On two problems concerning quasianalytic Denjoy–Carleman classes

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Abstract

Given a Denjoy–Carleman class $Q = Q_M$, consider the Hilbert space $H = H_M$ of some quasianalytic functions on the cube $(-1,1)^m$, introduced by Thilliez [5]. In our article [2], we posed the question whether polynomials are dense in H, and indicated that this open problem can be related to that of certain decompositions of functions from H with respect to their Taylor series at zero, which is of great geometric significance. In this paper we shall show that actually the latter assertion entails the former.

Let $M = (M_j)_{j \geq 0}$ be an increasing, logarithmically convex sequence of real numbers with $M_0 = 1$. An infinitely differentiable function f on an open subset $U \subseteq \mathbb{R}^m$ shall be called an ultradifferentiable function of class M, if on each compact subset $K \subset U$ there exist positive constants C and σ such that

$$|\partial^{|\alpha|} f / \partial x^{\alpha}(x)| \leq C \sigma^{|\alpha|} |\alpha|! \, M_{|\alpha|} \quad \text{for all} \quad x \in K.$$

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Logarithmical convexity ensures that the set $\mathcal{E}_M(U)$ of ultradifferentiable functions on U is an $\mathbb{R}$-algebra, and that such $\mathbb{R}$-algebras are closed under superposition of functions (a result due to Roumieu [3]).

By imposing analogous estimates, one can define the subalgebras $\mathcal{E}_m(M)$ and $\mathcal{F}_m(M)$ of the local $\mathbb{R}$-algebras $\mathcal{E}_m$ of the germs of smooth (i.e. infinitely differentiable) functions at $0 \in \mathbb{R}^m$ and $\mathcal{F}_m$ of formal power series in m variables, respectively. It is easy to check that $\mathcal{E}_m(M)$ and $\mathcal{F}_m(M)$ are both local rings. The classical Carleman theorem [1] says that, for every quasianalytic class $\mathcal{E}_m(M)$ which is larger than the ordinary algebra of analytic germs $\mathcal{O}_m$ at $0 \in \mathbb{R}^m$, the Borel mapping

$$\mathcal{E}_m(M) \ni f \longrightarrow \hat{f} \in \mathcal{F}_m(M)$$

is not surjective; here $\hat{f}$ denotes the Taylor series of f at $0 \in \mathbb{R}^m$.

Throughout the paper we shall confine ourselves only to the quasianalytic classes of ultradifferentiable functions, which amounts — due to the famous Denjoy–Carleman theorem — to the following condition imposed on the sequence $M = (M_j)_{j \geq 0}$:

$$\sum_{j=0}^{\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty.$$

We now recall the construction of a certain Hilbert space of some ultradifferentiable functions on the cube $(-1,1)^m$, introduced recently by Thilliez [5]. It is an analogue of Sobolev spaces of infinite order of type l_2, which allows one to handle simultaneously an infinite number of derivatives.

Consider the space of those smooth (i.e. of class C^∞) functions u on the cube $(-1,1)^m$ whose derivatives

$$u^\alpha := \partial^{|\alpha|} u / \partial x^\alpha, \quad \alpha \in \mathbb{N}^m,$$

extend continuously to $[-1,1]^m$; put

$$\|u\|_\infty := \sup\{|u(x)| : x \in [-1,1]^m, \quad \|u\|_2^2 := \int_{[-1,1]^m} |u(x)|^2 \, dx,$$

and

$$\|u\|^2 := \sum_{|\alpha|=0}^{\infty} \|u^\alpha\|_2^2 / (|\alpha|! \, M_{|\alpha|})^2.$$

Thilliez [5] introduced the Hilbert space $H = H_M$ of those functions u as above for which $\|u\| < \infty$. Denote by $(\cdot,\cdot)$ the associated scalar product.

Remark 1. It can be checked by induction with respect to m that there are positive constants $c, C > 0$ for which

$$c\|u\|_2 \leq \|u\|_\infty \leq C(\|u\|_2 + \|\partial u/\partial x_1\|_2 + \|\partial^2 u/\partial x_1 \partial x_2\|_2 \; + \; \cdots$$
$$\cdots \; + \|\partial^m u/\partial x_1 \ldots \partial x_m\|_2).$$

Denote by $B_{m,\sigma}(M)$ the Banach space of such ultradifferentiable functions $u : (-1,1)^m \longrightarrow \mathbb{R}$ that

$$|\partial^{|\alpha|} u/\partial x^\alpha(x)| \leq C\sigma^{|\alpha|}|\alpha|!\, M_{|\alpha|} \quad \text{for all} \;\; x \in (-1,1)^m,$$

with norm

$$\|u\|_{m,\sigma} := \sup\{\frac{|\partial^{|\alpha|} u/\partial x^\alpha(x)|}{\sigma^{|\alpha|}|\alpha|!\, M_{|\alpha|}} \;:\; x \in (-1,1)^m, \; \alpha \in \mathbb{N}^m\}.$$

By Ascoli's theorem, for any $0 < \sigma < \sigma'$, the canonical inclusions

$$B_{m,\sigma}(M) \subset B_{m,\sigma'}(M)$$

are compact operators. Consequently, for any $\eta > 0$, we have the following topological inclusions

$$B_{m,1-\eta}(M) \subset H_M \subset B_{m,1+\eta}(M')$$

with

$$M' = (M'_j)_{j\geq 0}, \quad M'_j := M_{j+m},$$

which are compact operators too.

Remark 2. The Hilbert space $H = H_M$ is an analogue of Sobolev spaces of infinite order of type l_2, which allows one to handle simultaneously an infinite number of derivatives.

We now turn to the following two problems concerning a quasianalytic Denjoy-Carleman class, posed in our previous paper [2] in connection with some significant geometric applications.

Problem I. *Let $f \in H$ and $\hat{f}$ be its Taylor series at $0 \in \mathbb{R}^m$. Split the set $\mathbb{N}^m$ of exponents into two disjoint subsets A and B, $\mathbb{N}^m = A \cup B$, and decompose the formal series $\hat{f}$ into the sum of two formal series G and H, supported by A and B, respectively. Do there exist $g, h \in H$ with Taylor series at zero G and H, respectively?*

Problem II. *Are polynomial mappings dense in the Hilbert space H?*

Here we shall prove that actually the former assertion entails the latter. Supposing, on the contrary, that polynomials are not dense in H, we shall construct disjoint subsets A and B of $\mathbb{N}^m = A \cup B$ which do not satisfy assertion I.

Let us adopt the following notation:
for each $\alpha \in \mathbb{N}^m$, e_α denotes the monomial $e_\alpha(x) := x^\alpha/\alpha!$; the mapping

$$H \longrightarrow \mathbb{R}, \quad u \mapsto \partial^{|\alpha|} u / \partial x^\alpha (0)$$

is a continuous linear form on H, and thus there is a unique element $h_\alpha \in H$ such that

$$\partial^{|\alpha|} u / \partial x^\alpha (0) = (u, h_\alpha) \quad \text{for all} \quad u \in H;$$

we have $(h_\alpha, e_\beta) = \delta_{\alpha,\beta}$ (Kronecker's delta). For simplicity, let us number the multi-indices $\alpha \in \mathbb{N}^m$ by the non-negative integers $i \in \mathbb{N}$. Then, in particular, we have $(h_i, e_j) = \delta_{i,j}$.

Further in the proof, we use the well-known fact that, given two subspaces E_1 and E_2 of a topological vector space E with $\dim E_2 < \infty$, if E_1 is closed, so is the sum $E_1 + E_2$ (see e.g. [4], Th. 1.42). It follows from the assumption of quasi-analyticity that the h_i's span a dense linear subspace of H; in other words, the Hilbert subspace $\langle h_i : i \in \mathbb{N} \rangle$ generated by the h_i is H. By the two facts above, we get

$$H_{>n} := \langle h_i : i > n \rangle = \langle e_1, \dots, e_n \rangle^\perp.$$

Hence

$$\langle e_j : j \in \mathbb{N} \rangle = \bigcap_{n \in \mathbb{N}} H_{>n}.$$

But our assumption means that $\langle e_j : j \in \mathbb{N} \rangle$ is a proper subspace of H, and therefore $\langle e_j : j \in \mathbb{N} \rangle^\perp \neq (0)$. Take thus an element

$$u \in \bigcap_{n \in \mathbb{N}} H_{>n} \quad \text{with} \quad \|u\| = 1.$$

For each $n \in \mathbb{N}$, we have

$$u = \lim_{l \to \infty} u_{n,l} \quad \text{with some} \quad u_{n,l} \in \operatorname{lin}\{h_i : i > n\};$$

one can, of course, assume that $\|u - u_{n,l}\| < 1/2^{n+l}$. Then

$$\|u - u_{n,n}\| < 1/2^{n+n}, \quad \text{whence} \quad u_{n,n} \to u \;\text{ when }\; n \to \infty.$$

We shall now define recursively an increasing sequence $(n_k)_{k \in \mathbb{N}}$ of integers as follows. Put $n_1 := 0$; having already defined integers $n_1, \ldots, n_k$, we consider the element u_{n_k,n_k}, which is, by construction, a linear combination of some elements $h_{i_1}, \ldots, h_{i_r}$ with $i_1, \ldots, i_r > n_{k-1}$. Pick as n_{k+1} any integer larger than the integers $i_1, \ldots, i_r$.

Putting $v_k := u_{n_k,n_k}$, and further

$$A := ([n_1, n_2) \cup [n_3, n_4) \cup \ldots) \cap \mathbb{N}, \quad B := ([n_2, n_3) \cup [n_4, n_5) \cup \ldots) \cap \mathbb{N},$$

we get

$$v_{2k+1} \in H_1 := \langle h_i : i \in A \rangle \quad \text{and} \quad v_{2k} \in H_2 := \langle h_i : i \in B \rangle \quad \text{for all} \;\; k \in \mathbb{N}.$$

Therefore

$$u = \lim_{k \to \infty} v_{2k+1} = \lim_{k \to \infty} v_{2k} \in H_1 \cap H_2,$$

and thus $H_1 \cap H_2 \neq (0)$. Hence

$$H_1^{\perp} + H_2^{\perp} \subseteq (H_1 \cap H_2)^{\perp} \subset H$$

are proper subspaces of the Hilbert space H. This completes the proof, because the subspaces $H_1^{\perp}$ and $H_2^{\perp}$ coincide with the subspaces of those functions from H whose Taylor series at $0 \in \mathbb{R}^m$ are supported by the sets B and A, respectively.

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