On two problems concerning quasianalytic Denjoy–Carleman classes

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Abstract

Given a Denjoy–Carleman class $Q = Q_M$, consider the Hilbert space $H = H_M$ of some quasianalytic functions on the cube $(-1, 1)^m$, introduced by Thilliez [5]. In our article [2], we posed the question whether polynomials are dense in $H$, and indicated that this open problem can be related to that of certain decompositions of functions from $H$ with respect to their Taylor series at zero, which is of great geometric significance. In this paper we shall show that actually the latter assertion entails the former.

Let $M = (M_j)_{j \geq 0}$ be an increasing, logarithmically convex sequence of real numbers with $M_0 = 1$. An infinitely differentiable function $f$ on an open subset $U \subseteq \mathbb{R}^m$ shall be called an ultradifferentiable function of class $M$, if on each compact subset $K \subset U$ there exist positive constants $C$ and $\sigma$ such that

$$|\partial^{[\alpha]} f / \partial x^\alpha(x)| \leq C \sigma^{|\alpha|} |\alpha|! M_{|\alpha|}$$

for all $x \in K$.

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Logarithmical convexity ensures that the set $E_M(U)$ of ultradifferentiable functions on $U$ is an $\mathbb{R}$-algebra, and that such $\mathbb{R}$-algebras are closed under superposition of functions (a result due to Roumieu [3]).

By imposing analogous estimates, one can define the subalgebras $E_m(M)$ and $F_m(M)$ of the local $\mathbb{R}$-algebras $E_m$ of the germs of smooth (i.e. infinitely differentiable) functions at $0 \in \mathbb{R}^m$ and $F_m$ of formal power series in $m$ variables, respectively. It is easy to check that $E_m(M)$ and $F_m(M)$ are both local rings. The classical Carleman theorem [1] says that, for every quasianalytic class $E_m(M)$ which is larger than the ordinary algebra of analytic germs $\mathcal{O}_m$ at $0 \in \mathbb{R}^m$, the Borel mapping

$$E_m(M) \ni f \longrightarrow \hat{f} \in F_m(M)$$

is not surjective; here $\hat{f}$ denotes the Taylor series of $f$ at $0 \in \mathbb{R}^m$.

Throughout the paper we shall confine ourselves only to the quasianalytic classes of ultradifferentiable functions, which amounts — due to the famous Denjoy–Carleman theorem — to the following condition imposed on the sequence $M = (M_j)_{j \geq 0}$:

$$\sum_{j=0}^{\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty.$$

We now recall the construction of a certain Hilbert space of some ultradifferentiable functions on the cube $(-1,1)^m$, introduced recently by Thilliez [5]. It is an analogue of Sobolev spaces of infinite order of type $l_2$, which allows one to handle simultaneously an infinite number of derivatives.

Consider the space of those smooth (i.e. of class $C^\infty$) functions $u$ on the cube $(-1,1)^m$ whose derivatives

$$u^\alpha := \frac{\partial^{|\alpha|} u}{\partial x^\alpha}, \quad \alpha \in \mathbb{N}^m,$$

extend continuously to $[-1,1]^m$; put

$$\|u\|_{\infty} := \sup \{|u(x)| : x \in [-1,1]^m\}, \quad \|u\|_2^2 := \int_{[-1,1]^m} |u(x)|^2 \, dx,$$

and

$$\|u\|_2^2 := \sum_{|\alpha|=0}^{\infty} \|u^\alpha\|_2^2 / (|\alpha|! M_{|\alpha|})^2.$$

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Thilliez [5] introduced the Hilbert space $H = H_M$ of those functions $u$ as above for which $\|u\| < \infty$. Denote by $(\cdot, \cdot)$ the associated scalar product.

**Remark 1.** It can be checked by induction with respect to $m$ that there are positive constants $c, C > 0$ for which

$$c\|u\|_2 \leq \|u\|_\infty \leq C(\|u\|_2 + \|\partial u/\partial x_1\|_2 + \|\partial^2 u/\partial x_1 \partial x_2\|_2 + \cdots$$

$$+ \|\partial^m u/\partial x_1 \ldots \partial x_m\|_2).$$

Denote by $B_{m,\sigma}(M)$ the Banach space of such ultradifferentiable functions $u : (-1, 1)^m \to \mathbb{R}$ that

$$|\partial^{[\alpha]} u/\partial x^{\alpha}(x)| \leq C \sigma^{[\alpha]} \| \alpha \| M_{[\alpha]} \quad \text{for all} \quad x \in (-1, 1)^m,$$

with norm

$$\|u\|_{m,\sigma} := \sup \left\{ \frac{|\partial^{[\alpha]} u/\partial x^{\alpha}(x)|}{\sigma^{[\alpha]} \| \alpha \| M_{[\alpha]}} : x \in (-1, 1)^m, \alpha \in \mathbb{N}^m \right\}.$$  

By Ascoli’s theorem, for any $0 < \sigma < \sigma'$, the canonical inclusions

$$B_{m,\sigma}(M) \subset B_{m,\sigma'}(M)$$

are compact operators. Consequently, for any $\eta > 0$, we have the following topological inclusions

$$B_{m,1-\eta}(M) \subset H_M \subset B_{m,1+\eta}(M')$$

with

$$M' = (M'_j)_{j \geq 0}, \quad M'_j := M_{j+m},$$

which are compact operators too.

**Remark 2.** The Hilbert space $H = H_M$ is an analogue of Sobolev spaces of infinite order of type $l_2$, which allows one to handle simultaneously an infinite number of derivatives.

We now turn to the following two problems concerning a quasianalytic Denjoy-Carleman class, posed in our previous paper [2] in connection with some significant geometric applications.
**Problem I.** Let \( f \in H \) and \( \hat{f} \) be its Taylor series at \( 0 \in \mathbb{R}^m \). Split the set \( \mathbb{N}^m \) of exponents into two disjoint subsets \( A \) and \( B \), \( \mathbb{N}^m = A \cup B \), and decompose the formal series \( \hat{f} \) into the sum of two formal series \( G \) and \( H \), supported by \( A \) and \( B \), respectively. Do there exist \( g, h \in H \) with Taylor series at zero \( G \) and \( H \), respectively?

**Problem II.** Are polynomial mappings dense in the Hilbert space \( H \)?

Here we shall prove that actually the former assertion entails the latter. Supposing, on the contrary, that polynomials are not dense in \( H \), we shall construct disjoint subsets \( A \) and \( B \) of \( \mathbb{N}^m = A \cup B \) which do not satisfy assertion I.

Let us adopt the following notation: for each \( \alpha \in \mathbb{N}^m \), \( e_\alpha \) denotes the monomial \( e_\alpha(x) := x^\alpha / \alpha! \); the mapping

\[
H \to \mathbb{R}, \quad u \mapsto \partial |\alpha| \frac{u}{\partial x^\alpha}(0)
\]

is a continuous linear form on \( H \), and thus there is a unique element \( h_\alpha \in H \) such that

\[
\partial |\alpha| \frac{u}{\partial x^\alpha}(0) = (u, h_\alpha) \quad \text{for all } \quad u \in H;
\]

we have \((h_\alpha, e_\beta) = \delta_{\alpha,\beta} \) (Kronecker’s delta). For simplicity, let us number the multi-indices \( \alpha \in \mathbb{N}^m \) by the non-negative integers \( i \in \mathbb{N} \). Then, in particular, we have \((h_i, e_j) = \delta_{i,j} \).

Further in the proof, we use the well-known fact that, given two subspaces \( E_1 \) and \( E_2 \) of a topological vector space \( E \) with \( \dim E_2 < \infty \), if \( E_1 \) is closed, so is the sum \( E_1 + E_2 \) (see e.g. [4], Th. 1.42). It follows from the assumption of quasi-analyticity that the \( h_i \)'s span a dense linear subspace of \( H \); in other words, the Hilbert subspace \( \langle h_i : i \in \mathbb{N} \rangle \) generated by the \( h_i \) is \( H \). By the two facts above, we get

\[
H_{>n} := \langle h_i : i > n \rangle = \langle e_1, \ldots, e_n \rangle^\perp.
\]

Hence

\[
\langle e_j : j \in \mathbb{N} \rangle = \bigcap_{n \in \mathbb{N}} H_{>n}.
\]

But our assumption means that \( \langle e_j : j \in \mathbb{N} \rangle \) is a proper subspace of \( H \), and therefore \( \langle e_j : j \in \mathbb{N} \rangle^\perp \neq (0) \). Take thus an element

\[
u \in \bigcap_{n \in \mathbb{N}} H_{>n} \quad \text{with} \quad \|u\| = 1.
\]
For each \( n \in \mathbb{N} \), we have
\[
   u = \lim_{l \to \infty} u_{n,l} \text{ with some } u_{n,l} \in \text{lin } \{ h_i : i > n \};
\]
one can, of course, assume that 
\[
   \| u - u_{n,l} \| < 1/2^{n+l}.
\]
Then
\[
   \| u - u_{n,n} \| < 1/2^{n+n}, \text{ whence } u_{n,n} \to u \text{ when } n \to \infty.
\]
We shall now define recursively an increasing sequence \( (n_k)_{k \in \mathbb{N}} \) of integers as follows. Put \( n_1 := 0 \); having already defined integers \( n_1, \ldots, n_k \), we consider the element \( u_{n_k,n_k} \), which is, by construction, a linear combination of some elements \( h_{i_1}, \ldots, h_{i_r} \) with \( i_1, \ldots, i_r > n_{k-1} \). Pick as \( n_{k+1} \) any integer larger than the integers \( i_1, \ldots, i_r \).

Putting \( v_k := u_{n_k,n_k} \), and further
\[
   A := ([n_1,n_2) \cup [n_3,n_4) \cup \ldots) \cap \mathbb{N}, \quad B := ([n_2,n_3) \cup [n_4,n_5) \cup \ldots) \cap \mathbb{N},
\]
we get
\[
   v_{2k+1} \in H_1 := \langle h_i : i \in A \rangle \quad \text{and} \quad v_{2k} \in H_2 := \langle h_i : i \in B \rangle \quad \text{for all } k \in \mathbb{N}.
\]
Therefore
\[
   u = \lim_{k \to \infty} v_{2k+1} = \lim_{k \to \infty} v_{2k} \in H_1 \cap H_2,
\]
and thus \( H_1 \cap H_2 \neq (0) \). Hence
\[
   H_1^\perp + H_2^\perp \subseteq (H_1 \cap H_2)^\perp \subset H
\]
are proper subspaces of the Hilbert space \( H \). This completes the proof, because the subspaces \( H_1^\perp \) and \( H_2^\perp \) coincide with the subspaces of those functions from \( H \) whose Taylor series at \( 0 \in \mathbb{R}^m \) are supported by the sets \( B \) and \( A \), respectively.
References


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