A NOTE ON SESHADRI CONSTANTS AND PACKING NUMBERS

HALSZKA TUTAJ-GASIŃSKA

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Abstract. We present another proof of Lazarsfeld's theorem from [9], connecting Seshadri constants with packing numbers.

1. Seshadri constants

Let X be a projective algebraic manifold with an ample line bundle L. Let $P_1, ..., P_r$ be r different points on X. Seshadri constants, introduced introduced by Demailly in [5], are defined as follows.

Definition 1. *Seshadri constant of L in $P_1, ..., P_r$ is defined as the number*

$$\varepsilon(L, P_1, ..., P_r) := inf\,\{\frac{LC}{mult_{P_1}C + ... + mult_{P_r}C} \mid C \text{ is a curve on } X\},$$

or, equivalently

$$\varepsilon(L, P_1, ..., P_r) := \ sup\ \{\varepsilon \mid \pi^* L - \varepsilon(E_1 + ... + E_r) \text{ is numerically effective}\},$$

where $\pi : \tilde{X} \longrightarrow X$ is the blow up of X in $P_1, ..., P_r$.

Remark 2. This definition is stated for an algebraic manifold of any dimension $\dim X \geq 2$. In this note we are interested only in algebraic surfaces, so from now on we restrict our considerations to the case $\dim X = 2$.

For $P_1, ..., P_r$ general on X we will write $\varepsilon(L, r)$ instead of $\varepsilon(L, P_1, ..., P_r)$.

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Remark 3. *It follows from the definition that for an ample line bundle L on X we have*

$$0 < \varepsilon(L, P_1, ..., P_r) \leq \sqrt{\frac{L^2}{r}}.$$

Finding the exact value of Seshadri constants is in most cases a difficult problem. For $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ the exact values of $\varepsilon(L, r)$ are known only if $r \leq 9$ or $r = k^2, k \in \mathbb{N}$. The famous conjecture of Nagata states that $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1), r) = \sqrt{\frac{1}{r}}$ (so is maximal possible) for $r \geq 10$. (cf [8]). The generalized conjecture, called Nagata-Biran Conjecture, says that for any algebraic surface X with an ample line bundle L there exists a number N, such that for all $r \geq N$ $\varepsilon(L, r) = \sqrt{\frac{L^2}{r}}$. (cf eg [14]).
So far, all known values of Seshadri constants are rational. In general, it is hard to find the value of a Seshadri constant even in one point. In case we can prove the existence of so called submaximal curves, ie curves C on X, such that

$$\frac{LC}{\mathrm{mult}_{P_1} C + ... + \mathrm{mult}_{P_r} C} < \sqrt{\frac{L^2}{r}},$$

we obtain that the Seshadri constant is necessarily rational and less than the maximal value $\sqrt{\frac{L^2}{r}}$, cf eg [13].
On the other hand, there are (so far) not many ways of proving nonexistence of submaximal curves. This makes very difficult proving that the Seshadri constants are maximal. One way to attack the problem is to give a lower bound on the Seshadri constants. A result proved by Lazarsfeld in [9] allows us to give a lower bound on a Seshadri constant by means of so called packing numbers, for the definition see next paragraph.
In this note we are going to give another proof of his result.

2. Packing numbers

Let us remind that a symplectic manifold is a smooth real manifold (of dimension $2n$) with a closed nondegenerate differential $2-$form ω. The volume form in X is given by $\frac{1}{n!}\omega^{\wedge n}$. The classical example is $\mathbb{R}^{2n}$ with the 2-form $\omega_0 := dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$.
Another example is given by an algebraic surface X with an ample line bundle L. This surface may be treated as a real four dimensional manifold with the closed nondegenerate differential 2-form given by the first Chern class of L, $\omega_L = c_1(L)$. Thus, X is a symplectic manifold, with the volume given as $\mathrm{vol}(X) = \frac{1}{2}L^2$.

If (X_1, ω_1) and (X_2, ω_2) are two symplectic manifolds, we define a symplectic embedding of X_1 to X_2 as follows.

Definition 4. *We say that $f : X_1 \longrightarrow X_2$ is a symplectic embedding if f is a $\mathcal{C}^\infty$-diffeomorphism onto the image and*

$$f^* \omega_2 = \omega_1.$$

We will use the notation

$$f : (X_1, \omega_1) \xrightarrow{s} (X_2, \omega_2).$$

Let (X, ω) be a symplectic manifold (of dimension $2n$) and let $(B^{2n}(R), \omega_0)$ be a ball of radius R in $\mathbb{R}^{2n}$ with the standard symplectic form $\omega_0 = dx_1 \wedge dy_1 + ... + dx_n \wedge dy_n$. We may consider the so called symplectic packing problem: find a maximal radius R such that there exists a symplectic embedding of the disjoint sum of r balls of radius R into a given symplectic manifold (X, ω),

$$f : \coprod_{i=1}^{r} (B^{2n}(R), \omega_0) \xrightarrow{s} (X, \omega).$$

If the volume of X is finite, than there is an obvious upper bound on R, $r\mathrm{vol}(B^{2n}(R)) \leq \mathrm{vol}(X)$. However, even if the volume of X is infinite, there may be obstructions in packing balls into X, for example the famous Gromov Nonsqueezing Theorem (see [7]) says that if there exists a symplectic embedding of a ball $B^{2n}(R)$ into $(B^2(\epsilon) \times \mathbb{R}^{2n-2}, \omega_0)$, then $R \leq \epsilon$.
Assume now, that the volume of a symplectic manifold X is finite. To measure how much of the volume of (X, ω) we may pack with the symplectic images of balls we define so called packing numbers (cf [2],[10]).

Definition 5. *Let (X, ω) be a symplectic manifold and let r be a natural number. A (symplectic) packing number is defined as*

$$v_r := \sup \left\{ \frac{r\,vol(B^{2n}(R))}{vol(X)} \right\},$$

where the supremum is taken over all R, such that there exists a symplectic packing $f : \coprod_{i=1}^{r} (B^{2n}(R), \omega_0) \xrightarrow{s} (X, \omega)$.
If $v_r = 1$ we say that full packing exists.

From now on we restrict our considerations to the case when X is an algebraic surface with ample line bundle L, so with the symplectic form ω_L given by the first Chern class of L. Then, X is four dimensional over reals and has a finite volume, $\mathrm{vol}(X) = \frac{1}{2} L^2$.
As our X is now symplectic and complex manifold we may define similar constants for embeddings being both symplectic and holomorphic:

Definition 6. *Let (X,ω) be a symplectic and holomorphic manifold and let r be a natural number. A symplectic and holomorphic packing number is defined as*

$$v_r^h := \sup\left\{\frac{r\,vol(B^4(R))}{vol(X)}\right\},$$

where the supremum is taken over all R, such that there exists a symplectic and holomorphic packing $f : \coprod_{i=1}^r (B^4(R),\omega_0) \overset{s,hol}{\longrightarrow} (X,\omega)$.

There are many interesting results about the constants v_r, cf eg [2], [3], [10]. In his famous paper [3], Biran proved the following theorem (here quoted in the version restricted to algebraic surfaces with the symplectic form ω_L):

Theorem 7. *Let (X,L) be projective algebraic surface, treated as a four dimensional symplectic manifold with the symplectic form ω_L. Then there exists a number N_0, such that for any $r \geq N_0$ there exists full packing, ie $v_r = 1$. Moreover, this N_0 can be taken equal $k_0^2L^2$ where k_0 is such, that the linear system k_0L contains a curve C of genus at least one.*

It seems that there exists a close connection between Seshadri constants and packing numbers. This connection was first stated in [10] and then in [2], [3], [9] and others.
Consider the following example. Let $X = \mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$. For $r = 1,...,9$ we have $\varepsilon(L,r) = 1, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{2}{5}, \frac{2}{5}, \frac{3}{8}, \frac{6}{17}, \frac{1}{3}$ respectively. In the same range of r, we have (cf [2]): $v_r = 1, \frac{1}{2}, \frac{3}{4}, 1, \frac{20}{25}, \frac{24}{25}, \frac{63}{64}, \frac{288}{289}, 1$, so $\varepsilon(L,r) = \sqrt{\frac{L^2v_r}{r}}$ here. For $r \geq 10$ we know by the results of Biran, [3, 2], that $v_r = 1$, whereas $\varepsilon(L,r)$ is still unknown (unless r is a square of a natural number, and then $\varepsilon(L,r) = \frac{1}{\sqrt{r}}$, cf eg [8]). As we have mentioned above, Nagata conjecture says that $\varepsilon(L,r) = \frac{1}{\sqrt{r}}$ for all $r > 9$, (cf eg [12, 8, 14]), so conjecturally, for any r $\varepsilon(L,r) = \sqrt{v_r\frac{L^2}{r}}$ for $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$. Perhaps this conjecture is true in general, ie $\varepsilon(L,r) = \sqrt{v_r\frac{L^2}{r}}$, for any polarized algebraic surface.
In [4] Biran and Cieliebak proved that we have the following upper bound on Seshadri constants by means of symplectic packing numbers:

Theorem 8. *With the notation as above*

$$\sqrt{v_r\frac{L^2}{r}} \geq \varepsilon(L,r).$$

On the other hand holomorphic and symplectic packing numbers give the lower bound. Lazarsfeld in [9] proved that

Theorem 9. $\varepsilon(L,r) \geq \sqrt{\frac{v_r^hL^2}{r}}$.

Remark 10. *Lazarsfeld's proof of this result is based on the construction of symplectic blowing up, cf* [10]. *The theorem in* [9] *is actually stated for* $r = 1$, *but the proof for any* r *is analogous.*

3. Proof

In this section we present the proof of theorem 9, using some facts form Geometric Measure Theory. (cf [11], [6], [1]).

Definition 11. *Let* S *be a surface in* $\mathbb{R}^4$. *We say that* S *is minimal if the mean curvature of* S *is zero.*

Remark 12. *Any analytic curve in* $\mathbb{C}^2$ *is a minimal surface.*

Let now S be a minimal surface in $\mathbb{R}^4$, being in the same time an analytic curve in $\mathbb{C}^2$. Let S pass through a point $P \in \mathbb{R}^4$. As S is analytic, the multiplicity of S in P is defined. Assume that $\text{mult}_P S = m$. Take then a ball $B^4(R)$, with the center P. By the volume of $S \cap B^4(R)$ we mean the area of S (in the Euclidean metric). Wirtinger's Theorem says that in this situation, the volume of a surface equals the integral from the symplectic form on S:

Remark 13. *(Wirtinger's Theorem, see* [6]*).*
1. In the situation as above, $vol(S \cap B^4(R)) = \int_{S \cap B^4(R)} \omega_0$.
2. If C *is an analytic curve on a polarized surface* (X, L), *then* $volC = LC$.

The following fact will be crucial for us.

Theorem 14. *(Monotonicity Lemma, see* [11], *Theorem 9.3). In the situation described above*

$$vol(S \cap B^4(R)) \geq m\pi R^2.$$

Let now (X, L) be our algebraic surface, with an ample line bundle L and symplectic form ω_L. Take R, such that there exists f, a symplectic and holomorphic embedding of r disjoint balls of radius R into X.
Let $f(P_1), ..., f(P_r)$ be the images of the centers of these balls. Take C, an algebraic curve on X, passing through $f(P_1), ..., f(P_r)$ with multiplicities $m_1, ..., m_r$ respectively. Let $S_i := f^{-1}(C \cap f(B(P_i, R)))$, where $B(P_i, R)$ denotes the ball of radius R with the center P_i. As f is symplectic and holomorphic, S_i is an analytic curve in $B(P_i, R)$. Moreover, $\text{mult}_{P_i} S_i = m_i$. From Monotonicity Lemma it follows that $\text{vol}(S_i) \geq m_i \pi R^2$.
Thus,

$$LC = \text{vol}(C) \geq \sum_{i=1}^{r} \text{vol}(C \cap f(B(P_i, R))) \overset{f \text{is symplectic}}{=} \sum_{i=1}^{r} \text{vol}(S_i)$$

$$\overset{\text{Monotonicity Lemma}}{\geq} \sum_{i=1}^{r} m_i \pi R^2.$$

From this,

$$\frac{LC}{\sum_{i=1}^{r} m_i} \geq \pi R^2,$$

for any R such, that symplectic and holomorphic embedding exists. Thus

$$\varepsilon(L,r) \geq \pi R^2$$

and from the definition of v_r^h, and the fact that the volume of the unit ball in $(\mathbb{R}^4, \omega_0)$ is $\frac{\pi^2}{2}$, we get the required inequality

$$\varepsilon(L,r) \geq \sqrt{\frac{L^2 v_r^h}{r}}.$$

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Halszka Tutaj-Gasińska
Jagiellonian University,
Institute of Mathematics,
Reymonta 4, 30 075 Kraków, Poland
htutaj@im.uj.edu.pl