathbf{C}^n \times \mathbf{C}^k$ be a natural projection. Then there exists a holomorphic mapping $\psi : B\theta \to I$ such that $Y = pr(X(\psi, \theta))$ (cp. the proof of Theorem 1.1 in [5]). In the proof of Theorem 1.1 of [5] it is shown that if a sequence $\{\psi_\nu : B\theta \to I\}$ of holomorphic mappings converges to ψ uniformly then the sequence $\{pr(X(\psi_\nu, \theta))\}$ converges to $pr(X(\psi, \theta))$ in the sense of chains. On the other hand, if ψ_ν are Nash mappings then, by Theorem 2.1, $pr(X(\psi_\nu, \theta))$ are Nash sets of pure dimension n.

Step 3. Without loss of generality we assume that U_1, U_2 above are polydiscs and put $B = U_1 \times U_2$. Now let $\psi : B \to I$ be a holomorphic mapping such that $X = pr(X(\psi, 1))$, where X is our set to be approximated. In order to prove Theorem 1.1 we show that there are $\theta \leq 1$ and a sequence $\psi_\nu : B\theta \to I$ of Nash mappings converging to $\psi|_{B\theta}$ uniformly such that certain conditions which will be expressed in terms of Nash equations are fulfilled. To formulate these conditions we need the following simple lemma from [6] (Lemma 3).

Lemma 3.5 *For every* $\mathbf{C}$*-linear form* $L : \mathbf{C}^\iota \to \mathbf{C}$ *there are* $P_{L,1}, \ldots, P_{L,n_L} \in \mathbf{C}[y_1, \ldots, y_N, z]$ *such that for any holomorphic mapping* $\psi : B \to I \subset \mathbf{C}^N$ *the following holds.*

$$\Phi_L(X(\psi, \theta)) = \{(x, z) \in B\theta \times \mathbf{C} : P_{L,1}(\psi(x), z) = \ldots = P_{L,n_L}(\psi(x), z) = 0\},$$

for $\theta < 1$, *where* $\Phi_L(u, v) = (u, L(v))$, *for* $u \in \mathbf{C}^n$, $v \in \mathbf{C}^\iota$.

We return to the proof of Theorem 1.1. For every $j \in \{1, \ldots, p\}$ define the form $\tilde{L}_j : \mathbf{C}^\iota \to \mathbf{C}$ by the formula $\tilde{L}_j = L_j \circ \tilde{pr}$, where $\tilde{pr} : \mathbf{C}^{\iota-k} \times \mathbf{C}^k \to \mathbf{C}^k$ is a natural projection. Let $P_{j,1}, \ldots, P_{j,n_j}$, for $j = 1, \ldots, p$, denote polynomials obtained by applying Lemma 3.5 to $\Phi_{\tilde{L}_j}(X(\psi, 1))$ (recall that $pr(X(\psi, 1)) = X$). Then, for every $j \in \{1, \ldots, p\}$ we have

$$P_{j,i}(\psi(x), z) = H_{j,i}(x, z)(W_j(x, z))^{k_{j,i}}, \text{ for } i = 1, \ldots, n_j.$$

Here W_j is a unitary polynomial in z with holomorphic coefficients and non-zero discriminant, describing $\Phi_{\tilde{L}_j}(X(\psi, 1))$, $H_{j,i}$ is a polynomial in z and $k_{j,i}$ is an integer such that $\{W_j = 0\}$ is not contained in $\{H_{j,i} = 0\}$.

The system of Nash equations mentioned in the first paragraph of Section 3, which will be denoted by (S) consists of the following equations:

(a) algebraic equations describing the set I (recall that $I = \phi((\mathbf{C}^k)^{\tilde{m}}_{sym})$ is an algebraic subset of $\mathbf{C}^N$, cf. Section 2.4),
(b) equations $P_{j,i}(\psi(x), z) = H_{j,i}(x, z)(W_j(x, z))^{k_{j,i}}$, for $i = 1, \ldots, n_j$, $j = 1, \ldots, p$, from the previous paragraph with ψ and the holomorphic coefficients of W_j and the coefficients of $H_{j,i}$ replaced by new variables,
(c) equations $F_{j,i} = 0$ where $F_{j,i}$ are obtained by applying Lemma 3.2 with $\tilde{p} = W_j$, $S = \Phi_{L_j}(E)$ for $j = 1, \ldots, p$, (since $\Phi_{\tilde{L}_j}(X(\psi, 1)) = \Phi_{L_j}(X)$, by what we have assumed about X it follows that W_j satisfies the hypotheses of Lemma 3.2).

Step 4. In the sequel, holomorphic solutions to the system (S) will be approximated by Nash functions. To do this we need the following theorem which is due to L. Lempert (see [20], Theorem 3.2, pp. 338-339).

Theorem 3.6 *Let* $K \subset \mathbf{C}^n$ *be a compact polydisc and* $f : K \to \mathbf{C}^k$ *a holomorphic mapping that satisfies a system of equations* $Q(z, f(z)) = 0$ *for* $z \in K$. *Here* Q *is a Nash mapping from a neighborhood* $U \subset \mathbf{C}^n \times \mathbf{C}^k$ *of the graph of* f *into some* $\mathbf{C}^q$. *Then* f *can be uniformly approximated by a Nash mapping* $F : K \to \mathbf{C}^k$ *satisfying* $Q(z, F(z)) = 0$.

We use Theorem 3.6 in the situation where $Q = 0$ are the equations of the system (S) whereas the components of f are the components of ψ (the equations (a) are satisfied by these functions), the coefficients of W_j, the coefficients of $H_{j,i}$ (which satisfy (b) together with functions ψ) and the holomorphic functions obtained from Lemma 3.2 (note that in Lemma 3.2 the functions $a_1, \ldots, a_r$, $g_1, \ldots, g_t$ satisfy the equations $F_i = 0$, $i = 1, \ldots, s$). Formally we should remove the variable z in (b) before using Theorem 3.6: this can be done by replacing every equation from (b) by an equivalent system of equations as every function appearing in (b) is a polynomial in z.

Let ψ_ν, $W_{j,\nu}$ be the components of a mapping from the obtained sequence of Nash mappings, converging uniformly to ψ, W_j respectively in some neighborhood of $0_n \in \mathbf{C}^n$. By *Step 2,* in view of the fact that ψ_ν satisfy the equations (a) and $X = pr(X(\psi, 1))$, we know that $X_\nu \rightarrowtail X \cap ((B\theta) \times \mathbf{C}^k)$, for some $0 < \theta \leq 1$, where $X_\nu = pr(X(\psi_\nu, \theta))$. Next, by the fact that the equations (b) are satisfied, it is easy to see that the unitary polynomial $W_{j,\nu}$ in z with Nash coefficients and non-zero discriminant describes $\Phi_{\tilde{L}_j}(X(\psi_\nu, \theta)) = \Phi_{L_j}(X_\nu)$. Finally, by the fact that the coefficients of $W_{j,\nu}$ (together with some other Nash functions) satisfy (c) and by Lemma 3.2, we see that the sets $X \cap ((B\theta) \times \mathbf{C}^k)$, X_ν, for $\nu \in \mathbf{N}$, satisfy the hypotheses (3) and (4) of Lemma 3.3. (Recall that, by *Step 1* of the proof of Theorem 1.1, X satisfies (0), (1) and (2) of Lemma 3.3.) In order to apply Lemma 3.3 it is sufficient to check that $E \subset X_\nu$ for $\nu \in \mathbf{N}$. This in turn follows by Lemma 3.2(4) and by the fact that for every $(\tilde{m}+1)$-element $A \subset \mathbf{C}^k$ there is $i \in \{1, \ldots, p\}$ such that $L_i|_A$ is injective. Applying Lemma 3.3 finishes the proof of Theorem 1.1.

Let us mention that if E was not irreducible at x_0 then the only difference in the proof would be to make the preparations in *Step 1* simultaneously for every irreducible component of E and then in the construction of the system (S) in (c) include equations coming from every component of E.$\blacksquare$

Proof of Proposition 1.2. It is sufficient to modify the proof of Theorem 1.1 so we give only a sketch. First, every analytic curve is locally biholomorphically equivalent to an algebraic curve (see [12]), therefore we may assume that E is algebraic. We also may assume that X satisfies all the assumptions made in the first paragraph of *Step 1* of the proof of Theorem 1.1.

Next, the conditions (0), (1) and (2) of Lemma 3.3 may be assumed satisfied for X for every $a \in E$ (in Theorem 1.1 we had these conditions for a from a dense subset of E). Indeed, (0) can be clearly obtained. As for (1) and (2), by a, close to the identity, change of the coordinates in a neighborhood of 0_{n+k} we obtain the conditions for $a = 0_{n+k}$ (recall that $x_0 = 0_{n+k}$). Then we proceed as in the second paragraph of *Step 1* of the proof of Theorem 1.1. (So another perturbation of the coordinate system is applied. Since it can be again arbitrarily close to the identity, we may assume that (1) and (2) remain true for 0_{n+k}.) Now that (1) and (2) hold both outside a nowhere dense analytic subset of E and for 0_{n+k}, these conditions must hold in some neighborhood of 0_{n+k} in E as E is one-dimensional.

Similarly, (1) and (2) of Lemma 3.2 may be assumed satisfied with Y, S replaced by $\Phi_{L_i}(X), \Phi_{L_i}(E)$ respectively for every $i = 1, \ldots, p$, with the additional assumption that F is empty. This completes *Step 1.*

Next observe that if in Lemma 3.2 we assume that S is a curve and $F = \emptyset$ then we may take $S_\nu = \{0_{n+1}\}$ for every ν. Indeed, it is sufficient to make sure that $\frac{\partial^\mu P_\nu}{\partial z^\mu}(a) \neq 0$ for $a \in \gamma(S) \cap (\tilde{U} \times \mathbf{C}) \setminus \{0_{n+1}\}$ (see the proof). Yet, since $\frac{\partial^\mu P}{\partial z^\mu}(a) \neq 0$ for $a \in \gamma(S) \cap (\tilde{U} \times \mathbf{C}) \setminus \{0_{n+1}\}$ it suffices to ensure that the order of zero of $P - P_\nu$ at 0_{n+1} is high enough (because every irreducible component of the germ of E at 0_{n+1} is the image of an analytic homeomorphism defined on a neighborhood of zero in $\mathbf{C}$: the Puiseux theorem. Composing $\frac{\partial^\mu P}{\partial z^\mu}$ and $\frac{\partial^\mu P_\nu}{\partial z^\mu}$ with this mapping one observes that the claim is a simple consequence of the Rouché theorem). This in turn can be easily achieved by substituting a_j, $j = 1, \ldots, r$, in the equations (c) (in the proof of Lemma 3.2), by $\acute{a}_j + \bar{a}_j$ where $\bar{a}_j$ is the sum of sufficiently large finite number of the terms of the Taylor expansion of a_j about 0_n, whereas $\acute{a}_j = a_j - \bar{a}_j$. Then every $\acute{a}_j(x)$ can be expressed as $x_1^{\beta_1} \cdot \ldots \cdot x_n^{\beta_n} \tilde{a}_j(x)$, where $\tilde{a}_j(0_n) \neq 0$. Now in the equations (c) the $\tilde{a}_j$'s are replaced by new variables instead of the a_j's.

Finally, the construction of the system (S) in *Step 3* differs in point (c) from what was presented in the proof of Theorem 1.1. Here we use Lemma 3.2 in a just strengthened version and obtain two systems of equations: one for $S = E$ and the other for $S = \{0_{n+k}\}$, both included in (S). Approximating the holomorphic solutions of (S) by Nash solutions yields sets X_ν satisfying the hypotheses of Lemma 3.3 with $F = F^\nu = \emptyset$. Application of this lemma finishes the proof.■

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