Two illustrative examples of spaces with maximal projection constant

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Abstract

Let V be an n-dimensional real Banach space and let $\lambda(V)$ denote its absolute projection constant. For any $N \in \mathbb{N}$, $N \geq n$ define

$$\lambda_n^N = \sup\{\lambda(V) : dim(V) = n, V \subset l_\infty^{(N)}\}$$

and

$$\lambda_n = \sup\{\lambda(V) : dim(V) = n\}.$$

A well-known Grünbaum conjecture ([6], p. 465) says that

$$\lambda_2 = 4/3.$$

In this paper we show that

$$\lambda_3^5 = \frac{5 + 4\sqrt{2}}{7}$$

and we determine a three-dimensional space $V \subset l_\infty^{(5)}$ satisfying $\lambda_3^5 = \lambda(V)$. In particular, this shows that Prop. 3.1 from ([11], p. 259)

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is incorrect. Hence the proof of the Grünbaum conjecture given in ([11]) which is based on Prop. 3.1 is incomplete. In the second part of this paper an alternative proof of the Grünbaum conjecture will be presented.

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1 Introduction

Let X be a real Banach space and let $V \subset X$ be a finite-dimensional subspace. A linear, continuous mapping $P : X \to V$ is called *a projection* if $P|_V = id|_V$. Denote by $\mathcal{P}(X,V)$ the set of all projections from X onto V. Set

$$\lambda(V,X) = \inf\{\|P\| : P \in \mathcal{P}(X,V)\}$$

and

$$\lambda(V) = \sup\{\lambda(V,X) : V \subset X\}.$$

The constant $\lambda(V,X)$ is called the *relative projection constant* and $\lambda(V)$ the *absolute projection constant.* General bounds for absolute projection constants were studied by many authors (see e.g. [2, 3, 8, 9, 10, 12, 14]). It is well-known (see e.g [15]) that if V is a finite-dimensional space then

$$\lambda(V) = \lambda(I(V), l_\infty\},$$

where $I(V)$ denotes any isometric copy of V in l_∞. Denote for any $n \in \mathbb{N}$

$$\lambda_n = \sup\{\lambda(V) : dim(V) = n\}$$

and for any $N \in \mathbb{N}$, $N \geq n$

$$\lambda_n^N = \sup\{\lambda(V) : V \subset l_\infty^{(N)}\}.$$

By the Kadec-Snobar Theorem (see [7]) $\lambda(V) \leq \sqrt{n}$ for any $n \in \mathbb{N}$. However, determination of the constant λ_n seems to be difficult. In ([6], p.465) it was conjectured by B. Grünbaum that

$$\lambda_2 = 4/3.$$

In ([11], Th. 1.1) an attempt has been made to prove the Grübaum conjecture (and a more general result). The proof presented in this paper is mainly based on ([11]), Proposition 3.1, p. 259 and ([11], Lemma 5.1, p. 273). Unfortunately, the proof of Proposition 3.1 is incorrect. In fact the formula (3.19) from ([11], p. 263) is false. This can be easily checked differentiating formula (3.12) on page 262 with respect to the proper variable. Because of this error, the part of the proof of [11], Proposition 3.1 on p. 265 is incorrect and as a result, the proof of [11], Th. 1.1 is incomplete.
In the first part of this paper we show that

$$\lambda_3^5 = \frac{5 + 4\sqrt{2}}{7}$$

and we determine a three-dimensional space $V \subset l_\infty^{(5)}$ satisfying $\lambda_3^5 = \lambda(V)$. In particular, this shows that not only the proof of Proposition 3.1 from ([11]) is incorrect but also the statement of Proposition 3.1 is incorrect.
In the second part of this paper we present an alternative proof of the Grünbaum conjecture, which is based on the proof given for λ_3^5.
Now we briefly describe the structure of the paper.
In Section 1 we demonstrate some preliminary results useful as well as for determination of λ_3^5 and the proof of the Grünbaum conjecture.
In Section 2 after proving some preliminary lemmas we determine the constant λ_3^5.
Section 3 contains a proof of the Grünbaum conjecture based on the proof given in Section 2.
The main tools applied in both proofs are the Lagrange Multiplier Theorem and the Implicit Function Theorem.

2 Preliminary results

In this section mainly we consider the following problem. For a fixed $u_1 \in [0, 1]$ maximize a function $f_{u_1} : \mathbb{R}^{N-1} \times (\mathbb{R}^N)^n \to R$ defined by:

$$f_{u_1}((u_2, ..., u_N), x^1, ..., x^n) = \sum_{i,j=1}^{N} u_i u_j | < x_i, x_j >_n | \tag{1}$$

under constraints:

$$< x^i, x^j >_N = \delta_{ij}, 1 \leq i \leq j \leq n; \tag{2}$$

$$\sum_{j=2}^{N} u_j^2 = 1 - u_1^2. \qquad (3)$$

Here for $j = 1, ..., N$, $x_j = ((x^1)_j, ..., (x^n)_j)$, $< w, z >_n = \sum_{j=1}^n w_j z_j$ for any $w = (w_1, ..., w_n), z = (z_1, ..., z_n) \in \mathbb{R}^n$ and $< p, q >_N = \sum_{j=1}^N p_j q_j$ for any $p = (p_1, ..., p_N), q = (q_1, ..., q_N) \in \mathbb{R}^N$. Also we will work with

$$f_{u_1,A}((u_2, ..., u_N), x^1, ..., x^n) = \sum_{i,j=1}^{N} u_i u_j a_{ij} < x_i, x_j >_n, \qquad (4)$$

where $A = \{a_{ij}\}$ is a fixed $N \times N$ symmetric matrix.

LEMMA 2.1 *Let $C = (c_{ij})_{i,j=1,...,n})$ be a real $n \times n$ orthonormal matrix. Then for any $x^1, ..., x^n, u \in \mathbb{R}^N$ satisfying (2, 3),*

$$f_{u_1}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1}((u_2, ..., u_N), C(x^1), ..., C(x^n)),$$

and

$$f_{u_1,A}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1,A}((u_2, ..., u_N), C(x^1), ..., C(x^n))$$

for any $N \times N$ matrix A. Here $C(x^i) = \sum_{j=1}^n c_{ij} x^j$.

Proof. Note that

$$< Cx^i, Cx^j >_N = < \sum_{k=1}^{n} c_{ik} x^k, \sum_{l=1}^{n} c_{jl} x^l >_N$$

$$= \sum_{k,l=1}^{n} c_{ik} c_{jl} < x^k, x^l >_N = \sum_{k,l=1}^{n} a_{ik} a_{jl} \delta_{kl} = \sum_{k=1}^{n} c_{ik} c_{jk} = \delta_{ij},$$

which shows that u and Cx^i $i = 1, ..., n$ satisfy (2, 3). Note that for $i = 1, ..., N$ and $j = 1, ..., n$

$$(Cx^j)_i = \sum_{k=1}^{n} c_{jk} (x^k)_i.$$

Denote for $i = 1, ..., N$ $(Cx)_i = ((Cx^1)_i, ..., (Cx^n)_i)$. Notice that for $i, j = 1, ..., N$,

$$< (Cx)_i, (Cx)_j >_n = \sum_{l=1}^{n} (\sum_{k=1}^{n} c_{lk} (x^k)_i)(\sum_{u=1}^{n} c_{lu} (x^u)_j)$$

$$= \sum_{k,u=1}^{n} \sum_{l=1}^{n} (c_{lk}(x^k)_i c_{lu}(x^u)_j) = \sum_{k,u=1}^{n} (x^k)_i (x^u)_j \sum_{l=1}^{n} c_{lk} c_{lu}$$

$$= \sum_{k,u=1}^{n} (x^k)_i (x^u)_j \delta_{ku} = \sum_{k=1}^{n} (x^k)_i (x^k)_j =< x_i, x_j >_n .$$

By (1) and (4)

$$f_{u_1}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1}((u_2, ..., u_N), C(x^1), ..., C(x^n))$$

and

$$f_{u_1,A}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1,A}((u_2, ..., u_N), C(x^1), ..., C(x^n))$$

which shows our claim. ■

Now we recall the following well-known

LEMMA 2.2 *Let* $(X, < \cdot, \cdot >)$ *be a finite-dimensional Hilbert space with an orthonormal basis* $x^1, ..., x^n$. *Let* $T : X \to X$ *be a linear isometry. If* C *is an* $n \times n$ *matrix with columns* $c_j = (c_{1j}, ..., c_{nj})$ *defined by*

$$Tx^j = \sum_{i=1}^{n} c_{ji} x^i,$$

then C *is an orthonormal matrix.*

Proof. Notice that for any $j = 1, ..., n$,

$$1 =< x^j, x^j >=< Tx^j, Tx^j >=< \sum_{i=1}^{n} c_{ij} x^i, \sum_{l=1}^{n} c_{lj} x^l >= \sum_{i,l=1}^{n} c_{ij} c_{lj} < x^i, x^l >$$

$$= \sum_{i,l=1}^{n} c_{ij} c_{lj} \delta_{ij} = \sum_{i=1}^{n} (c_{ij})^2.$$

Also for any $i, j \in \{1, ..., n\}$, $i \neq j$

$$2 =< x_i + x_j, x_i + x_j >=< Tx_i + Tx_j, Tx_i + Tx_j >$$

$$=< Tx_i, Tx_i > + < Tx_j, Tx_j > +2 < Tx_i, Tx_j >= 2 + 2 < Tx_i, Tx_j > .$$

Hence

$$0 =< Tx_i, Tx_j >= \sum_{k,u=1}^{n} c_{ki}c_{uj} < x^k, x^u >= \sum_{k,u=1}^{n} c_{ki}c_{uj}\delta_{ku} = \sum_{k=1}^{n} c_{ki}c_{kj},$$

which shows our claim. ∎

LEMMA 2.3 *Let $x^1, ..., x^n \in \mathbb{R}^N$ and $u \in \mathbb{R}^N$ satisfy (2, 3). Set $V = span[x^1, ..., x^n]$. Assume $v^1, ..., v^n$ is an orthonormal basis of V (with respect to $< \cdot, \cdot >_N$). Then*

$$f_{u_1}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1}((u_2, ..., u_N), v^1, ..., v^n)$$

and

$$f_{u_1,A}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1,A}((u_2, ..., u_N), v^1, ..., v^n)$$

for any $N \times N$ matrix A.

Proof. It is well-known that for any $x, y \in \mathbb{R}^N$, $< x, x >_N = < y, y >_N = 1$, there exists a linear isometry (with respect to the Euclidean norm in $\mathbb{R}^N$) $T_{x,y} : \mathbb{R}^N \to \mathbb{R}^N$ such that $Tx = y$. Applying this fact and the induction argument with respect to n we get that there exists a linear isometry $T : \mathbb{R}^N \to \mathbb{R}^N$ such that $Tx^i = v^i$ for $i = 1, ..., n$. By Lemma (2.2) there exists an orthonormal matrix C such that $Cx^i = \sum_{j=1}^{n} C_{ij}x^j = v^i$. By Lemma (2.1),

$$f_{u_1}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1}((u_2, ..., u_N), v^1, ..., v^n),$$

and

$$f_{u_1,A}((u_2, ..., u_N), x^1, ..., x^n) = f_{u_1,A}((u_2, ..., u_N), v^1, ..., v^n),$$

which completes the proof. ∎

LEMMA 2.4 *Let $n, N \in \mathbb{N}$, $N \geq n$. Fix $u = (u_1, ..., u_N) \in \mathbb{R}^N$ with non-negative coordinates. Let us consider a function $f : \mathbb{R}^{nN} \to \mathbb{R}$ given by*

$$f(x^1, ..., x^n) = \sum_{i,j=1}^{N} u_i u_j | < x_i, x_j > |_n,$$

where $x^i \in \mathbb{R}^N$ *for* $i = 1, ..., n,$ *Assume that* $y^1, ..., y^n \in \mathbb{R}^N$ *are so chosen that*

$$f(y^1, ..., y^n) = \max\{f(x^1, ..., x^n) : (x^1, ..., x^n) \text{ satisfying } (2)\}$$

Let $A \in \mathbb{R}^{N\times N}$ *be a matrix defined by*

$$a_{ij} = sgn(< y_i, y_j >_n) \tag{5}$$

for $i, j = 1, ..., N$. *(sgn(0) = 1 by definition). Define* $B \in \mathbb{R}^{N\times N}$ *by*

$$b_{ij} = u_i u_j a_{ij} \tag{6}$$

for $i, j = 1, ..., N$. *Let*

$$b_1 \geq b_2 \geq ... \geq b_N$$

denote the eigenvalues of B *(Since* B *is symmetric all of them are real.) Then there exist orthonormal (with respect to* $< \cdot, \cdot >_N$*) eigenvectors of* B $w^1, ..., w^n \in \mathbb{R}^N$ *corresponding to* $b_1, ..., b_n$ *such that*

$$f(w^1, ..., w^n) = f(y^1, ..., y^n) = \sum_{j=1}^{n} b_j.$$

Set

$$f_1(x^1, ..., x^n) = \sum_{i,j=1}^{N} b_{ij} < x_i, x_j >_n .$$

If $y^1, ..., y^n \in \mathbb{R}^N$ *are such that*

$$f_1(y^1, ..., y^n) = max\{f_1, \text{ under constraint (2) }\} = max\{f, \text{ under constraint (2) }\}$$

and $b_n > b_{n+1}$ *then* $span[y^i : i = 1, ..., n] = span[w^i : i = 1, ..., n]$.

Proof. Since u_j are nonnegative,

$$f_1(x^1, ..., x^n) \leq f(x^1, ..., x^n)$$

for any $x^1, ..., x^n \in \mathbb{R}^N$. Moreover,

$$f_1(y^1, ..., y^n) = f(y^1, ..., y^n).$$

Hence f_1 attains its maximum under constraints (2) at $(y^1, ..., y^n)$. We now apply the Lagrange Multiplier Theorem to the function f_1. This is possible since f_1 is a C^∞ function. Notice that by ([11], p. 261) $rank(G'(y^1, ..., y^n)) = n(n+1)/2$ where G is the $n(n+1)/2 \times nN$ matrix associated with conditions (2). Consequently there exist Lagrange multipliers k_{ij}, $1 \leq i \leq j \leq n$ such that

$$\frac{\partial(f_1 - \sum_{1\leq i\leq j\leq n} k_{ij} G_i)}{\partial(x^i)_j}(y^1, ..., y^n) = 0 \tag{7}$$

for $i = 1, ..., n$, $j = 1, ..., N$, where $G_i(x^1, ...x^n) =< x^i, x^j >_N$. Let us define for $i, j \in \{1, ..., n\}$, $\gamma_{ij} = k_{ij}/2$ if $i < j$, $\gamma_{ij} = k_{ji}/2$, if $j < i$ and $\gamma_{ii} = k_{ii}$. Hence the system (7) can be rewritten (compare with [11], p.262, formula(3.14)) as:

$$B(y^m) = \sum_{i=1}^{n} \gamma_{mi} y^i \tag{8}$$

for $m = 1, ..., n$. Let $\Gamma = \{\gamma_{ij}, i, j = 1, ..., n\}$. Observe that Γ is a symmetric $n \times n$ matrix . Hence it has real eigenvalues $a_1, ..., a_n$. Without loss of generality we can assume that

$$a_1 \geq a_2 \geq ... \geq a_n. \tag{9}$$

Let $V = [v_{ij}]$ be the $n \times n$ orthonormal matrix consisting of eigenvectors of Γ. Then

$$V^T \Gamma V = D, \tag{10}$$

where D is a diagonal matrix with $d_{ii} = a_i$ for $i = 1, ..., n$. Now we show that

$$a_i = b_i \tag{11}$$

for $i = 1, ..., n$. First we prove that a_m, $m = 1, ..., n$, are also eigenvalues of B. To do this, fix $m \in \{1, ..., n\}$. Define

$$w^m = \sum_{j=1}^{n} v_{jm} y^j. \tag{12}$$

We show that $Bw^m = a_m w^m$. Note that

$$Bw^m = B(\sum_{j=1}^{n} v_{jm} y^j) = \sum_{j=1}^{n} v_{jm} B(y^j) = \sum_{j=1}^{n} v_{jm} (\sum_{i=1}^{n} \gamma_{ji} y^i)$$

$$= \sum_{i=1}^{n}(\sum_{j=1}^{n} v_{jm}\gamma_{ji})y^i = \sum_{i=1}^{n}(\sum_{j=1}^{n} v_{jm}\gamma_{ij})y^i = \sum_{i=1}^{n}(\Gamma V)_{im}y^i$$

(by(10))

$$= \sum_{i=1}^{n}(VD)_{im}y^i = \sum_{i=1}^{n} v_{im}a_m y^i = a_m(\sum_{i=1}^{n} v_{im}y^i) = a_m w^m.$$

Hence for $m = 1, ..., n$ a_m are eigenvalues of B with the corresponding vectors w^m. By the proof of Lemma(2.1), $< w^i, w^j >_N = \delta_{ij}$. Notice that by (12) and Lemma(2.3)

$$f_1(y^1, ..., y^n) = f_1(w^1, ..., w^n).$$

Since for any $m = 1, ..., n$ and $i = 1, ..., N$,

$$(Bw^m)_i = a_m(w^m)_i,$$

multiplying each of the above equations by $(w^m)_i$ and summing them up we get that

$$\sum_{j=1}^{n} a_m = f_1(w^1, ..., w^n) = f_1(y^1, ..., y^n) = f(y^1, ..., y^n).$$

If $a_i \neq b_i$ for some $i \in \{1, ..., n\}$, let $v^1, ..., v^n$ be the orthonormal eigenvectors of B corresponding to $b_1, ..., b_n$. Reasoning as above, we get

$$f(v^1, ..., v^n) \geq \sum_{i,j=1}^{N} u_i u_j sgn(< y_i, y_j >_n) < v_i, v_j >_n$$

$$= \sum_{i=1}^{n} b_i > \sum_{i=1}^{n} a_i = f(y^1, ..., y^n);$$

a contradiction. The fact that span$[y^i : i = 1, ..., n]$=span$[w^i : i = 1, ..., n]$ follows from (12) and invertibility of the matrix V. ■

Reasoning as in the proof of Lemma(2.4) we can show

THEOREM 2.1 *Let $\mathcal{A}$ denote the set of all $N \times N$ symmetric matrices (a_{ij}) such that $a_{ij} = \pm 1$ and $a_{ii} = 1$ for $i, j = 1, ..., N$. Let f_{u_1} be given by (1). Then*

$$\max\{f_{u_1} : ((u_2, ..., u_N), x^1, ..., x^n) \textit{ satisfying (2,3) }\}$$

$$= \max\{\sum_{i=1}^{n} b_i(v,A) : A \in \mathcal{A}, v = (v_1, ...v_n) \in \mathbb{R}^N, \sum_{i=1}^{N} v_i^2 = 1, v_1 = u_1\},$$

where $b_1(v,A) \geq b_2(v,A) \geq ... \geq b_n(v,A)$ *denote the biggest eigenvalues of an* $N \times N$ *matrix* $(v_i v_j a_{ij})_{ij=1}^N$. *Analogously for any* $A = (a_{ij}) \in \mathcal{A}$,

$$\max\{\sum_{i,j=1}^{N} u_i u_j a_{ij} < x_i, x_j >_n : (x^1, ..., x^n) \text{ satisfying (2)}$$

$$u_j = \sqrt{(1-u_1^2)/(N-1)}, j = 2, ..., N\}$$

$$= \max\{\sum_{i=1}^{n} b_i(v,A) : A \in \mathcal{A}, v = (u_1, c(u_1), ..., c(u_1))\},$$

where $c(u_1) = \sqrt{(1-u_1^2)/(N-1)}$. *Also*

$$\max\{\sum_{i,j=1}^{N} u_i u_j | < x_i, x_j > |_n : (x^1, ..., x^n) \text{ satisfying (2)}, \sum_{j=1}^{N} u_j^2 = 1\}$$

$$= \max\{\sum_{i=1}^{n} b_i(v,A) : A \in \mathcal{A}, v = (v_1, ...v_n) \in \mathbb{R}^N, \sum_{i=1}^{N} v_i^2 = 1\}.$$

Now for $n, N \in \mathbb{N}$, $N \geq n$ define

$$\lambda_n^N = \sup\{\lambda(V, l_\infty^{(N)}) : V \subset l_\infty^{(N)}, dim(V) = n\}. \tag{13}$$

LEMMA 2.5 *For any* $n, N \in \mathbb{N}$, $2 \leq n \leq N$,

$$\lambda_{n-1}^{N-1} \leq \lambda_n^N.$$

Proof. Let $V \subset l_\infty^{(N-1)}$ be an $n-1$-dimensional subspace with a basis $w^1, ..., w^{n-1}$. Define

$$V_1 = span[e_1, (0, w^j) : j = 1, ..., n-1] \subset l_\infty^N.$$

Let $P \in \mathcal{P}(l_\infty^{(N)}, V_1)$ be such that

$$\|P\| = \lambda(V_1, l_\infty^{(N)})$$

(Since V_1 is finite-dimensional such a projection exists.). Define $Q \in \mathcal{L}(l_\infty^{(N-1)}, V)$ by

$$Qx = ((P(0,x)_2, ..., (P(0,x)_n).$$

It is clear that $Q(l_\infty^{(N-1)}) \subset V$ and $Qw^j = w^j for$ j=1,...,n-1. Hence $Q \in \mathcal{P}(l_\infty^{(N-1)}, V)$. Moreover, $\|Q\| \le \|P\|$. Taking supremum over V we get that

$$\lambda_{n-1}^{N-1} \le \lambda_n^N,$$

as required. ■

THEOREM 2.2 *Let $n, N \in \mathbb{N}$, $N \ge n$. Then*

$$\lambda_n^N = \max\{\sum_{i,j=1}^N u_i u_j | < x_i, x_j > |_n : (x^1, ..., x^n) \textit{ satisfying (2)}, \sum_{j=1}^N u_j^2 = 1\}.$$

Proof. By ([11], Prop. 2.2 and (3.7), p.260),

$$\lambda_n^N \le \max\{\sum_{i,j=1}^N u_i u_j | < x_i, x_j > |_n : (x^1, ..., x^n) \text{ satisfying (2)}, \sum_{j=1}^N u_j^2 = 1\}.$$

To prove a converse assume that there exist $n, N \in \mathbb{N}$, $N \ge n$ such that

$$\lambda_n^N < \phi_n^N = \max\{\sum_{i,j=1}^N u_i u_j | < x_i, x_j > |_n : (x^1, ..., x^n) \text{ satisfying (2)}, \sum_{j=1}^N u_j^2 = 1\}.$$

Without loss of generality we can assume that

$$n = \min\{m \in \mathbb{N} : \lambda_m^M < \phi_m^M \text{ for some } M \ge m\}$$

and

$$N = \min\{M \in \mathbb{N}, M \ge n : \lambda_n^M < \phi_n^M\}$$

Let us define

$$f(u, x^1, ..., x^n) = \sum_{i,j=1}^N u_i u_j | < x_i, x_j > |_n.$$

Let $y^1, ..., y^n \in \mathbb{R}^N$ satisfying (2) and $u^o \in \mathbb{R}^N$ with $\sum_{j=1}^N (u_j^o)^2 = 1$, be such that

$$f(u^o, y^1, ..., y^n) = \phi_n^N.$$

Define as in Lemma(2.4)

$$a_{ij} = sgn(< y_i, y_j >_n) \tag{14}$$

for $i, j = 1, ..., N$. Also let $B \in \mathbb{R}^{N \times N}$ be given by

$$b_{ij} = u_i^o u_j^o a_{ij} \tag{15}$$

for $i = 1, ..., N$. By Lemma(2.4) and Theorem(2.1) we can get that

$$f(u^o, y^1, ..., y^n) = \sum_{i=1}^{n} b_i(u^o, A)$$

where $b_1(u^o, A) \geq b_2(u^o, A) \geq ... \geq b_n(u^o, A)$ denote the biggest eigenvalues of the above defined matrix B. First suppose that $u_j^o = 0$ for some$j \in \{1, ..., N\}$. Without loss of generality we can assume that $u_1^o = 0$. Let B_1 be an $(N-1) \times (N-1)$ matrix given by

$$B_1 = \{b_{ij}\}_{i,j=2,...,N}$$

(the part of B without the first row and the first column). Let $d_1 \geq ... \geq d_{N-1}$ be the eigenvalues of B_1 and $z^1, ..., z^{N-1}$ the corresponding orthonormal eigenvectors. Since $u_1^o = 0$, $v^j = (0, z^j)$, $j = 1, ..., N-1$ are the orthonormal eigenvectors of B corresponding to d_j. Also $d_o = 0$ is an eigenvalue of B with e_1 as an eigenvector. Consequently

$$b_j(u^o, A) \in \{0, d_k, k = 1, ...N-1\}$$

for $j = 1, ..., n$. If $b_j(u^o, A) > 0$ for $j = 1, ..., n$, then $b_j(u^o, A)$ are also the eigenvalues of B_1. By Theorem(2.1),

$$\sum_{i=1}^{n} b_i(u^o, A) = \phi_n^N = \phi_n^{N-1} = \lambda_n^{N-1} \leq \lambda_n^N;$$

a contradiction with the definition of N. If $b_j(u^o, A) = 0$ for some $j \in \{1, ..., n\}$, then again by Theorem(2.1)

$$\phi_n^N \leq \sum_{i \neq j} b_i(u^o, A) \leq \phi_{n-1}^{N-1} = \lambda_{n-1}^{N-1}.$$

Consequently by Lemma(2.5),

$$\lambda_n^N \geq \lambda_{n-1}^{N-1} = \phi_{n-1}^{N-1} \geq \phi_n^N,$$

which again leads to a contradiction. Now assume that $u_j^o > 0$ for $j = 1, ..., N$. Let $w^1, ..., w^n$ be the orthonormal eigenvectors corresponding to $b_i(u^o, A)$ for $i = 1, ..., n$. By the proof of Lemma(2.4)

$$f_1(u^o, w^1, ..., w^n) = \phi_n^N.$$

Define, for $j = 1, ..., n$,

$$z^j = (w_1^j/u_1^o, ..., w_N^j/u_N^o)$$

and let

$$V = span[z^j : j = 1, ..., n] \subset l_\infty^{(N)}.$$

We show that $\lambda(V, l_\infty^{(N)}) = \sum_{j=1}^n b_j(u^o, A) = \phi_n^N$.. Define, for $j = 1, ..., n$,

$$f^j = (w_1^j u_1^o, ..., w_N^j u_N^o)$$

and let $P \in \mathcal{L}(l_\infty^{(N)}, V)$ be given by

$$Px = \sum_{j=1}^n < f^j, x >_N z^j.$$

Since the vectors w^j are orthogonal with respect to $< \cdot, \cdot >_N$, $P \in \mathcal{P}(l_\infty^{(N)}, V)$. Now we show that

$$\|P\| = \lambda(V, l_\infty^{(N)}) = \phi_n^N.$$

Since the function f_1 attains its conditional maximum at $u^o, w^1, ..., w^n$ (compare with the proof of Lemma(2.4)) by the Lagrange Multiplier Theorem there exist $k_{ij} \in \mathbb{R}$, $1 \leq i \leq j \leq n$ and $d \in \mathbb{R}$ such that

$$\frac{\partial(f_1 - \sum_{1 \leq i \leq j \leq n} k_{ij} G_i - d(\sum_{j=1}^N u_j^2 - 1))}{\partial(u_j)}(u^o, w^1, ..., w^n) = 0 \tag{16}$$

It is easy to see that (16) reduces to (compare with ([11], (3.12), p.262))

$$\sum_{j=1}^N u_j^o a_{ij} < w_i, w_j >_n = d u_i^o$$

for $i = 1, ..., N$. Multiplying the above equalities by u_i^o and summing them up, we get that

$$d = f_1(u^o, w^1, ..., w^n) = \phi_n^N.$$

Also since $u_i^o > 0$ for $i = 1, ..., N$, (16) reduces to

$$(\sum_{j=1}^{N} u_j^o a_{ij} < w_i, w_j >_n)/u_i^o = d.$$

Consequently, by definition of $< \cdot, \cdot >_n$, we get, for $i = 1, ..., N$,

$$d = \sum_{k=1}^{n} (\sum_{j=1}^{N} a_{ij} u_j^o w^k)_j w_i^k / u_i^o = \sum_{k=1}^{n} (\sum_{j=1}^{N} a_{ij} f_j^k) z_i^k = (P(a_{i1}, ..., a_{iN}))_i. \quad (17)$$

Since $\|(a_{i1}, ..., a_{iN})\|_\infty = 1$, $\|P\| \geq d$. On the other hand, for any $x = (x_1, ..., x_N) \in l_\infty^{(N)}$, $\|x\|_\infty = 1$ and $i \in \{1, ..., N\}$,

$$|(Px)_i| = |\sum_{j=1}^{n} < f^j, x >_N z_i^j| = |\sum_{k=1}^{N} x_k (\sum_{j=1}^{n} f_k^j z_i^j)|$$

$$= |\sum_{k=1}^{N} x_k (\sum_{j=1}^{n} w_k^j u_k^o w_i^j / u_i^o)| \leq (\sum_{k=1}^{N} u_k^o | < w_k, w_i >_n |)/u_i^o$$

$$= (\sum_{k=1}^{N} u_k^o a_{ik} < w_k, w_i >_n)/u_i^o = d,$$

since $a_{ij} = sgn(< y_j, y_i >_n) = sgn(< w_j, w_i >_n)$ for $i, j = 1, ..., N$. Hence

$$\|P\| = d = \phi_n^N.$$

Now we show that

$$\|P\| = \lambda(V, l_\infty^{(N)}).$$

To do this set for $i = 1, ..., N$ $a^i = (a_{i1}, ..., a_{iN})$ and define an operator $E_p : l_\infty^{(N)} \to l_\infty^{(N)}$ by

$$E_p(x) = \sum_{i=1}^{N} (u_i^o)^2 x_i a^i.$$

We show that $E_p(V) \subset V$. Note that for any $k = 1, ..., N$, and $j = 1, ..., n$

$$(E_p(z^j))_k = \sum_{i=1}^{N} (u_i^o)^2 (w_i^j / u_i^o)(a^i)_k = \sum_{i=1}^{N} u_i^o w_i^j a_{ki}$$

$$= b_j(u^o, A) w_k^j / u_k^o = b_j(u^o, A) z_k^j,$$

since w^j is an eigenvector associated to $b_j(u^o, A)$. Observe that by (17)

$$(Pa^i)_i = d = \|P\|$$

for $i = 1, ..., N$ and $\sum_{i=1}^{N} (u_i^o)^2 = 1$. By [4] (see also [13], Th. 1.3), P is a minimal projection in $\mathcal{P}(l_\infty^{(N)}, V)$. Finally

$$\lambda_n^N \geq \lambda(V, l_\infty^{(N)}) = \|P\| = d = \phi_n^N$$

which leads to a contradiction. The proof is complete. ■

LEMMA 2.6 *For any $n \geq 2$,*

$$\lambda_n^{n+1} = 2 - 2/(n+1).$$

Moreover, $\lambda_n^{n+1} = \lambda(ker(f), l_\infty^{(n+1)})$ if and only if $f = c(\pm 1, ..., \pm 1)$, where c is a positive constant.

Proof. It is clear that

$$\lambda_n^{n+1} = \max\{\lambda(ker(f), l_\infty^{(n+1)}) : f \in l_1^{(n+1)} \setminus \{0\}, \|f\|_1 = 1\}.$$

By ([1]), if $f = (f_1, ..., f_{n+1}) \in l_1^{(n+1)}$, $\|f\|_1 = 1$ is so chosen that $\lambda(ker(f), l_\infty^{(n+1)}) > 1$, then $|f_j| < 1/2$ for any $j = 1, ..., n+1$ and

$$\lambda(ker(f), l_\infty^{(n+1)}) = 1 + (\sum_{i=1}^{n+1} \frac{|f_j|}{(1 - 2|f_j|)})^{-1}.$$

Hence it is easy to see that

$$\lambda_n^{n+1} = \max\{1 + (\sum_{i=1}^{n+1} \frac{f_j}{(1 - 2f_j)})^{-1}\}$$

under constraints

$$\{\sum_{j=1}^{n+1} f_j = 1,\ 1/2 \geq f_j \geq 0, j = 1, ..., n+1\}. \tag{18}$$

Now we show by induction argument that

$$\lambda_n^{n+1} = 2 - 2/(n+1).$$

If $n = 2$, by the Lagrange Multiplier Theorem the only functional $f = (f_1, f_2, f_3)$ which can maximize the function $\phi_2(f) = 1 + (\sum_{i=1}^{3} \frac{f_j}{(1-2f_j)})^{-1}$ under constraint (18) is $f = (1/3, 1/3, 1/3)$ and $\phi_2(1/3, 1/3, 1/3) = 4/3$. Now assume that $\lambda_k^{k+1} = 2 - 2/(k+1)$ for any $k \leq n$. Then by the Lagrange Multiplier Theorem the only functional $f = (f_1, ..., f_{n+1})$ which can maximize the function $\phi_n(f) = 1 + (\sum_{i=1}^{n+1} \frac{f_j}{(1-2f_j)})^{-1}$ under constraint (18) is $f = (1/(n+1), ..., 1/(n+1))$ and $\phi_n((1/(n+1), ..., 1/(n+1)) = 2 - 2/(n+1)$. Notice that $\phi_{n+1}(1/(n+2), ..., 1/(n+2)) = 2 - 2/(n+2)$, where $\phi_{n+1}(f) = 1 + (\sum_{i=1}^{n+1} \frac{f_j}{(1-2f_j)})^{-1}$ Consequently, by the induction hypothesis,

$$\lambda_{n+1}^{n+2} = \max\{1 + (\sum_{i=1}^{n+2} \frac{f_j}{(1-2f_j)})^{-1}\}$$

under constraints

$$\{\sum_{j=1}^{n+2} f_j = 1,\ 1/2 > f_j > 0, j = 1, ..., n+2\}. \tag{19}$$

Again by the Lagrange Multiplier Theorem the only $f = (f_1, ..., f_{n+2})$ which can maximize ϕ_{n+1} under constraints (19) is $f = (1/(n+2), ..., 1/(n+2))$. Hence $\lambda_{n+1}^{n+2} = 2 - 2/(n+2)$, as required. By the above proof, any functional f satisfying $\lambda(ker(f), l_\infty^{(n+1)}) = \lambda_n^{n+1}$ is of the form $c(\pm 1/(n+1), ..., \pm 1/(n+1))$. The proof is complete. ■

LEMMA 2.7 *Let us consider problem (1) with* $u_1 = 0$ *and fixed* $N \geq n+2$. *Assume that* $\lambda_n^{N-1} > \lambda_{n-1}^{N-1}$. *Then the maximum of* f_{u_1} *under constraints (2, 3) is equal to* λ_n^{N-1}.

Proof. By ([11], Th. 1.2) and Theorem(2.2) for any $n, N \in \mathbb{N}$, $N \geq n+1$,

$$\lambda_n^N = \max\{\sum_{i,j=1}^{N} u_i u_j | < x_i, x_j >_n |\} \tag{20}$$

under constraints:

$$< x^i, x^j >_N = \delta_{ij}, 1 \leq i \leq j \leq n; \tag{21}$$

$$\sum_{j=1}^{N} u_j^2 = 1. \tag{22}$$

Moreover, if $u, y^1, ..., y^n \in \mathbb{R}^N$ satisfying (21, 22) are such that

$$\sum_{i,j=1}^{N} u_i u_j | < y_i, y_j >_n | = \lambda_n^N,$$

then by Lemma(2.4) and Theorem(2.1),

$$\lambda_n^N = \sum_{j=1}^{n} b_j, \tag{23}$$

where $b_1 \geq b_2 \geq, ..., \geq b_n$ are the biggest eigenvalues of the $N \times N$ matrix $B = (b_{ij})_{i,j=1,...,N}$ defined by $b_{ij} = u_i u_j sgn(< y_i, y_j >_n)$.
Now, assume

$$f_{u_1}((v_2, ..., v_n), y^1, ..., y^n) = \max\{f_{u_1}((u_2, ..., u_n), x^1, ..., x^n) :$$
$$(u_1, ..., u_n), (x^1, ..., x^n) \text{ satisfying } (2,3)\}$$

Since $u_1 = 0$, by (20), and Theorem(2.2),

$$f_{u_1}((v_2, ..., v_n), y^1, ..., y^n) \geq \phi_n^{N-1} = \lambda_n^{N-1}.$$

To prove the opposite inequality, let B be an $N \times N$ matrix defined by

$$b_{ij} = v_i v_j sgn(< y_i, y_j >_n).$$

Let $b_1 \geq b_2 \geq ... \geq b_N$ be the eigenvalues of B (with multiplicities). By Lemma(2.4),

$$f_{u_1}((v_2, ..., v_n), y^1, ..., y^n) = \sum_{j=1}^{n} b_j.$$

Let $C = \{b_{ij}\}_{i,j=2,...,N}$ and let $c_1 \geq c_2 \geq ..., \geq c_{N-1}$ be the eigenvalues of C. Since $u_1 = 0$,

$$\{c_1, ..., c_{N-1}\} \cup \{0\} = \{b_1, ..., b_N\}.$$

If $b_{j_o} = 0$ for some $j_o \in \{1, ..., n\}$, then again by ([11], Th. 1.2), (20), Theorem(2.1) and Lemma(2.6)

$$\lambda_n^{N-1} \leq f_{u_1}((v_2, ..., v_n), y^1, ..., y^n)$$

$$= \sum_{j=1}^{n} b_j \leq \sum_{j<j_o} b_j \leq \lambda_{n-1}^{N-1};$$

a contradiction with our assumptions. Hence $b_i = c_i$ for $i = 1, ..., n$. Now let $z^1, ..., z^n \in \mathbb{R}^{N-1}$ be the corresponding to $b_1, ..., b_n$ orthonormal eigenvectors of C. Hence for any $j = 1, ..., n$ and $i = 1, ..., N-1$

$$(Cz^j)_i = c_j(z^j)_i.$$

Multiplying each of the above equations by $(z^j)_i$ and summing them up we get

$$\max\{f_{u_1}\} = \sum_{j=1}^{n} c_j = \sum_{i,j=2}^{N} b_{ij} < z_{i-1}, z_{j-1} >_n$$

$$= \sum_{i,j=2}^{N} v_i v_j sgn(< y_i, y_j >_n) < z_{i-1}, z_{j-1} >_n \leq \lambda_n^{N-1}.$$

The proof is complete. ■

LEMMA 2.8 *Let $u = (u_1, ..., u_N) \in \mathbb{R}^N$ and let $z = (z_2, ..., z_n) \in \{-1, 1\}^{N-1}$. Let A_z be $N \times N$ matrix defined by $z_j = a_{1j} \in \{\pm 1\}$ for $j = 2, ..., N$, $a_{ij} = -1$ for $i, j = 2, ..., N$ $i \neq j$ and $a_{ii} = 1$ for $i = 1, ..., N$. Let $B_z = \{(b_z)_{ij}, i, j = 1, ..., N\}$ where $(b_z)_{ij} = u_i u_j (A_z)_{ij}$. Hence*

$$B_z = \begin{pmatrix} u_1^2 & z_2 u_1 u_2 & z_3 u_1 u_3 & ... & z_N u_1 u_N \\ z_2 u_1 u_2 & u_2^2 & -u_2 u_3 & ... & -u_2 u_N \\ z_3 u_1 u_2 & -u_2 u_3 & u_3^2 & ... & -u_2 u_N \\ ... & ... & ... & ... & ... \\ z_N u_1 u_N & -u_2 u_N & ... & ... & u_N^2 \end{pmatrix}. \qquad (24)$$

Let σ be a permutation of $\{1, ..., N\}$ such that $\sigma(1) = 1$ and let for any $x = (x_1, ..., x_N), \in \mathbb{R}^N$, $x_- = (x_1, -x_2, ..., -x_N)$. Then the matrices

$$B_{\sigma(z)} = \{u_{\sigma(i)}u_{\sigma(j)}(A\sigma(z))_{ij}, i, j = 1, ..., N\},$$

$$B_{z_-} = \{(u_i u_j (A_{z_-})_{ij}), i, j = 1, ..., N\}$$

and B_z have the same eigenvalues.

Proof. Let b be an eigenvalue of B_z with an eigenvector $x = (x_1, ..., x_N)$. Define $x_\sigma = (x_1, x_{\sigma(2)}, ..., x_{\sigma(N)})$ and $x_- = (x_1, -x_2, ..., -x_N)$. Notice that

$$(B_{\sigma(z)}x_{\sigma(z)})_1 = u_1^2 x_1 + \sum_{j=2}^{n} x_{\sigma(j)}u_{\sigma(j)}u_1 = u_1^2 x_1 + \sum_{j=2}^{n} x_j u_j u_1 = bx_1.$$

Analogously, for $i = 2, ..., N$,

$$(B_{\sigma(z)}x_\sigma)_i = u_1 u_{\sigma(i)}x_{\sigma(i)}x_1 + \sum_{j=2, j\neq i}^{n} -u_{\sigma(j)}u_{\sigma(i)}x_{\sigma(j)} + u_{\sigma(i)}^2 x_{\sigma(i)}$$

$$= bx_{\sigma(i)} = b(x_\sigma)_i.$$

Also notice that

$$(B_{z_-}x_-)_1 = u_1^2 x_1 + \sum_{j=2}^{N}(-x_j)u_1 u_j(-x_j) = bx_1 = b(x_-)_1$$

and for $i = 2, ..., N$

$$(B_{z_-}x_-)_i = u_1 u_i(-x_i)x_1 + \sum_{j=2}^{n} a_{ij}u_1 u_j(-x_j) = -bx_i = b(x_-)_i.$$

This shows that any eigenvalue of B is an eigenvalue of B_{z_-} and $B_{\sigma(z)}$ with the same multiplicity. By the same reasoning, any eigenvalue of B_{z_-} and $B_{\sigma(z)}$ is also an eigenvalue of B, which completes the proof. ■

THEOREM 2.3 *Let $n = 3$ and $N = 5$. Let $z = (z_2, z_3, z_4, z_5)$ be such that $z_i = \pm 1$, for $i = 2, ..., 5$ and $z_j = -1$ for exactly one $j \in \{2, 3, 4, 5\}$. Assume that $A_z = (a_{ij}(z))$ is a 5×5 matrix defined by*

$$A_z = \begin{pmatrix} 1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & 1 & -1 & -1 & -1 \\ z_3 & -1 & 1 & -1 & -1 \\ z_4 & -1 & -1 & 1 & -1 \\ z_5 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{25}$$

Let

$$M_A = \max\{\sum_{i,j=1}^{5} u_i u_j a_{ij}(z) < x_i, x_j >_3 : (x^1, x^2, x^3) \in (\mathbb{R}^5)^3 \text{ satisfying (2)}, \sum_{i=1}^{5} u_i^2 = 1\}.$$

Then $M_A = 3/2$.

Proof. By Lemma(2.8), we can assume that $z_2 = -1$. Fix $u \in \mathbb{R}^5$, $\sum_{i=1}^{5} u_i^2 = 1$. Let B_u denote the 5×5 matrix defined by

$$(b_u)_{ij} = u_i u_j a_{ij}(z)$$

for $i, j = 1, ..., 5$. By Lemma(2.4),

$$M_A = \max\{\sum_{j=1}^{3} b_j(u, A) : u \in \mathbb{R}^5, \sum_{i=1}^{5} u_i^2 = 1\},$$

where $b_1(u, A) \geq b_2(u, A) \geq b_3(u, A)$ denote the three biggest eigenvalues of B_u. Put for $i = 1, ..., 5$, $v_i = u_i^2$. After elementary but tedious calculations (we advise to check them by the symbolic Mathematica program) we get that

$$det(B_u - tId) = -t^5 + t^4(\sum_{i=1}^{5} v_i) + 16tv_3v_4v_5(v_1 + v_2)$$

$$-4t^2(v_3 - v_4v_5 + (v_1 + v_2)(v_4v_5 + v_3(v_4 + v_5))).$$

Define $w = (w_1, ..., w_5)$ by $w_1 = 0$, $w_2 = \sqrt{u_1^2 + u_2^2}$, $w_i = u_i$ for $i = 3, 4, 5$. Observe that by the above formula B_u and B_w have the same eigenvalues.

Since $w_1 = 0$, by Lemma(2.7), Theorem(2.1), Theorem(2.2) and Lemma(2.6) applied to $n = 3$ and $N = 5$ we get

$$\sum_{j=1}^{3} b_j(u, A) \leq \lambda_3^4 = 3/2,$$

which completes the proof. ■

THEOREM 2.4 *Let $n = 2$ and $N = 4$. Let $z = (z_2, z_3, z_4)$ be such that $z_i = \pm 1$, for $i = 2, 3, 4$ and $z_j = -1$ for exactly one $j \in \{2, 3, 4\}$. Assume that $A_z = (a_{ij}(z))$ is a 4×4 matrix defined by*

$$A_z = \begin{pmatrix} 1 & z_2 & z_3 & z_4 \\ z_2 & 1 & -1 & -1 \\ z_3 & -1 & 1 & -1 \\ z_4 & -1 & -1 & 1 \end{pmatrix}. \tag{26}$$

Let

$$M_A = \max\{\sum_{i,j=1}^{4} u_i u_j a_{ij}(z) < x_i, x_j >_2 : (x^1, x^2) \in (\mathbb{R}^4)^2 \text{ satisfying (2)}, \sum_{i=1}^{4} u_i^2 = 1\}.$$

Then $M_A = 4/3$.

Proof. By Lemma(2.8), we can assume that $z_4 = -1$. Fix $u \in \mathbb{R}^4$, $\sum_{i=1}^4 u_i^2 = 1$. Let B_u denote the 4×4 matrix defined by

$$(b_u)_{ij} = u_i u_j a_{ij}(z)$$

for $i, j = 1, ..., 4$. By Lemma(2.4),

$$M_A = \max\{b_1(u, A) + b_2(u, A) : u \in \mathbb{R}^4, \sum_{i=1}^{5} u_i^2 = 1\},$$

where $b_1(u, A) \geq b_2(u, A)$ denote the two biggest eigenvalues of B_u. Put for $i = 1, ..., 4$, $v_i = u_i^2$. After elementary calculations (we advise to check them by the symbolic Mathematica program) we get that

$$det(B_u - tId) = t^4 - t^3(\sum_{i=1}^{4} v_i) + 4tv_3v_2(v_1 + v_4).$$

Define $w = (w_1, ..., w_4)$ by $w_1 = 0$, $w_4 = \sqrt{u_1^2 + u_4^2}$, $w_i = u_i$ for $i = 2, 3$. Observe that by the above formula B_u and B_w have the same eigenvalues. Since $w_1 = 0$ by Lemma(2.7), Theorem(2.1), Theorem(2.2) and Lemma(2.6) applied to $n = 2$ and $N = 4$ we get

$$b_1(u, A) + b_2(u, A) \leq \lambda_2^3 = 4/3,$$

which completes the proof. ■

LEMMA 2.9 *Let $n = 2$ and $N = 4$ and let $u \in [0, 1/\sqrt{3})$. Assume that $B = B(u)$ is a 4×4 matrix defined by*

$$B = \begin{pmatrix} u^2 & u/\sqrt{3} & u/\sqrt{3} & -u\sqrt{1/3 - u^2} \\ u/\sqrt{3} & 1/3 & -1/3 & -\sqrt{1/3 - u^2}/\sqrt{3} \\ u/\sqrt{3} & -1/3 & 1/3 & -\sqrt{1/3 - u^2}/\sqrt{3} \\ -u\sqrt{1/3 - u^2} & -\sqrt{1/3 - u^2}/\sqrt{3} & -\sqrt{1/3 - u^2}/\sqrt{3} & 1/3 - u^2 \end{pmatrix}. \tag{27}$$

Then the eigenvalues of B are $2/3$ (with multiplicity 2), $-1/3$ and 0. Moreover,

$$w^1 = (\sqrt{2}u, 1/\sqrt{6}, 1/\sqrt{6}, -\sqrt{2(1 - 3u^2)}/\sqrt{3}$$

$$w^2 = (0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$$

are orthonormal eigenvectors corresponding to $2/3$ and

$$w^3 = (1, 0, 0, u/(\sqrt{1/3 - u^2})$$

is an eigenvector corresponding to 0.

Proof. It can be done by elementary calculations. We advise to check them by a symbolic Mathematica program. ■

LEMMA 2.10 *Let $n = 2$ and $N = 4$ and let $u \in [0, 1)$. Assume that $B = B(u)$ is a 4×4 matrix defined by*

$$B = \begin{pmatrix} u^2 & u\sqrt{1 - u^2}/\sqrt{2} & u\sqrt{1 - u^2}/\sqrt{2} & 0 \\ u\sqrt{1 - u^2}/\sqrt{2} & (1 - u^2)/2 & (u^2 - 1)/2 & 0 \\ u\sqrt{1 - u^2}/\sqrt{2} & (u^2 - 1)/2 & (1 - u^2)/2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{28}$$

Then the eigenvalues of B are

$$0, (u^2 + \sqrt{4u^2 - 3u^4})/2, 1 - u^2 \textit{ and } (u^2 - \sqrt{4u^2 - 3u^4})/2.$$

Moreover,

$$w^2 = (z/\sqrt{z^2+2}, 1/\sqrt{z^2+2}, 1/\sqrt{z^2+2}, 0),$$

where

$$z = (u^2 + \sqrt{4u^2 - 3u^4})/u(\sqrt{2 - 2u^2}),$$

is an eigenvector corresponding to $(u^2 + \sqrt{4u^2 - 3u^4})/2$ *and*

$$w^3 = (0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$$

is an eigenvector corresponding to $1 - u^2$*. Also*

$$M = \max\{1 - u^2 + (u^2 + \sqrt{4u^2 - 3u^4})/2 : u \in [1/\sqrt{3}, 1]\} = 4/3.$$

Proof. It can be verified by elementary calculations that the above defined numbers are the eigenvalues of B. We advise to check them by a symbolic Mathematica program. Also notice that if

$$f(v) = 1 - v/2 + \sqrt{4v - 3v^2}/2,$$

then

$$f'(v) = -1/2 + (4 - 6v)/(4\sqrt{4v - 3v^2}).$$

Notice that $f'(v) = 0$ if and only if $3v^2 - 4v + 1 = 0$. Hence $f'(1) = f'(1/3) = 0$. Since $f(1) = 1$, $M = f(1/3) = 4/3$. Notice that if $u = 1/\sqrt{3}$ then $v = 1/3$, which shows our claim. ■

LEMMA 2.11 *Let $n = 2$ and $N = 4$ and let $c \in [0, 1/\sqrt{3})$. Assume that $B = B(c)$ is a 4×4 matrix defined by*

$$B = \begin{pmatrix} 1-3c^2 & c\sqrt{1-3c^2} & c\sqrt{1-3c^2} & c\sqrt{1-3c^2} \\ c\sqrt{1-3c^2} & c^2 & -c^2 & -c^2 \\ c\sqrt{1-3c^2} & -c^2 & c^2 & -c^2 \\ c\sqrt{1-3c^2} & -c^2 & -c^2 & c^2 \end{pmatrix}. \tag{29}$$

Then the eigenvalues of B are $2c^2$ (with multiplicity 2),

$$(1 - 4c^2 + \sqrt{1 + 8c^2 - 32c^4})/2, \textit{ and } (1 - 4c^2 - \sqrt{1 + 8c^2 - 32c^4})/2.$$

Moreover,

$$w^1 = (0, 1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}),$$

and

$$w^2 = (0, -1/\sqrt{2}, 1/\sqrt{2}, 0)$$

are the orthonormal eigenvectors corresponding to $2c^2$.

Proof. It can be done by elementary calculations. We advise to check them by a symbolic Mathematica program. ■

LEMMA 2.12 *Let* B *be a* 5×5 *matrix defined by*

$$B = \begin{pmatrix} u_{o1}^2 & z_2 u_{o1} c & z_3 u_{o1} c & z_4 u_{o1} c & z_5 u_{o1} c \\ z_2 u_{o1} c & c^2 & -c^2 & -c^2 & -c^2 \\ z_3 u_{o1} c & -c^2 & c^2 & -c^2 & -c^2 \\ z_4 u_{o1} c & -c^2 & -c^2 & c^2 & -c^2 \\ z_5 u_{o1} c & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}, \tag{30}$$

where $z_j \in \{\pm 1\}$ *for* $j = 2, 3, 4, 5$. *Then* $2c^2$ *is an eigenvalue of* B *with multiplicity at least 2.*

Proof. Let C be defined by

$$C = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & c^2 & -c^2 & -c^2 & -c^2 \\ 0 & -c^2 & c^2 & -c^2 & -c^2 \\ 0 & -c^2 & -c^2 & c^2 & -c^2 \\ 0 & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}. \tag{31}$$

Since $2c^2$ is a eigenvalue of C with the multiplicity 3 with the eigenvectors v^j, $j = 2, 3, 4$ given by (34), there exist 2 orthonormal vectors w^1, w^2 in $span[v^2, v^3, v^4]$ which are orthogonal to the first row of B, which completes the proof. ■

THEOREM 2.5 *Let* $n = 3$ *and* $N = 5$. *Fix* $u_{o1} \in [0, 1]$. *Assume* $A = (a_{ij})$ *is a* 5×5 *matrix defined by*

$$A = \begin{pmatrix} 1 & 1 & 1 & -1 & -1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ -1 & -1 & -1 & 1 & -1 \\ -1 & -1 & -1 & -1 & 1 \end{pmatrix}. \tag{32}$$

Let

$$M_A(u_1) = \max\{\sum_{i,j=1}^{5} u_i u_j a_{ij} < x_i, x_j >_3 : (x^1, x^2, x^3) \in (\mathbb{R}^5)^3$$

$$\textit{satisfying (2)}, u_1 = u_{o1}, u_i = \sqrt{1-u_1^2}/2, i = 2,3,4,5\}.$$

Then

$$M_A(u_1) = \frac{1 + 6c^2 + \sqrt{(6c^2-1)^2 + 16(1-4c^2)c^2}}{2}$$

where $c = \sqrt{1-u_{o1}^2}/2$. *Moreover,*

$$M_A = \max\{M_A(u) : u \in [0,1]\} = \frac{5+4\sqrt{2}}{7} = M_A\left(\sqrt{(5-3\sqrt{2})/7}\right).$$

Proof. Notice that by Theorem(2.1),

$$M_A = \sum_{j=1}^{3} b_j(B),$$

where $b_1(B) \geq b_2(B) \geq ... \geq b_5(B)$ denote the eigenvalues of the matrix B given by

$$B = \begin{pmatrix} u_{o1}^2 & u_{o1}c & u_{o1}c & -u_{o1}c & -u_{o1}c \\ u_{o1}c & c^2 & -c^2 & -c^2 & -c^2 \\ u_{o1}c & -c^2 & c^2 & -c^2 & -c^2 \\ -u_{o1}c & -c^2 & -c^2 & c^2 & -c^2 \\ -u_{o1}c & -c^2 & -c^2 & -c^2 & c^2 \end{pmatrix}, \tag{33}$$

where $c = \sqrt{1-u_{o1}^2}/2$. Hence we should calculate the eigenvalues of B. To do this, let C be given by (31). It is easy to see that the eigenvalues of C are: 0 (with the eigenvector $v^1 = (1,0,0,0,0)$, $2c^2$ (with the orthonormal eigenvectors

$$v^2 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0, 0), v^3 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}), \tag{34}$$

$$v^4 = (0, 1/2, 1/2, -1/2, -1/2))$$

and $-2c^2$ (with the eigenvector $v^5 = (0, 1/2, 1/2, 1/2, 1/2)$). Hence our theorem is proved for $u_{o1} = 0$ (in this case $c = 1/2$). If $u_{o1} > 0$, since the vectors v^2, v^3 and v^5 are orthogonal to the first row of B, by Lemma(2.12), $2c^2$ (with multiplicity 2) and $-2c^2$ (with multiplicity 1) are also eigenvalues of B. Now we find the other 2 eigenvalues of B. To do this, we show that an element $(a, 1/2, 1/2, -1/2, -1/2)$ for a properly chosen a is an eigenvector of B. Let us consider a system of equations:

$$u_{o1}^2 a + 2u_{o1}c = \lambda a \tag{35}$$

and

$$u_{o1}ca + c^2 = \lambda/2 \tag{36}$$

with unknown variables a and λ. Hence we easily get that

$$u_{o1}^2 a + 2u_{o1}c = 2(u_{o1}ca + c^2)a.$$

The last equation has two solutions. Namely:

$$a_1 = \frac{u_{o1}^2 - 2c^2 + \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{4u_{o1}c}$$

and

$$a_2 = \frac{u_{o1}^2 - 2c^2 - \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{4u_{o1}c}$$

Since a_1, λ_1 and a_2, λ_2 are the solutions of (35) and (36),) it is easy to check that $(a_1, 1/2, 1/2, -1/2, -1/2)$ is an eigenvector of B corresponding to the eigenvalue

$$\lambda_1 = 2u_{o1}ca_1 + 2c^2 = 2c^2 + \frac{u_{o1}^2 - 2c^2 + \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{2}$$

and $(a_2, 1/2, 1/2, -1/2, -1/2)$ is an eigenvector of B corresponding to the eigenvalue

$$\lambda_2 = 2u_{o1}ca_2 + 2c^2 = 2c^2 + \frac{u_{o1}^2 - 2c^2 - \sqrt{(u_{o1}^2 - 2c^2)^2 + 16u_{o1}^2c^2}}{2}.$$

It is clear that $\lambda_1 > 2c^2$ and $\lambda_2 < 2c^2$. Hence by Theorem(2.1),

$$M_A = \lambda_1 + 2c^2 + 2c^2.$$

Since $u_{o1}^2 = 1 - 4c^2$,

$$\lambda_1 + 4c^2 = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2},$$

which completes this part of the proof.
Now define for $c \in [0, 1/2]$,

$$h(c) = \frac{1 + 6c^2 + \sqrt{(6c^2-1)^2 + 16(1-4c^2)c^2}}{2}.$$

Notice that $h(0) = 1$ and $h(1/2) = 3/2$. After elementary calculations (substituting c^2 by x), we get that

$$c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$$

is the only point in $[0, 1/2]$ such that $h'(c_o) = 0$. Since

$$h(c_o) = \frac{5+4\sqrt{2}}{7} > 3/2,$$

$$M_A = h(c_o) = \frac{5+4\sqrt{2}}{7}.$$

Note $u_1 = \sqrt{(5-3\sqrt{2})/7}$ satisfies

$$u_1^2 + 4c_o^2 = 1.$$

The proof is complete. ■

REMARK 2.1 *Notice that* $\lambda_1 \geq max\{2c^2, u_{o1}^2\}$. *Indeed if* $2c^2 \geq u_{o1}^2$, *this has been proven in Theorem(2.5). If* $u_{o1}^2 > 2c^2$,

$$\lambda_1 \geq 2c^2 + \frac{u_{o1}^2 - 2c^2 + (u_{o1}^2 - 2c^2)}{2} = u_{o1}^2.$$

LEMMA 2.13 *Let* B *be defined by (30). Assume that* $c \in (0, 1/2)$ *is so chosen that there exist* $b_4(B) \geq b_5(B)$ *eigenvalues of* B *satisfying,* $b_4(B) < 2c^2$. *Let* w^1, w^2, w^3 *be the orthonormal eigenvectors corresponding to the three biggest eigenvalues of* B. *Assume that*

$$\sum_{i,j=1}^{5} b_{ij} < w_i, w_j >_3 = M = \max\{\sum_{i,j=1}^{5} u_i u_j | < z_i, z_j > |_3 : z^1, z^2, z^3 \in \mathbb{R}^5\}, \tag{37}$$

under constraint (2) with $u_1 = \sqrt{1-4c^2}$ *and* $u_j = c$ *for* $j = 2, 3, 4, 5$. *Then the matrix* B_o *determined by* $1 = z_2 = z_3 = -z_4 = -z_5$ *satisfies (37).*

Proof. By Theorem(2.1) we need to calculate the sum of the three biggest eigenvalues of any matrix B satisfying (30). If $z_i = z_j = 1$ for exactly two indices $i, j \in \{2, 3, 4, 5\}$ then applying Theorem(2.5) and Lemma(2.8), we can show that B has the same eigenvalues as B_o. Now assume that $z_i = -1$ for exactly one $i \in \{2, 3, 4, 5.\}$ Then by Theorem(2.3),

$$b_1(B) + b_2(B) + b_3(B) \leq 3/2$$

where $b_1(B) \geq b_2(B) \geq b_3(B)$ denote the three biggest eigenvalues of B. Notice that by Theorem(2.5),

$$M \geq M_A > 3/2.$$

By Lemma(2.8) the same conclusion holds true if $z_i = 1$ for exactly one $i \in \{2, 3, 4, 5\}$.
Now assume that $z_i = 1$ for $i = 2, 3, 4, 5$. Then, reasoning as in Theorem(2.5), we get that the eigenvalues of B are: $2c^2$ with the multiplicity 3,

$$1/2 - 3c^2 + \sqrt{1 + 12c^2 - 60c^4}/2 \text{ and } 1/2 - 3c^2 - \sqrt{1 + 12c^2 - 60c^4}/2.$$

After elementary calculations we obtain that

$$1/2 - 3c^2 + \sqrt{1 + 12c^2 - 60c^4}/2 \geq 2c^2$$

if and only if $1/2 \geq c \geq 1/\sqrt{5}$. If B_o satisfies (37), by Theorem(2.1), we should have:

$$b_1(B) \leq b_1(B_o),$$

which by the above calculations and Theorem(2.5) is equivalent to

$$\sqrt{1 + 12c^2 - 60c^4}/2 < 2c^2 + \sqrt{1 + 8c^2 - 32c^4}/2$$

or

$$2c^2 < 1/2 - c^2 + \sqrt{1 + 4c^2 - 28c^4}/2.$$

After elementary calculations we get that both inequalities are equivalent to

$$0 < c < 1/2,$$

which shows our claim. If $z_i = -1$ for $i = 2, 3, 4, 5$, by Lemma(2.8) the conclusion is the same. Finally, by Theorem(2.1), B_o satisfies (37). ■

LEMMA 2.14 *Let $A = \{a_{ij}, i, j = 1, ..., 5\}$ be a 5×5 symmetric matrix such that $a_{ij} \in \{\pm 1\}$ for $i, j = 1, ..., 5$ and $a_{ii} = 1$ for $i = 1, ..., 5$. Consider a function*

$$f_{u_1,A}((u_2, ..., u_5), x^1, x^2, x^3) = \sum_{i,j=1}^{5} u_i u_j a_{ij} < x_i, x_j >_3 \tag{38}$$

under constraints (2) and (3). Then there exist $x^1, x^2, x^5 \in \mathbb{R}^5$ satisfying (2) and (u_2, u_3, u_4, u_5) satisfying (3) maximizing the function $f_{u_1,A}$ such that $x_4^3 = x_5^3 = 0$, $x_2^3 \geq 0$, $x_2^2 = 0$, $x_4^2 \geq 0$ and $x_2^1 \geq 0$.

Proof. Let y^1, y^2, y^3 and (u_2, u_3, u_4, u_5) be any vectors satisfying (2) and (3) maximizing $f_{u_1,A}$. Let $V = span[y^1, y^2, y^3]$. Since $\dim(V) = 3$, there exist linearly independent $f, g \in \mathbb{R}^5$ such that $V = ker(f) \cap ker(g)$. Hence we can find $d^3 \in V \setminus \{0\}$, which is orthogonal to e_4, e_5 such that $d_2^3 \geq 0$. Set $x^3 = d^3/\|d^3\|_2$. Analogously we can find $d^2 \in V \setminus \{0\}$, orthogonal to x^3 and e_2 satisfying $d_4^2 \geq 0$. Define $x^2 = d^2/\|d^2\|_2$. Finally we can find $d^1 \in V \setminus \{0\}$, orthogonal to x^3 and x^2 with $d_2^1 \geq 0$. Set $x^1 = d^1/\|d^1\|_2$. Note that $x^i \in V$ for $i = 1, 2, 3$ and they are orthonormal. By Lemma(2.3), x^1, x^2, x^3 and (u_2, u_3, u_4, u_5) maximize the function $f_{u_1,A}$, which completes the proof. ■

LEMMA 2.15 *Let $A = \{a_{ij}, i, j = 1, ..., N\}$ be an $N \times N$ symmetric matrix such that $a_{ij} \in \{\pm 1\}$ for $i, j = 1, ..., N$ and $a_{ii} = 1$ for $i = 1, ..., N$. Consider a function*

$$f_{u_1,A}^N((u_2, ..., u_N), x^1, x^2) = \sum_{i,j=1}^{N} u_i u_j a_{ij} < x_i, x_j >_2 \tag{39}$$

under constraints (2) and (3). Then there exist $x^1, x^2 \in \mathbb{R}^N$ satisfying (2) and $(u_2, ..., u_N)$ satisfying (3) maximizing the function $f_{u_1,A}^N$ such that $x_{N-1}^2 \geq 0$, $x_N^2 = 0$, and $x_{N-2}^1 \geq 0$.

Proof. Let y^1, y^2 and $(u_2, ..., u_N)$ be any vectors satisfying (2) and (3) maximizing $f_{u_1,A}^N$. Let $V = span[y^1, y^2]$. Since $\dim(V) = 2$, there exist linearly independent $f^1, ...f^{N-2} \in \mathbb{R}^N$ such that $V = \bigcap_{j=1}^{N-2} ker(f^j)$. Hence we can find $d^2 \in V \setminus \{0\}$, which is orthogonal to e_N such that $d_{N-1}^2 \geq 0$. Set $x^2 = d^/\|d^2\|_2$. Analogously, we can find $d^1 \in V \setminus \{0\}$, orthogonal to x^2 with $d_{N-2}^1 \geq 0$. Set

$x^1 = d^1/\|d^1\|_2$. Note that $x^i \in V$ for $i = 1, 2$ and they are orthonormal. By Lemma(2.3), x^1, x^2 and $(u_2, ..., u_N)$ maximize the function $f^N_{u_1,A}$, which completes the proof. ■

LEMMA 2.16 *Let A be a fixed 5×5 matrix given by*

$$A = \begin{pmatrix} 1 & z_2 & z_3 & z_4 & z_5 \\ z_2 & 1 & -1 & -1 & -1 \\ z_3 & -1 & 1 & -1 & -1 \\ z_4 & -1 & -1 & 1 & -1 \\ z_5 & -1 & -1 & -1 & 1 \end{pmatrix}, \tag{40}$$

where $z_i \in \{\pm 1\}$ for $i = 2, 3, 4, 5$. Let

$$g_{t,u_1,A}((u_2, ..., u_5), x^1, x^2, x^3) = f_{u_1,A}((u_2, ..., u_5), x^1, x^2, x^3)$$

$$+t(\sum_{i=2}^{5} u_i + x^2_4 - x^2_5 + x^3_2 - x^3_3)$$

where $t > 0$ is fixed and $(u_2, ..., u_5)$, (x^1, x^2, x^3) satisfy (2) and (3). Let $u_1 = 0$ and let $(u_2, ..., u_5)$ and $(x, y, z) \in \mathbb{R}^{15}$ satisfying (2) (3) maximize $g_{t,u_1,A}$. Assume that $x_2 \geq 0$. Then $u_i = 1/\sqrt{2}$, for $i = 2, 3, 4, 5$, $x = (0, 1/2, 1/2, 1/2, 1/2)$, $y = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2})$, and $z = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$.

Proof. By Lemma(2.7), the above mentioned x, y, z and $(u_2, ..., u_5)$ maximize $f_{0,A}$ and

$$f_{0,A}(u_2, ..., u_5), x^1, ..., x^3) = 3/2.$$

Since the maximum of $\sum_{i=2}^5 u_i + x^2_4 - x^2_5 + x^3_2 - x^3_3$ under restrictions $\sum_{i=2}^5 u_i^2 = 1 - u_1^2$, $\sum_{j=1}^5 (x^i_j)^2 = 1$ for $i = 2, 3$ is attained only for $u_i = \sqrt{(1 - u_1^2)/2}$ for $i = 2, 3, 4, 5$, $x^2 = y$ and $x^3 = z$),

$$g_{t,0,A}((u_2, ..., u_5), x^1, ..., x^3) = 3/2 + t(4/\sqrt{2} + 2).$$

Now assume that $(v_2, ..., v_5)$ and (x^1, y^1, z^1) maximize the function $g_{t,0,A}$. Hence in particular, $\sum_{i=1}^5 v_i = 4/\sqrt{2}$, which shows that $v_i = 1/\sqrt{2} = u_i$ for $i = 2, ..., 5$. Analogously, $y = y^1$ and $z = z^1$. By Lemma(2.4),

$$span[x^1, y^1, z^1] = span[x, y, z].$$

Assume that $x^1 = px + qy + rz$. Since $y = y^1$, $z = z^1$ and x^1, y^1, z^1 are orthonormal, we get $q = r = 0$. Hence $p = \pm 1$. Since $x^1_2 \geq 0$ and $x_2 > 0$, $x^1 = x$, which completes the proof. ■

The next lemma is a simple consequence of the Implicit Function Theorem.

LEMMA 2.17 *Let $U \subset \mathbb{R}^l$ be an open, non-empty set and let $f : U \times \mathbb{R}^n \to \mathbb{R}$ and $G_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, ..., k$ be fixed C^2 functions. Let $g : U \times \mathbb{R}^{n+k} \to \mathbb{R}$ be defined by*

$$g(u, x, d) = f(u, x) - \sum_{i=1}^{k} d_i G_i(x)$$

for $u \in U$, $x \in \mathbb{R}^n$ and $d \in \mathbb{R}^k$. Assume that $\frac{\partial g}{\partial z_j}(u^o, x^o, d^o) = 0$ for $j = 1, ..., n + k$ and

$$det(\frac{(\partial^2 g)}{\partial z_i \partial z_j}(u^o, x^o, d^o)) \neq 0$$

for some $(u^o, x^o, d^o) \in U \times R^{n+k}$ and $i, j = 1, ..., n+k$ (We do not differentiate with respect to the coordinates of u.) Assume that $(u^m, x^m, d^m) \in U \times R^{n+k}$, and $(w^m, y^m, z^m) \in U \times R^{n+k}$, are such that $(u^m, x^m, d^m) \to (u^o, x^o, d^o)$ and $(w^m, y^m, z^m) \to (u^o, x^o, d^o)$ with respect to any norm in $\mathbb{R}^{l+n+k}$. If, for any $m \in \mathbb{N}$, $\frac{\partial g}{\partial z_j}(u^m, x^m, d^m) = 0$ and $\frac{\partial g}{\partial z_j}(w^m, y^m, z^m) = 0$ for $j = 1, ..., n+k$ then

$$(u^m, x^m, d^m) = (w^m, y^m, z^m)$$

for $m \geq m_o$.

Proof. It suffices to apply the Implicit Function Theorem to the function

$$G(u, x, d) = (\frac{\partial g}{\partial z_1}(u, x, d), ..., \frac{\partial g}{\partial z_{n+k}}(u, x, d))$$

and $(u, x, d) = (u^o, x^o, d^o)$.

3 Determination of λ^5_3

In this section we will work with functions f_{u_1} and $f_{u_1,A}$ defined by(1) and (4). The next two theorems show how look like candidates for maximizing the function $f_{u_1,A}$.

THEOREM 3.1 *Let A be defined by (32). Fix $t \in \mathbb{R}$ and $u_1 \in [0,1)$. Let us consider a function $h_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \times \mathbb{R}^6 \times \mathbb{R}$ defined by:*

$$h_{u_1,A,t}((v_2,v_3,v_4,v_5),z^1,z^2,z^3,d_1,d_2,d_3,d_{12},d_{13},d_{23},d_7) \tag{41}$$

$$= \sum_{i,j=1}^{5} a_{ij} v_i v_j < z_i, z_j >_3 + t(\sum_{i=2}^{5} v_i + z_4^2 - z_5^2 + z_2^3 - z_3^3)$$

$$-(\sum_{j=1}^{3} d_j < z^j, z^j >_5 -1) - \sum_{i,j=1,i\leq j}^{3} d_{ij} < z^i, z^j >_5 - d_7 < (u_1,v),(u_1,v) >_5,$$

where $v = (v_2,v_3,v_4,v_5)$. Define for $i = 2,...,5$ $u_i = \sqrt{(1-u_1^2)}/2 = c$,

$$w = w(u_1) = \frac{4u_1 c}{\sqrt{(u_1^2 - 2c^2)^2 + 16c^2u_1^2} + 2c^2 - u_1^2},$$

$x_1^1 = w/\sqrt{1+w^2}$, $x_i^1 = \frac{1}{2\sqrt{1+w^2}}$, $i = 2,3$, $x_i^1 = \frac{-1}{2\sqrt{1+w^2}}$, $i = 4,5$,

$$x^2 = (0,0,0,1/\sqrt{2},-1/\sqrt{2}), x^3 = (0,1/\sqrt{2},-1/\sqrt{2},0,0),$$

$d_1 = 1/2 - c^2 + \sqrt{1+4c^2-28c^4}/2$, $d_2 = d_3 = 2c^2 + (1/\sqrt{2})t$, $d_{ij} = 0$ *for* $i,j = 1,2,3$, $i \leq j$ *and*

$$d_7 = 1 + t/(2c) + 2(x_2^1)^2 + (x_1^1 x_2^1 u_1)/c.$$

Then the above defined $x^1,x^2,x^3,u_2,u_3,u_4,u_5,d_1,d_2,d_3,d_{12},d_{13},d_{23},d_7$ satisfy the system of equations:

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}(x^1,x^2,x^3,u_2,u_3,u_4,u_5,d_1,d_2,d_3,d_{12},d_{13},d_{23},d_7) = 0$$

for $j = 1,...,26$, where

$$w_j \in \{v_2,v_3,v_4,v_5,z_k^i, k = 1,...,5, i = 1,2,3\}$$

and

$$w_j \in \{d_{ik}, i,k \in \{1,2,3\}, i \leq k, d_i, i = 1,2,3,7\}.$$

(We do not differentiate with respect to u_1).

Proof. Notice that the equations

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7) = 0$$

for

$$w_j \in \{z_k^i, k = 1, ..., 5, i = 1, 2, 3\}$$

follow from the fact that x^i, $i = 1, 2, 3$, are the orthonormal eigenvectors of the matrix B defined by (33) corresponding to the eigenvalues d_i, $i = 1, 2, 3$, which has been established in the proof of Theorem(2.5). Also the equations

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7) = 0$$

where

$$w_j \in \{d_1, d_2, d_3, d_{ik}, i, k \in \{1, 2, 3\}, i \leq k, d_7\}$$

follows immediately from the fact that $< x^i, x^j >_5 = \delta_{ij}$ for $i, j = 1, 2, 3, i \leq j$ and $< (u_1, u), (u_1, u) >_5 = 1$, where $u = (u_2, u_3, u_4, u_5)$. To end the proof, we show that

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7) = 0$$

for

$$w_j \in \{v_2, v_3, v_4, v_5\}.$$

Notice that for $i = 2, 3, 4, 5$

$$\frac{\partial h_{u_1,A,t}}{\partial w_i}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$$

$$= 2\sum_{j=1}^{5} u_j a_{ij} < x_i, x_j >_3 + t - 2u_i d_7.$$

Since $u_1 < 1$, $u_i = \sqrt{(1 - u_1^2)}/2 = c > 0$ for $i = 2, 3, 4, 5$. Hence

$$\frac{\partial h_{u_1,A,t}}{\partial w_i}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7) = 0$$

if and only if

$$(\sum_{j=1}^{5} u_j a_{ij} < x_i, x_j >_3)/c + t/(2c) = d_7.$$

Notice that for $i = 2, 3, 4, 5$,

$$(\sum_{j=1}^{5} a_{ij}u_j < x_i, x_j >_3)/c = x_i^1(\sum_{j=1}^{5} a_{ij}u_j x_j^i)/c$$

$$+x_i^2(\sum_{j=1}^{5} a_{ij}u_j x_j^2)/c + x_i^3(\sum_{j=1}^{5} a_{ij}u_j x_j^3)/c$$

$$= (u_1 a_{i1} x_1^1 x_i^1)/c + 2(x_i^1)^2 + 1/\sqrt{2}((1/\sqrt{2})c + (-1)(-1/\sqrt{2})c)/c$$

$$= 1 + (u_1 a_{i1} x_1^1 x_i^1)/c + 2(x_i^1)^2.$$

Hence for $i = 2, 3, 4, 5$,

$$d_7 = 1 + t/(2c) + 2(x_i^1)^2 + (x_1^1 a_{i1} x_i^1 u_1)/c.$$

Since $x_2^1 = x_3^1 = -x_4^1 = -x_5^1$, $1 = a_{21} = a_{31} = -a_{41} = -a_{51}$, and $u_i = c$ for $i = 2, 3, 4, 5$,

$$d_7 = 1 + t/(2c) + 2(x_2^1)^2 + (x_1^1 x_2^1 u_1)/c,$$

as required. ■

Reasoning as in Theorem(3.1), we can show

THEOREM 3.2 *Let A be defined by*

$$A = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 & -1 \\ 1 & -1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 & -1 \\ 1 & -1 & -1 & -1 & 1 \end{pmatrix} \qquad (42)$$

Fix $t \in \mathbb{R}$ and $u_1 \in [0, 1)$. Let us consider a function $h_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \times \mathbb{R}^6 \times \mathbb{R}$ given by (41) with A defined as above. Define for $i = 2, ..., 5$ $u_i = \sqrt{(1 - u_1^2)}/2 = c$,

$$x^1 = (0, 1/2, 1/2, -1/2, -1/2),$$

$$x^2 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}), x^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0),$$

$d_1 = 2c^2$ $d_2 = d_3 = 2c^2 + (1/\sqrt{2})t$, $d_{ij} = 0$ *for* $i, j = 1, 2, 3,$ $i \leq j$ *and*

$$d_7 = 3c + t/2c.$$

Then the above defined $x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7$ *satisfy the system of equations:*

$$\frac{\partial h_{u_1,A,t}}{\partial w_j}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7) = 0$$

for $j = 1, ..., 26,$ *where*

$$w_j \in \{v_2, v_3, v_4, v_5, z_k^1, k = 1, ..., 5, i = 1, 2, 3\}$$

and

$$w_j \in \{d_{ik}, i, j \in \{1, 2, 3\}, i \leq j, d_1, d_2, d_3, d_7\}.$$

(We do not differentiate with respect to u_1*).*

LEMMA 3.1 *Let* A *be defined by (32). For a fixed* $u_1 \in (0, 1)$ *and* $t > 0$ *let* $g_{u_1,A,t} : \mathbb{R}^4 \times (\mathbb{R}^5)^3 \to R$ *defined by*

$$g_{u_1,A,t}((v_2, ..., v_5), y^1, y^2, y^3) = \sum_{i,j=1}^{5} v_i v_j a_{ij} < y_i, y_j >_3 + t(\sum_{i=2}^{5} v_i + z_4^2 - z_5^2 + z_2^3 - z_3^3)$$

Let $M_{u_1,A,t} = \max g_{u_1,A,t}$ *under constraints:*

$$< y^i, y^j >_5 = \delta_{ij}, 1 \leq i \leq j \leq 3;$$

and

$$\sum_{j=2}^{5} v_j^2 = 1 - u_1^2.$$

Assume that $u_1 \in (0, 1)$ *is so chosen that*

$$M_{u_1,A,0} = f_{u_1,A}((u_2, u_3, u_4, u_5), x^1, x^2, x^3, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7),$$

where $u_2, u_3, u_4, u_5, x^1, x^2, x^3, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7$ *are as in Theorem(3.1) (for* $c = \sqrt{1 - u_1^2}/2$*). Set*

$$D_{u_1} = \{(v_2, v_3, v_4, v_5, y^1, y^2, y^3) : y_4^3 = y_5^3 = y_2^2 = 0, y_2^1 \geq 0\}. \quad (43)$$

Then

$$X_{u_1} = (u_2, u_3, u_4, u_5, x^1, x^2, x^3) \quad (44)$$

is the only point maximizing $g_{u_1,A,t}$ *satisfying (2) and (3) belonging to* D_{u_1}*.*

Proof. Let

$$Y_{u_1} = (v_2, v_3, v_4, v_5, y^1, y^2, y^3) \in D_{u_1}$$

maximize $g_{u_1,A,t}$ and satisfy (2) and (3). Since $t > 0$, and the maximum of $f_{u_1,A}$ is attained at X_{u_1}, we have $v_i = u_i = \sqrt{1 - u_1^2}/2$ for $i = 2, 3, 4, 5$, $y^2 = x^2$ and $x^3 = y^3$. Since x^1, x^2, x^3 are the eigenvectors of A, by Lemma(2.4), span$[y^i : i = 1, 2, 3]$=span$[x^i : i = 1, 2, 3]$. Note that

$$< x^1, x^i >_5 = < y^1, x^i >_5 = 0$$

for $i = 2, 3$. Since span$[y^i : i = 1, 2, 3]$=span$[x^i : i = 1, 2, 3]$, $y^1 = dx^1$. Since $< y^1, y^1 >_5 = 1$, $y_2^1 \geq 0$ and $x_2^1 > 0$, $x^1 = y^1$, as required. ■

THEOREM 3.3 *Let A be defined by (32). For a fixed $u_1 \in [0, 1)$ and $t \in \mathbb{R}$ let $g_{u_1,A,t}$ and $M_{u_1,A,t}$ be as in Lemma(3.1). Assume that $u_1 \in [0, 1)$ is so chosen that*

$$M_{u_1,A,0} = g_{u_1,A,t}(u_2, u_3, u_4, u_5, x^1, x^2, x^3)$$

where $u_2, u_3, u_4, u_5, x^1, x^2, x^3$ are as in Theorem(3.1) (for $c = \sqrt{1 - u_1^2}/2$). Let the function $h_{u_1,A,t}$ be defined by (41). Assume furthermore that the 23×23 matrix $D_{u,A,t}$ defined by

$$D_{u,A,t} = \frac{\partial h_{u_1,A,t}}{\partial w_i, \partial w_j}(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7), \quad (45)$$

where

$$w_i, w_j \in \{v_2, v_3, v_4, v_5, y_k^1, k = 1, ..., 5, y_1^2, y_3^2, y_4^2, y_5^2, y_1^3, y_2^3, y_3^3,$$

$$d_i, i = 1, 2, 3, 7, d_{ik}, 1 \leq i \leq k \leq 3\},$$

(we do not differentiate with respect to u_1, y_4^3, y_5^3, y_2^2) is such that

$$Det(D_{u,A,t}) = \sum_{j=o}^{k} a_j(u) t^j$$

and $a_j(u_1) \neq 0$ for some $j \in \{1, ..., k\}$. (Here $(d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$ are such as in Theorem(3.1) for $c = \sqrt{1 - u_1^2}/2$ and $t \in \mathbb{R}$.) Then there exists an open interval $U \subset [0, 1)$, ($U = [0, w)$ if $u_1 = 0$) such that $u_1 \in U$ and for *any $u \in U$ the function $f_{u,A}$ attains its global maximum under constraints (2) and (3) at*

$$X_u = (u_2, u_3, u_4, u_5, x^1, x^2, x^3),$$

where $u_i = c_u = \sqrt{1-u^2}/2$ for $i = 2,3,4,5$ and x^1, x^2, x^3, are defined in Theorem(3.1) (with $c = c_u$.) The same result holds true if A will be defined by (42). (In this case

$$(x^1, x^2, x^3, u_2, u_3, u_4, u_5, d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$$

are such as in Theorem (3.2).)

Proof. Fix $u_1 \in [0,1)$ satisfying our assumptions and let $c_1 = \sqrt{1-u_1^2}/2$. Let $j_o = min\{j \in \{0,...,k\} : a_j(u_1) \neq 0\}$. Set for $(u,t) \in [0,1) \times \mathbb{R}$,

$$h(t,u) = \sum_{j=j_o}^{k} a_j(u) t^{j-j_o}.$$

Since $a_{j_o}(u_1) \neq 0$, and a_j are continuous there exists an open interval $U \subset [0,1)$ and $\delta > 0$ such that $u_1 \in U$ and

$$h(t,u) \neq 0$$

for $u \in U$ and $|t| < \delta$. Fix $t_o \in (0,\delta)$. Set

$$U_{t_o} = \{u \in U : M_{u,A,t_o} \text{ is attained at } X_u\}.$$

Note that $u_1 \in U_{t_o}$. Now we show that U_{t_o} is an open set. Let $u_o \in U_{t_o}$. Assume on the contrary that there exist $\{u_n\} \in U \setminus U_{t_o}$ such that $u_n \to u_o$. Let for any $u \in U$,

$$Z_{u,t_o} = Z_u = (v_{2u}, v_{3u}, v_{4u}, v_{5u}, x^{1u}, x^{2u}, x^{3u})$$

be a point maximizing g_{u,A,t_o} under constraints (2) and (3). Since the function $f_{u,A} - g_{t,u,A}$ is independent of x^1 and by Lemma(2.14), the function $g_{u,A,t}$ can be considered as a function of 16 variables from $\mathbb{R}^5 \times \mathbb{R}^4 \times \mathbb{R}^3 \times \mathbb{R}^4$. Consequently, can assume that

$$Z_u \in D_u.$$

(see(43)). By (2) and (3), passing to a subsequence, if necessary, we can assume that $Z_{u_n} \to Z$. By definition of D_{u_o}, $Z \in D_{u_o}$. Also by the continuity of the function

$$(v, X) \to (\sum_{i,j=1}^{5} v_i v_j a_{ij} < y_i, y_j >_3 + t_o(\sum_{i=2}^{5} v_i + z_4^2 - z_5^2 + z_2^3 - z_3^3))$$

$$g_{u_o,A,t_o}(Z) = M_{u_o,A,t_o}.$$

By Lemma(2.16) and Lemma(3.1) X_{u_o} is the only point in D_{u_o} which maximizes $g_{u,A,t}$ and $Z \in D_{u_o}$. Hence $Z = X_{u_o}$. Moreover, since $X_{u_o} \in int(D_{u_o})$, by the Lagrange Multiplier Theorem, there exists

$$M_{u_n} = M_{u_n}(t_o) = \{d_i^n, i = 1, 2, 3, 7, d_{ij}^n, 1 \leq i \leq j \leq 3\} \subset \mathbb{R}^7$$

such that

$$\frac{\partial h_{u,A,t_o}}{\partial w_i}(Z_{u_n}, M_{u_n}) = 0, \tag{46}$$

for $w_i \in X \cup DD$. Here $h_{u,A,t}$ is defined by (41) and

$$DD = \{d_i, i = 1, 2, 3, 7, d_{ij}, 1 \leq i \leq j \leq 3\}.$$

Also by (2), (3),(7),(8) (see the proof of Lemma(2.4)) and (46)

$$M_n \to L_{u_o} = L_{u_o}(t_o) = (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7),$$

where L_{u_o} is defined in Theorem(3.1) for $c = \sqrt{1 - u_o^2}/2$ and $t = t_o$. Now we apply Lemma(2.17). Let us consider a function $G : U \times \mathbb{R}^{12} \times \mathbb{R}^4 \times \mathbb{R}^7 \to \mathbb{R}^{23}$ defined by

$$G(u, x, v, Q) = (\frac{\partial h_{u,A,t_o}}{\partial w_1}(u, x, v, Q)), ..., \frac{\partial h_{t_o,u}}{\partial w_{23}}(u, x, v, Q))/(t_o)^{j_o/23}$$

for for $w_i \in X \cup DD$. Notice that by (46)

$$G(u_n, Z_{u_n}, M_{u_n}) = 0.$$

Also $G(u_n, X_{u_n}, L_{u_n}(t_o)) = 0$, where $(X_{u_n}, L_{u_n}(t_o))$ are defined for u_n in Theorem(3.1). Moreover,

$$(u_n, Z_{u_n}, M_{u_n}) \to (u_o, X_{u_o}, L_{u_o})$$

and

$$(u_n, X_{u_n}, L_{u_n}) \to (u_o, X_{u_o}, L_{u_o}).$$

Notice that

$$Det(\frac{\partial G}{\partial w_j}(u_o, X_{u_o}, L_{u_o}))$$

$$= \frac{det(D_{u_o,A,t_o})}{(t_o^{j_o/23})^{23}} = \sum_{j=j_o}^{k} a_j(u_o) t_o^{j-j_o} = h(t_o, u_o) \neq 0,$$

by definition of j_o and t_o. By Lemma(2.17) applied to the function G, $Z_{u_n} = X_{u_n}$ and $M_{u_n} = L_{u_n}$ for $n \geq n_o$. Hence $u_n \in U_1$ for $n \geq n_o$; a contradiction. This shows that U_{t_o} is an open set. It is clear that U_{t_o} is closed. Since $u_1 \in U_{t_o}$ and U is connected, $U_{t_o} = U$. Consequently for any $n \in \mathbb{N}$, $n \geq n_o$ and $u \in U$, the functions $g_{u,A,1/n}$ achieve their maximum at $u_2, u_3, u_4, u_5, x^1, x^2, x^3$, where $u_i = c_u = \sqrt{1-u^2}/2$ for $i = 2,3,4,5$ and x^1, x^2, x^3, are defined in Theorem(3.1) (with $c = c_u$). Since $g_{u,A,1/n}$ tends uniformly to $g_{u,A,0} = f_{u,A}$, on the set defined by (2) and (3), with $u \in U$ fixed, $f_{u,A}$ attains its maximum at $u_2, u_3, u_4, u_5, x^1, x^2, x^3$ for any $u \in U$.
By Theorem(3.2), reasoning exactly in in the same way as above we can deduce our conclusion for the function $f_{u,A}$ determined by A given by (42). The proof is complete. ■

Now we show that the assumptions of Theorem(3.3) concerning $D_{u,A,t}$ are satisfied. This is the most important technical result which permits us to determine the constant λ_3^5.

THEOREM 3.4 *Let A be defined by (32) and let $D_{u,A,t}$ be given by (45). Then for any $u \in [0,1)$ and $t \in \mathbb{R}$,*

$$Det(D_{u,A,t}) = \sum_{j=0}^{7} a_j(u) t^j,$$

where the functions a_j is continuous for $j = 0, ..., 7$ and $a_7(u) \neq 0$ for any $u \in [0,1)$.

Proof. Set

$$X = (x_1, b, b-b, -b, 0, 0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 1/\sqrt{2}, -1/\sqrt{2}),$$

$$B = (b_1, d, d, 0, 0, 0, b_7)$$

and

$$v = (c, c, c, c).$$

Assume that we will differentiate $h_{u,A,t}$ in the following manner:

$$(w_1, ..., w_5) = (x_1^1, ...x_5^1), (w_6, ..., w_{11}) = (b_1, b_2, b_3, b_{12}, b_{13}, b_{23})$$

$$(w_{12}, ..., w_{18}) = (x_1^2, x_3^2, x_4^2, x_5^2, x_1^3, x_2^3, x_3^3), (w_{19}, ..., w_{23}) = (u_2, u_3, u_4, u_5, b_7).$$

(Recall that we do not differentiate with respect to u_1, x_2^2, x_4^3 and x_5^3.) Notice that by elementary but very tedious calculations (which we verified by a symbolic Mathematica program) we get that the 23×23 symmetric matrix $C = D_{u,A,t}(X, B, v)$ is given by

$$C = \begin{pmatrix} A_1 & B_1 \\ (B_1)^T & A_2 \end{pmatrix}. \tag{47}$$

Here

$$A_1 = \begin{pmatrix} 2(u^2-b_1) & 2cu & 2cu & -2cu & -2cu & -2x_1 & 0 & 0 & 0 & 0 & 0 \\ 2cu & 2(c^2-b_1) & -2c^2 & -2c^2 & -2c^2 & -2b & 0 & 0 & 0 & -1/\sqrt{2} & 0 \\ 2cu & -2c^2 & 2(c^2-b_1) & -2c^2 & -2c^2 & -2b & 0 & 0 & 0 & 1/\sqrt{2} & 0 \\ -2cu & -2c^2 & -2c^2 & 2(c^2-b_1) & -2c^2 & 2b & 0 & 0 & -1/\sqrt{2} & 0 & 0 \\ -2cu & -2c^2 & -2c^2 & -2c^2 & 2(c^2-b_1) & 2b & 0 & 0 & 1/\sqrt{2} & 0 & 0 \\ -2x_1 & -2b & -2b & 2b & 2b & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{48}$$

$A_2 = (A_{12}, A_{22})$, where

$$A_{12} = \begin{pmatrix} 2(u^2-d) & 2cu & -2cu & -2cu & 0 & 0 & 0 \\ 2cu & 2(c^2-d) & -2c^2 & -2c^2 & 0 & 0 & 0 \\ -2cu & -2c^2 & 2(c^2-d) & -2c^2 & 0 & 0 & 0 \\ -2cu & -2c^2 & -2c^2 & 2(c^2-d) & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(u^2-d) & 2cu & 2cu \\ 0 & 0 & 0 & 0 & 2cu & 2(c^2-d) & -2c^2 \\ 0 & 0 & 0 & 0 & 2cu & -2c^2 & 2(c^2-d) \\ 0 & 0 & 0 & 0 & \sqrt{2}u & 3\sqrt{2}c & -\sqrt{2}c \\ 0 & 0 & 0 & 0 & -\sqrt{2}u & \sqrt{2}c & -3\sqrt{2}c \\ -\sqrt{2}u & -\sqrt{2}c & 3\sqrt{2}c & -\sqrt{2}c & 0 & 0 & 0 \\ \sqrt{2}u & \sqrt{2}c & \sqrt{2}c & -3\sqrt{2}c & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix} \tag{49}$$

and

$$A_{22} = \begin{pmatrix} 0 & 0 & -\sqrt{2}u & \sqrt{2}u & 0 \\ 0 & 0 & -\sqrt{2}c & \sqrt{2}c & 0 \\ 0 & 0 & 3\sqrt{2}c & \sqrt{2}c & 0 \\ 0 & 0 & -\sqrt{2}c & -3\sqrt{2}c & 0 \\ \sqrt{2}u & -\sqrt{2}u & 0 & 0 & 0 \\ 3\sqrt{2}c & \sqrt{2}c & 0 & 0 & 0 \\ -\sqrt{2}c & -3\sqrt{2}c & 0 & 0 & 0 \\ 2b^2-2b_7+1 & 1-2b^2 & 2b^2 & 2b^2 & -2c \\ 1-2b^2 & 2b^2-2b_7+1 & 2b^2 & 2b^2 & -2c \\ 2b^2 & 2b^2 & 2b^2-2b_7+1 & 1-2b^2 & -2c \\ 2b^2 & 2b^2 & 1-2b^2 & 2b^2-2b_7+1 & -2c \\ -2c & -2c & -2c & -2c & 0 \end{pmatrix}; \tag{50}$$

$$B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2bu & 2bu & 2bu & 2bu & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 6bc+2ux_1 & -2bc & 2bc & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & 6bc+2ux_1 & 2bc & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & -2bc & -6bc-2ux_1 & 2bc & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2bc & -2bc & 2bc & -6bc-2ux_1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & \sqrt{2} & 0 & 0 & 0 & 0 & 0 \\ -x_1 & -b & b & b & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -x_1 & -b & -b & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{51}$$

Notice that in 11-st row of C the only non-zero element is $c_{11,13} = c_{13,11} = 1/\sqrt{2}$ and in 23-rd row of A the only elements which could be different from 0 are $c_{23,19} = c_{23,20} = c_{23,21} = c_{23,22} = -2c$. Also the only non-zero elements in 7-th row are $c_{7,14} = -\sqrt{2}$ and $c_{7,15} = \sqrt{2}$. Analogously, the only non-zero elements in 8-th row are $c_{8,17} = -\sqrt{2}$ and $c_{8,18} = \sqrt{2}$.Consequently, applying the symmetry of C, subtracting 19-th row from 20, 21 and 22-nd row, 19-th column from 20, 21 and 22-nd column, adding 15-th row to 14-th row and 15-th column to 14-th column and adding 18-th row to 17-th row and 18-th column to 17-th column we get that

$$det(C) = 8c^2 det(D),$$

where D is a 15×15 symmetric matrix defined by

$$D = \begin{pmatrix} D_1 & E \\ E^T & D_2 \end{pmatrix}. \tag{52}$$

Here

$$D_1 = \begin{pmatrix} 2(u^2-b_1) & 2cu & 2cu & -2cu & -2cu & -2x_1 & 0 & 0 \\ 2cu & 2(c^2-b_1) & -2c^2 & -2c^2 & -2c^2 & -2b & 0 & -1/\sqrt{2} \\ 2cu & -2c^2 & 2(c^2-b_1) & -2c^2 & -2c^2 & -2b & 0 & 1/\sqrt{2} \\ -2cu & -2c^2 & -2c^2 & 2(c^2-b_1) & -2c^2 & 2b & -1/\sqrt{2} & 0 \\ -2cu & -2c^2 & -2c^2 & -2c^2 & 2(c^2-b_1) & 2b & 1/\sqrt{2} & 0 \\ -2x_1 & -2b & -2b & 2b & 2b & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\sqrt{2} & 1\sqrt{2} & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 1/\sqrt{2} & 0 & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{53}$$

$D_2 = (D_{21}, D_{22})$, where

$$D_{21} = \begin{pmatrix} 2(u^2-d) & -4cu & 0 & 0 \\ -4cu & -4d & 0 & 0 \\ 0 & 0 & 2(u^2-d) & 4cu \\ 0 & 0 & 4cu & -4d \\ 0 & 0 & -2\sqrt{2}u & -4\sqrt{2}c \\ -\sqrt{2}u & 2\sqrt{2}c & -\sqrt{2}u & -2\sqrt{2}c \\ \sqrt{2}u & -2\sqrt{2}c & -\sqrt{2}u & -2\sqrt{2}c \end{pmatrix} \tag{54}$$

and

$$D_{22} = \begin{pmatrix} 0 & -\sqrt{2}u & \sqrt{2}u \\ 0 & 2\sqrt{2}c & -2\sqrt{2}c \\ -2\sqrt{2}u & -\sqrt{2}u & -\sqrt{2}u \\ -4\sqrt{2}c & -2\sqrt{2}c & -2\sqrt{2}c \\ 8b^2-4b_7 & 4b^2-2b_7 & 4b^2-2b_7 \\ 4b^2-2b_7 & 2-4b_7 & 2-4b^2-2b_7 \\ 4b^2-2b_7 & 2-4b^2-2b_7 & 2b^2-4b_7 \end{pmatrix}; \tag{55}$$

$$E = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -8bc-2ux_1 & -4bc-2ux_1 & -4bc-2ux_1 \\ 0 & 0 & 0 & 0 & 8bc+2ux_1 & 4bc & 4bc \\ 0 & 0 & 0 & 0 & 0 & -4bc-2ux_1 & 4bc \\ 0 & 0 & 0 & 0 & 0 & 4bc & -4bc-2ux_1 \\ 0 & 0 & 0 & 0 & 0 & 0 & \\ -x_1 & 2b & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -x_1 & -2b & 0 & 0 & 0 \end{pmatrix}; \tag{56}$$

Now we calculate the coefficient $a_7(u)$. Notice that

$$Det(C(t)) = Det(D_{u,A,t}(X,B,v)) = 8c^2 Det(D(t)),$$

where $C(t)$ and $D(t)$ denote the above written matrices C and D with b_7 replaced by $b_7 + t/2c$ and $d_2 = d_3 = d$ replaced by $d + (1/\sqrt{2})t$. By definition of determinant

$$det(D(t)) = \sum_{\sigma \in \Pi_{15}} sgn(\sigma)(\sum_{i=1}^{15} d_{i,\sigma(i)}),$$

where Π_{15} denotes the set of all permutations of $\{1, ..., 15\}$. Notice that by the above given formulas the variable t appears only in D_{21} and D_{22}. Consequently to calculate $a_7(u)$ it is enough to consider

$$\sum_{\sigma \in P_1} sgn(\sigma)(\sum_{i=1}^{15} d_{i,\sigma(i)}),$$

where

$$P_1 = \{\sigma \in \Pi : \sigma(\{13, 14, 15\}) = \{13, 14, 15\}, \sigma(j) = j, j = 9, 10, 11, 12\}.$$

Consequently, applying the formula on D_{21} and D_{22} we can deduce that

$$a_7(u) = (2^7/c) det(F) det(D_1),$$

where

$$F = -\begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}. \tag{57}$$

Note that $det(F) = -4$. Also $c = \sqrt{1-u_1^2}/2 > 0$ for $u_1 \in [0.1)$. Hence to end the proof we should demonstrate that $Det(D_1) \neq 0$ for (X, B, v) defined in Theorem(3.1). Let $E_1, ...E_8$ denote the rows of D_1. We show that

$E_1, ..., E_8$ are linearly independent. First assume that $u = u_1 = 0$. Then $x_1 = 0, c = b_1 = 1/2$. Let

$$\sum_{j=1}^{8} \alpha_j E_j = 0. \tag{58}$$

Since $b_1 = 1/2$, $\alpha_1 = 0$. Also $(\alpha_i - \alpha_{i+1})/\sqrt{2} = 0$ for $i = 2, 4$, which gives $\alpha_2 = \alpha_3$ and $\alpha_4 = \alpha_5$. Since $2b(2\alpha_4 - 2\alpha_2) = 0$, and $b = 1/2$, $\alpha_2 = \alpha_4$. Consequently,

$$\alpha_2(-4c^2 - 2b_1) = \alpha_i\sqrt{2} = -\alpha_i\sqrt{2}$$

for $i = 7, 8$, which gives, $\alpha_7 = \alpha_8 = 0$. Analogously,

$$\alpha_2(-4c^2 - 2b_1) = 2b\alpha_6 = -2b\alpha_6,$$

which implies $\alpha_6 = 0$ and $\alpha_2 = 0$. Consequently, $Det(D_1) \neq 0$. Now assume that $u = u_1 \in (0, 1)$. Reasoning as in the previous case we can show that $\alpha_2 = \alpha_3$ and $\alpha_4 = \alpha_5$. Also

$$\alpha_1 2cu_1 - \alpha_2 2b_1 - \alpha_4 4c^2 - 2b\alpha_6 = \alpha_8\sqrt{2} = -\alpha_8\sqrt{2}$$

and

$$-\alpha_1 2cu_1 - \alpha_2 4c^2 - \alpha_4 2b_1 + 2b\alpha_6 = \alpha_7\sqrt{2} = -\alpha_7\sqrt{2},$$

which implies $\alpha_7 = \alpha_8 = 0$. By the above equations

$$-\alpha_2 2b_1 - \alpha_4 4c^2 = 2b\alpha_6 - \alpha_1 2cu_1$$

and

$$-\alpha_2 4c^2 - \alpha_4 2b_1 = -2b\alpha_6 + \alpha_1 2cu_1.$$

Hence

$$-\alpha_2(2b_1 + 4c^2) = \alpha_4(2b_1 + 4c^2).$$

Since $b_1 > 0$, $\alpha_4 = -\alpha_2$. Consequently, applying (58) to 1-st, 5-th and 6-th column of D_1 we get

$$\alpha_1(2(u_1^2 - b_1), -2cu_1, -2x_1) + \alpha_2(8cu_1, 2d_1 - 4c^2, -8b) + \alpha_6(-2x_1, 2b, 0) = 0.$$

Let

$$G = \begin{pmatrix} 2(u_1^2 - b_1) & -2cu_1 & -2x_1 \\ 8cu_1 & 2b_1 - 4c^2 & -8b \\ -2x_1 & 2b & 0 \end{pmatrix}. \tag{59}$$

Note that

$$Det(G) = -8(4b^2(b_1 - u_1^2) + 8bcu_1x_1 + (b_1 - 2c^2)x_1^2).$$

By Theorem(2.5) and Remark(2.1), $b_1 = \lambda_1 > 2c^2$ and $b_1 = \lambda_1 > u_1^2$. Hence $Det(G) < 0$, which means that $\alpha_1 = \alpha_2 = \alpha_6 = 0$. This shows that $Det(D_1) \neq 0$ and consequently $a_7(u) \neq 0$, for any $u \in [0, 1)$. The proof is complete. ■

REMARK 3.1 *Applying a symbolic Mathematica program we can show that*

$$Det(D_{11}) = -64(b_1 - 2c^2)^2(2c^2 + b_1)(4b^2(b_1 - u_1^2) + 8bcu_1x_1 + (b_1 - 2c^2)x_1^2).$$

Now we will prove one of the main results of this section

THEOREM 3.5 *Let f_{u_1} be defined by (1), i.e.*

$$f_{u_1}(u_2, u_3, u_4, u_5, x^1, x^2, x^3) = \sum_{i,j=1}^{5} u_iu_j| < x_i, x_j > |_3.$$

Let $M_u = \max(f_u)$ under constraints (2) and (3). Then for any $u \in [0, 1]$

$$M_u = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2},$$

where $c = c(u) = \sqrt{1 - u^2}/2$.

Proof. Define

$$U = \{u \in [0, 1) : M_u = \frac{1 + 6c^2 + \sqrt{(6c^2 - 1)^2 + 16(1 - 4c^2)c^2}}{2}\}.$$

By Lemma(2.6) and Lemma(2.7), $0 \in U$, since $M_o = 3/2$. Now we show that U is an open set. Fix $u \in U$. First we consider the case $u = 0$. We apply Theorem(3.3) and Theorem(3.4). Let (X_v, L_v) where

$$X_u = (x^1, x^2, x^3, c(v), c(v), c(v), c(v)),$$

$(c(0) = 1/2)$ and

$$L_v(t) = (d_1, d_2, d_3, d_{12}, d_{13}, d_{23}, d_7)$$

are given by Theorem(3.1) for for fixed $v \in [0, 1)$ and $t \in \mathbb{R}$. Assume that $u_n \to 0$ and $u_n \notin U$. Let $(X_{u_n}, L_{u_n}(t))$ be such as in Theorem(3.3). Passing to a subsequence, if necessary, and reasoning as in Theorem(3.3), we can assume that $(X_{u_n}, L_{u_n}(t)) \to (X_o, L_o(t))$. Let

$$X_{u_n} = (x^{1n}, x^{2n}, x^{3n}, c(u_n), c(u_n), c(u_n), c(u_n)).$$

Since $X_{u_n} \to X_o$, we can assume that $sgn < x_{in}, x_{jn} >_3 = -1$ for $i, j = 2, 3, 4, 5$, $i \neq j$. Without loss of generality, passing to a subsequence if necessary we can assume that for $n \geq n_o$

$$sgn < x_{1n}, x_{jn} >_3 = z_j$$

for $j = 2, 3, 4, 5$, where $z_j = \pm 1$. By Lemma(2.8) we have to consider three cases:

a) $z_2 = -1, z_3 = z_4 = z_5 = 1$;
b) $z_2 = z_3 = z_4 = z_5 = 1$;
c) $z_2 = z_3 = -z_4 = -z_5 = 1$.

By Theorem(2.3) and Theorem(2.5) a) can be excluded. If b) holds true, then by Theorem(3.3), Theorem(3.2) and Theorem(3.4) applied to $u_1 = 0$ and $h_{t,A,0}$, where A is given by (42), we get that

$$M_{u_n} = 6c_u^2 \leq 3/2,$$

which by Theorem(2.5) leads to a contradiction. (Since $u_1 = 0$, $D_{o,A,t}$ is the same for the function $h_{o,A,t}$, determined by A given by (42). This permits us to apply Theorem(3.4) in this case.) If c) holds true, we get a contradiction with Theorem(3.3). Consequently, there exists an interval $[0, v) \subset U$.
Now assume that $u \in U$ and $u > 0$. Assume $u_n \to u$ and $u_n \notin U$ for $n \in \mathbb{N}$. Let $(X_{u_n}, L_{u_n}(t))$ be such as in Theorem(3.3). Without loss of generality we can assume that $(X_{u_n}, L_{u_n}(t)) \to (X_u, L_u(t))$, where $(X_u, L_u(t))$ is defined in Theorem(3.3). Since $X_{u_n} \to X_u$

$$sgn < x_{in}, x_{jn} >_3 = a_{ij}$$

for $i, j = 1, 2, 3, 4, 5$ for $n \geq n_o$, where the matrix $\{a_{ij}\}$ is given by (32). Applying Theorem(3.3), we get that $u_n \in U$ for $n \geq n_o$; a contradiction. Hence the set U is open. It is easy to see that U is also closed. Since $0 \in U$ and $[0, 1)$ is connected, $U = [0, 1)$. Since $M(1, 0) = 1$ the proof is complete. ■

THEOREM 3.6

$$\lambda_3^5 = \frac{5+4\sqrt{2}}{7}.$$

Moreover, $\lambda_3^5 = \lambda(V)$, *where* $V \subset l_\infty^{(5)}$ *is spanned by*

$$x^1 = (a/u_1, b/c_o, b/c_o, -b/c_o, -b/c_o),$$

$$x^2 = (0,0,0,1/\sqrt{2}, -1/\sqrt{2})/c_o$$

and

$$x^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)/c_o,$$

where

$$u_1 = \sqrt{(5-3\sqrt{2})/7},\ c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}.$$

and

$$a = \sqrt{(2\sqrt{2}-1)/7},\ b = \sqrt{1-a^2}/2,$$

Proof. Let $f_{3,5} : \mathbb{R}^5 \times (\mathbb{R}^5)^3 \to R$ be defined by

$$f_{3,5}((v_1, v_2, ..., v_5), y^1, y^2, y^3) = \sum_{i,j=1}^{5} v_i v_j | < y_i, y_j >_3 |$$

Let $M_{3,5} = \max f_{3,5}$ under constraints:

$$< y^i, y^j >_5 = \delta_{ij}, 1 \le i \le j \le 3;$$

and

$$\sum_{j=1}^{5} v_j^2 = 1.$$

By Theorem(2.2),

$$\lambda_3^5 = M_{3,5}.$$

By Theorem(3.5),

$$M_{3,5} = max\{h(c) = \frac{1+6c^2+\sqrt{(6c^2-1)^2+16(1-4c^2)c^2}}{2}\} : c \in [0,1/2]\}.$$

By Theorem(2.5), $c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$ and

$$M_{3,5} = h(c_o) = \frac{5+4\sqrt{2}}{7}.$$

By the proof of Theorem(2.2), and Theorem(2.5), the function $f_{3,5}$ attains its maximum at $z^1 = (a, b, b, -b, -b)$, $z^2 = (0, 0, 0, 1/\sqrt{2}, -1/\sqrt{2})$ and $z^3 = (0, 1/\sqrt{2}, -1/\sqrt{2}, 0, 0)$, $u = (u_1, c_o, c_o, c_o, c_o)$, where

$$u_1 = \sqrt{(5-3\sqrt{2})/7},\ c_o = \frac{\sqrt{(2+3\sqrt{2})/7}}{2}$$

and

$$a = \sqrt{(2\sqrt{2}-1)/7},\ b = \sqrt{1-a^2}/2.$$

By the proof of Theorem(2.2), x^1, x^2 and x^3, defined in the statement of our theorem, form a basis of a space V satisfying $\lambda(V) = \lambda_3^5$. ■

REMARK 3.2 *Note that (compare with [11], p. 259) $3/2 = \lambda_3^4 < \lambda_3^5$. Also $\lambda_2^3 = 4/3$ and by the Kadec-Snobar Theorem ([7]) $\lambda_2^4 \leq \sqrt{2} < 3/2$. If x^1, x^2, x^3, u are such as in Theorem(3.6), then after elementary calculations we get*

$$\|x_1\|_3 = \sqrt{\frac{2\sqrt{2}-1}{5-3\sqrt{2}}}$$

and

$$\|x_2\|_3 = \sqrt{\frac{22-2\sqrt{2}}{2+3\sqrt{2}}},$$

where $x_1 = (x_1^1, x_1^2, x_1^3)$, $x_2 = (x_2^1, x_2^2, x_2^3)$ and $\|\cdot\|_3$ is the Euclidean norm in $\mathbb{R}^3$. Hence it is easy to see that

$$\|x_1\|_3 = \|x_2\|_3$$

if and only if

$$77\sqrt{2} = 112,$$

which is false. Consequently, by the above calculations and Theorem(3.6) Proposition 3.1 from [11] is incorrect.

4 A proof of the Grünbaum conjecture

Our proof of the Grünbaum conjecture will be given by the induction argument. First we show that $\lambda_2^3 = \lambda_2^4 = 4/3$. The proof of this fact goes exactly in the same way as the proof presented in the previous section for determination λ_3^5. Then assuming that $\lambda_2^N = 4/3$, we show that $\lambda_2^{N+1} = 4/3$. In this section we will work with a function $f^N_{u_{N-3}}$ instead of f_{u_1} ($u_{N-3} \in [0,1]$ will be fixed). The next three theorems show how look like candidates for maximizing the function $f^N_{u_{N-3}}$ given by

$$f^N_{u_{N-3}}(v_1, v_2, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), z^1, z^2) = \sum_{i,j=1}^{N} v_i v_j | < z_i, z_j >_2 |. \quad (60)$$

Also define for any $N \times N$ matrix A, as in Section 1

$$f^N_{u_{N-3},A}(v_1, v_2, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), z^1, z^2) = \sum_{i,j=1}^{N} a_{ij} v_i v_j < z_i, z_j >_2 . \quad (61)$$

THEOREM 4.1 *Let A be an $N \times N$ symmetric matrix defined by*

$$A = \begin{pmatrix} 1 & a_{1,2} & \cdots & a_{1,N-3} & a_{1,N-2} & a_{1,N-1} & a_{1,N} \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ a_{N-3,1} & a_{N-3,2} & \cdots & 1 & 1 & 1 & a_{N-3,N} \\ a_{N-2,1} & a_{N-2,2} & \cdots & 1 & 1 & -1 & a_{N-2,N} \\ a_{N-1,1} & a_{N-1,2} & \cdots & 1 & -1 & 1 & a_{N-1,N} \\ a_{N,1} & a_{N,2} & \cdots & a_{N,N-3} & a_{N,N-2} & a_{N,N-1} & 1 \end{pmatrix}, \quad (62)$$

where $a_{ij} \in \{-1, 1\}$ for $i \neq j$. Assume additionally that

$$a_{j,N} = a_{N,j} = -1$$

for $j = N-3, N-2, N-1$. Fix $t \in \mathbb{R}$ and $u_{N-3} \in [0, 1/\sqrt{3})$. Let us consider a function $h^N_{u_{N-3},A,t} : \mathbb{R}^{N-1} \times (\mathbb{R}^N)^2 \times \mathbb{R}^3 \times \mathbb{R}$ defined by:

$$\begin{aligned} & h_{u_{N-3},A,t}((v_1, v_2, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), z^1, z^2, b_1, b_2, b_{12}, b_4) \quad (63) \\ & = f_{u_{N-3},A}((v_1, v_2, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), z^1, z^2) \\ & + t((v_{N-1} + v_{N-2})/\sqrt{1 - 3u^2_{N-3}} + v_N + z^2_{N-1} - z^2_{N-2}) \end{aligned}$$

$$-(b_1(< z^1, z^1 >_N -1) + b_2(< z^2, z^2 >_N -1))$$
$$-b_{12} < z^1, z^2 >_N -b_4(< (u_{N-3}, v), (u_{N-3}, v) >_N -1),$$

where $v = (v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N)$. *Let us define for fixed* $N \in \mathbb{N}$,

$$u^N = u = (0, ..., 0_{N-4}, u_{N-2}, u_{N-1}, u_N),$$

$$x^{1N} = x^1 = (0, ..., 0_{N-4}, x^1_{N-3}, x^1_{N-2}, x^1_{N-1}, x^1_N),$$
$$x^{2N} = x^2 = (0, ..., 0_{N-4}, x^2_{N-3}, x^2_{N-2}, x^2_{N-1}, x^2_N)$$

and

$$d^N = d = (d_1, d_2, d_{12}, d_4).$$

Here $u_{N-2} = u_{N-1} = 1/\sqrt{3}$, $u_N = \sqrt{1/3 - u^2_{N-3}}$, $x^1_{N-3} = \sqrt{2}u_{N-3}$, $x^1_{N-2} = x^1_{N-1} = 1/\sqrt{6}$, $x^1_N = -\sqrt{2(1 - 3u^2_{N-3})}/\sqrt{3}$, $x^2_{N-3} = 0$, $x^2_{N-2} = -x^2_{N-1} = -1/\sqrt{2}$, $x^2_N = 0$, $d_1 = 2/3$, $d_2 = 2/3 + t(/\sqrt{2})$, $d_{12} = 0$ *and* $d_4 = 4/3 + t/(2\sqrt{1/3 - u^2_{N-3}})$. *Then the above defined* x^1, x^2, u, d *satisfy the system of equations:*

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, v, d) = 0$$

for $j = 1, ..., 3N + 3$ *where*

$$w_j \in \{v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N, z^i_k, i = 1, 2, k = 1, ..., N\},\ j = 1, ..., 3N-1$$

and

$$w_j \in \{b_{12}, b_1, b_2, b_4\},\ j = 3N, ..., 3N + 3.$$

(We do not differentiate with respect to u_{N-3}*).*

Proof. Notice that the equations

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{z^i_k, i = 1, 2; k = 1, ..., N\}$$

follow from the fact that (for $N = 4,$) $x^{i4} = x^i$, $i = 1, 2$, are the orthonormal eigenvectors of the matrix B defined by (27) corresponding to the eigenvalues d_i, $i = 1, 2$ which has been established in Lemma(2.9). Also the equations

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{b_{12}, b_1, b_2, b_4\}.$$

follows immediately from the fact that $< x^i, x^j >_N = \delta_{ij}$ for $i, j = 1, 2, i \leq j$ and $< (u_{N-3}, u), (u_{N-3}, u) >_N = 1$. To end the proof, we show that

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N\}.$$

Notice that, for $i = 1, ..., N - 4$,

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_i}(x^1, x^2, u, d) = 2\sum_{j=1}^{N} u_j a_{ij} < x_i, x_j >_2 - 2u_i d_4 = 0$$

since $x_i = 0$ and $u_i = 0$ for $i = 1, ..., N - 4$. Now assume that $w_i = u_{N-2}$. Then

$$2\sum_{j=1}^{N} u_j a_{N-2,j} < x_{N-2}, x_j >_2 + t/\sqrt{1 - 3u^2_{N-3}} - 2u_{N-2} d_4$$

$$= 2(u^2_{N-3}/\sqrt{3} + 1/\sqrt{3} + (1/3\sqrt{3})(1 - 3u^2_{N-3}) + t/2\sqrt{1 - 3u^2_{N-3}} - u_{N-2} d_4)$$

$$= 2((4/3)/\sqrt{3} + t/2\sqrt{1 - 3u^2_{N-3}} - (4/3)u_{N-2} - t/(2\sqrt{1/3 - u^2_{N-3}})u_{N-2}) = 0.$$

The same calculation works for $i = N - 1$. If $i = N$, then

$$2\sum_{j=1}^{N} u_j a_{N,j} < x_N, x_j >_2 + t - 2u_N d_4$$

$$= 2(2u^2_{N-3}\sqrt{1 - 3u^2_{N-3}}/\sqrt{3} + 2\sqrt{1 - 3u^2_{N-3}}/3\sqrt{3} + t/2$$

$$+(2/3)(1-3u_{N-3}^2)\sqrt{(1/3)-u_{N-3}^2}-u_N d_4)$$

$$=2(2u_{N-3}^2\sqrt{(1/3)-u_{N-3}^2}+(4/3)\sqrt{(1/3)-u_{N-3}^2}+t/2$$

$$-2u_{N-3}^2\sqrt{(1/3)-u_{N-3}^2}-d_4u_N)=0,$$

which completes the proof. ■

THEOREM 4.2 *Let A be an $N\times N$ symmetric matrix defined by (62.) Fix $t\in\mathbb{R}$ and $u_{N-3}\in[1/\sqrt{3},1)$. Let us consider a function $h^N_{u_{N-3},A,t}$: $\mathbb{R}^{N-1}\times(\mathbb{R}^N)^2\times\mathbb{R}^3\times\mathbb{R}$ defined by:*

$$h^N_{u_{N-3},A,t}((v_1,v_2,...v_{N-4},v_{N-2},v_{N-1},v_N),z^1,z^2,b_1,b_2,b_{12},b_4)$$

$$=\sum_{i,j=1}^{N}a_{ij}v_iv_j<z_i,z_j>_2+t(u_{N-2}+u_{N-1}+z^2_{N-1}-z^2_{N-2})$$

$$-(b_1(<z^1,z^1>_N-1)+b_2(<z^2,z^2>_N-1))$$

$$-b_{12}<z^1,z^2>_N-b_4(<(u_{N-3},v),(u_{N-3},v)>_N-1)$$

where $v=(v_1,v_2,...,v_{N-4},v_{N-2},v_{N-1},v_N)$. Let us define for fixed $N\in\mathbb{N}$,

$$u^N=u=(0,...,0_{N-4},u_{N-2},u_{N-1},u_N),$$

$$x^{1N}=x^1=(0,...,0_{N-4},x^1_{N-3},x^1_{N-2},x^1_{N-1},x^1_N),$$

$$x^{2N}=x^2=(0,...,0_{N-4},x^2_{N-3},x^2_{N-2},x^2_{N-1},x^2_N)$$

and

$$d^N=d=(d_1,d_2,d_{12},d_4).$$

Here $u_{N-2}=u_{N-1}=\sqrt{(1-u^2_{N-3})/2}$, $u_N=0$, $x^1_{N-3}=0$, $x^1_{N-2}=-x^1_{N-1}=-1/\sqrt{2}$, $x^1_N=0$, $x^2_{N-2}=x^2_{N-1}=1/\sqrt{2+w^2}$, $x^2_{N-3}=w/\sqrt{2+w^2})$, $x^2_N=0$, $d_1=1-u^2_{N-3}$,

$$d_2=(u^2_{N-3}+\sqrt{4u^2_{N-3}-3u^4_{N-3}})/2+t/\sqrt{2}$$

$d_{12} = 0$ and $d_4 = 1 + \frac{u_{N-3}w}{u_{N-2}(2+w^2)} + t/(2u_{N-1})$. Here

$$w = \frac{u_{N-3}^2 + \sqrt{4u_{N-3}^2 - 3u_{N-3}^3}}{u_{N-3}\sqrt{2 - 2u_{N-3}^2}}.$$

Then the above defined x^1, x^2, u_1, d satisfy the system of equations:

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for $j = 1, ..., 3N + 3$ where

$w_j \in \{v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N, z^i_k, i = 1, 2, k = 1, ..., N\}$, $j = 1, ..., 3N-1$

and

$$w_j \in \{b_{12}, b_1, b_2, b_4\},\ j = 3N, ..., 3N + 3.$$

(We do not differentiate with respect to u_{N-3}).

Proof. Notice that the equations

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{z^i_k, i = 1, 2, k = 1, ..., N\}$$

follow from the fact that (for $N = 4$) $x^{i4} = x^i$, $i = 1, 2$, are the orthonormal eigenvectors of the matrix B defined by (28) corresponding to the eigenvalues d_i, $i = 1, 2$ which has been established in the proof of Lemma(2.10). Also the equations

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{b_{12}, b_1, b_2, b_4\}.$$

follows immediately from the fact that $< x^i, x^j >_N = \delta_{ij}$ for $i, j = 1, 2, i \leq j$ and $< (u_{N-3}, u), (u_{N-3}, u) >_N = 1$. To end the proof, we show that

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for

$$w_j \in \{v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N\}.$$

Notice that for $i = 1, ..., N-4, N$

$$\frac{\partial h_{u_{N-3},t}}{\partial w_i}(x^1, x^2, u, d) =$$

$$= 2\sum_{j=1}^{N} u_j a_{ij} < x_i, x_j >_2 + t - 2u_i d_4 = 0$$

since $x_i = 0$ and $u_i = 0$ for $i = 1, ..., N-4, N$. Now assume that $w_i = u_{N-2}$. Then

$$2(\sum_{j=1}^{N} u_j a_{N-2,j} < x_{N-2}, x_j >_2 + t/2 - u_{N-2} d_4)$$

$$= 2((u_{N-3} w)/(2 + w^2) + u_{N-2} + t/2 - u_{N-2} d_4) = 0$$

Since $u_{N-2} = u_{N-1}$, the same calculations work for $i = N-1$ which completes the proof. ■

Reasoning as in Theorem(4.1) and Theorem(4.2) and applying Lemma(2.11) we can show

THEOREM 4.3 *Let A be an $N \times N$ symmetric matrix defined by (62). Assume additionally that*

$$1 = a_{N,N-3} = -a_{N,N-2} = -a_{N,N-1}.$$

Fix $t \in \mathbb{R}$ and $u_{N-3} \in [0, 1)$. Let us consider a function $h^N_{u_{N-1},A,t} : \mathbb{R}^{N-1} \times (\mathbb{R}^N)^2 \times \mathbb{R}^3 \times \mathbb{R}$ defined by:

$$h^N_{u_{N-3},A,t}((v_1, v_2, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), z^1, z^2, b_1, b_2, b_{12}, b_4)$$

$$= \sum_{i,j=1}^{N} a_{ij} v_i v_j < z_i, z_j >_2 + t(u_N + u_{N-2} + u_{N-1} + z^2_{N-1} - z^2_{N-2})$$

$$-(b_1(< z^1, z^1 >_N - 1) + b_2(< z^2, z^2 >_N - 1))$$

$$-b_{12} < z^1, z^2 >_N - b_4(< (u_{N-3}, v), (u_{N-3}, v) >_N - 1),$$

where $v = (v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N)$*. Let us define for fixed* $N \in \mathbb{N}$,

$$u^N = u = (0, ..., 0_{N-4}, u_{N-2}, u_{N-1}, u_N),$$

$$x^{1N} = x^1 = (0, ..., 0_{N-4}, x^1_{N-3}, x^1_{N-2}, x^1_{N-1}, x^1_N),$$
$$x^{2N} = x^2 = (0, ..., 0_{N-4}, x^2_{N-3}, x^2_{N-2}, x^2_{N-1}, x^2_N)$$

and

$$d^N = d = (d_1, d_2, d_{12}, d_4).$$

Here $u_{N-2} = u_{N-1} = u_N = \sqrt{(1 - u^2_{N-3})/3}$, $x^1_{N-2} = x^1_{N-1} = 1/\sqrt{6}$, $x^1_N = -2/\sqrt{6}$, $x^2_{N-2} = -x^2_{N-1} = -1/\sqrt{2}$, $x^2_N = 0$ $d_1 = 2c^2$, $d_2 = 2c^2 + t/\sqrt{2}$ $d_{12} = 0$ *and* $d_4 = 4c^2 + t/(2u_N)$, *where* $c = \sqrt{(1 - u^2_{N-3})/3}$*. Then the above defined* x^1, x^2, u, d *satisfy the system of equations:*

$$\frac{\partial h^N_{u_{N-3},A,t}}{\partial w_j}(x^1, x^2, u, d) = 0$$

for $j = 1, ..., 3N + 3$ *where*

$$w_j \in \{v_1, v_2, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N, z^i_k, i = 1, 2; k = 1, ..., N\},\ j = 1, ..., 3N-1$$

and

$$w_j \in \{b_{12}, b_1, b_2, b_4\},\ j = 3N, ..., 3N + 3.$$

(We do not differentiate with respect to u_{N-3}*).*

LEMMA 4.1 *Let* A *be such as in Theorem(4.1). For a fixed* $u_{N-3} \in [0, 1/\sqrt{3})$ *and* $t > 0$ *let* $g^N_{u_{N-3},A,t} : \mathbb{R}^{N-1} \times (\mathbb{R}^N)^2 \to R$ *defined by*

$$g^N_{u_{N-3},A,t}((v_1, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), y^1, y^2) = \sum_{i,j=1}^{N} v_i v_j a_{ij} < y_i, y_j >_2$$

$$+ t g^{1,N}_{u_{N-3}}((v_1, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), y^1, y^2)$$

where

$$g^{1,N}_{u_{N-3}}((v_1, ...v_{N-4}, v_{N-2}, v_{N-1}, v_N), y^1, y^2)$$
$$= (v_N + (v_{N-2} + v_{N-1})/\sqrt{1 - 3v^2_{N-3}} + y^2_{N-1} - y^2_{N-2}),$$

Let $M^N_{u_{N-3},A,t} = \max g^N_{u_{N-3},A,t}$ *under constraints:*

$$< y^i, y^j >_N = \delta_{ij}, 1 \leq i \leq j \leq 2;$$

and

$$\sum_{j=1}^N v_j^2 = 1.$$

Assume that $u_{N-3} \in [0, 1/\sqrt{3})$ *is so chosen that*

$$M^N_{u_{N-3},A,0} = g_{u_{N-3},A,0}(u, x^1, x^2),$$

where u, x^1, x^2 *are as in Theorem(4.1). Set*

$$D^N_{u_{N-3}} = \{(v_1, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N, y^1, y^2) : y^2_N = 0, y^1_{N-2} \geq 0\}. \quad (64)$$

Then

$$X^N_{u_{N-3}} = (u, x^1, x^2)$$

is the only point maximizing $g^N_{u_{N-3},A,t}$ *satisfying (2) and (3) belonging to* $D^N_{u_{N-3}}$.

Proof. Let

$$Y^N_{u_{N-3}} = ((v_1, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N), y^1, y^2) \in D^N_{u_{N-3}}$$

maximizes $g^N_{u_{N-3},A,t}$ and satisfies (2) and (3). Notice that $g^{1,N}_{u_{N-3}}$ (as a function of $v = (v_1, ..., v_{N-4}, v_{N-2}, ..., v_n)$ and y^2) attains its maximum under constraints (2) and (3) only at

$$v = (0, ..., 0_{N-4}, 1/\sqrt{3}, 1/\sqrt{3}, \sqrt{1/3 - u^2_{N-3}})$$

and

$$y^2 = (0, ..., 0_{N-3}, -1/\sqrt{2}, 1/\sqrt{2}, 0).$$

Since $g^{1,N}_{u_{N-3}}$ does not depend on y^1, $t > 0$, and the maximum of $g^N_{u_{N-3},A,0}$ is attained at $X^N_{u_{N-3}}$, we have $v_i = 0$ for $i = 1, ..., N-4$, $v_{N-2} = v_{N-1} = 1/\sqrt{3}$, $v_N = u_N$ and $y^2 = x^2$. Since x^1, x^2 are the eigenvectors of A, by Lemma(2.4), span$[y^i : i = 1, 2]$=span$[x^i : i = 1, 2]$. Note that

$$< x^1, x^2 >_N = < y^1, x^2 >_N = 0$$

Hence $y^1 = dx^1$. Since $< y^1, y^1 >_N = 1$ and $y^1_{N-2} \geq 0$ and $x^1_{N-2} > 0$, $x^1 = y^1$, as required. ■

REMARK 4.1 *Lemma(4.1) remains true (with the same proof) if we replace the function* $g^{1,N}_{u_{N-3}}$ *by*

$$g^{2,N}_{u_{N-3}}((v_1,...v_{N-4},v_{N-2},v_{N-1},v_N),y^1,y^2) = v_{N-2}+v_{N-1}+y^2_{N-2}-y^2_{N-1},$$

and $X^N_{u_{N-3}}$*, A from Theorem(4.1) by* $X^N_{u_{N-3}}$ *and A from Theorem(4.2).Also the statement Lemma(4.1) remains true if we replace* $g^{1,N}_{u_{N-3}}$ *by*

$$g^{3,N}_{u_{N-3}}((v_1,...v_{N-4},v_{N-2},v_{N-1},v_N),y^1,y^2) = v_{N-2}+v_{N-1}+v_N+y^2_{N-2}-y^2_{N-1},$$

$X^N_{u_{N-3}}$ *and A from Theorem(4.1) by* $X^N_{u_{N-3}}$ *and A from Theorem(4.3).*

THEOREM 4.4 *Fix* $N \geq 4$ *and* $u_{N-3} \in [0, 1/\sqrt{3})$*. Let A,*

$$u, x^1, x^2, d = d(t)$$

be as in Theorem(4.1). Let $M^N_{u_{N-3},A,t} = \max g^N_{u_{N-3},A,t}$*, where* $g^N_{u_{N-3},A,t}$ *has been defined in Lemma(4.1), under constraints:*

$$< y^i, y^j >_N = \delta_{ij}, 1 \leq i \leq j \leq 2;$$

and

$$\sum_{j=1, j\neq N-3}^{N} v_j^2 = 1 - u^2_{N-3}.$$

Assume that $u_{N-3} \in [0, 1/\sqrt{3})$ *is so chosen that*

$$M^N_{u_{N-3},0} = f^N_{u_{N-3},A}(u, x^1, x^2).$$

Denote by $D^N_{u,A,t}$ *a* $3N+2 \times 3N+2$ *matrix defined by*

$$D^N_{u,A,t} = \frac{\partial h^N_{u_{N-3},A,t}}{\partial w_i, \partial w_j}(x^1, x^2, u, d_1(t), d_2(t), d_{12}(t), d_4(t)), \tag{65}$$

where

$$w_i, w_j \in \{v_1, ..., v_{N-4}, v_{N-2}, v_{N-1}, v_N, y^1_j, j = 1, ..., N, y^2_j, j = 1, ..., N-1, b_1, b_2, b_{1,2}, b_4\},$$

(we do not differentiate with respect to u_{N-3} *and* y^2_N*). Assume that*

$$det(D^N_{u,A,t}) = \sum_{j=o}^{k} c_{j,N}(u)t^j$$

and $c_{j,N}(u_{N-3}) \neq 0$ for some $j \in \{1, ..., k\}$. Then there exists an open interval $U_N \subset [0, 1/\sqrt{3})$, ($U_N = [0, w)$ if $u_{N-3} = 0$) such that $u_{N-3} \in U_N$ and for any $u \in U_N$ the function $f^N_{u,A}$ attains its global maximum under constraints (2) and (3) at (u, x^1, x^2) defined in Theorem(4.1). The same result holds true if we replace the function $g^N_{u,A,t}$ from Theorem(4.1)by the function $g^N_{u,A,t}$ from Theorem (4.2) and we assume that $u_{N-3} \in [1/\sqrt{3}, 1)$. In this case $(x^1, x^2, u, d_1(t), d_2(t), d_{12}(t), d_4(t))$ are as in Theorem (4.2).

Proof. The proof goes in exactly the same way as the proof of Theorem(3.3), so we omit it. ■

Now we prove the crucial technical result of this section, which shows that the assumptions of Theorem(4.4) concerning $D^N_{u,A,t}$ are satisfied.

THEOREM 4.5 *Let A, $d(t) = (d_1(t), d_2(t), d_{12}(t), d_4(t))$, and (u, x^1, x^2) be as in Theorem(4.1). Let $D^N_{u,A,t}$ be defined by (65). Then for any $u \in [0, 1/\sqrt{3})$ and $t \in \mathbb{R}$,*

$$det(D^N_{u,A,t}) = \sum_{j=0}^{2(N-4)+4} c_{j,N}(u)t^j,$$

where the functions $c_{j,N}$ are continuous for $j = 0, ..., 2(N-4) + 4$ and

$$c_{2N-4,N}(u) \neq 0$$

for any $u \in [0, 1/\sqrt{3})$. The same result holds true if we replace A, $(d(t), u, x^1, x^2)$ from Theorem(4.1) by A, $(d(t), u, x^1, x^2)$ from Theorem(4.2) and assume that $u \in [1/\sqrt{3}, 1)$.

Proof. First we assume that $N = 4$. Let $g^4_{u_1,A,t}$ be as in Theorem(3.1). We will differentiate our function $h^4_{u_1,A,t}$ in the following way:

$$(w_1, ..., w_8) = (x^1_1, x^1_2, x^1_3, x^1_4, b_1, b_2, b_{12}, b_4)$$

and

$$(w_9, ..., w_{14}) = (x^2_1, x^2_2, x^2_3, v_2, v_3, v_4).$$

Set

$$X = (x_1, b, b, x_4, 0, -1/\sqrt{2}, 1/\sqrt{2}, 0),$$

$$BB = (b_1, b_2, 0, z)$$

and

$$v = (u_1, c, c, u_4).$$

Notice that by elementary but very tedious calculations (which we have checked applying a symbolic Mathematica program) we get that the 14×14 symmetric matrix $C = D^4_{u_1,A,t}(X, BB, v)$ is given by

$$C = \begin{pmatrix} A_1 & B \\ B^T & A_2 \end{pmatrix}. \tag{66}$$

Here

$$A_1 = \begin{pmatrix} 2(u_1^2 - b_1) & 2cu_1 & 2cu_1 & -2u_1u_4 & -2x_1 & 0 & 0 \\ 2cu_1 & 2(c^2 - b_1) & -2c^2 & -2cu_4 & -2b & 0 & 1/\sqrt{2} \\ 2cu_1 & -2c^2 & 2(c^2 - b_1) & -2cu_4 & -2b & 0 & -1/\sqrt{2} \\ -2u_1u_4 & -2cu_4 & -2cu_4 & 2(u_4^2 - b_1) & -2x_4 & 0 & 0 \\ -2x_1 & -2b & -2b & -2x_4 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 & 0 \end{pmatrix}; \tag{67}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 & -2c & -2c & -2u_4 \\ 0 & 2(u_1^2 - b_2) & 2cu_1 & 2cu_1 & -\sqrt{2}u_1 & \sqrt{2}u_1 & 0 \\ 0 & 2cu_1 & 2(c^2 - b_2) & -2c^2 & -3\sqrt{2}c & -\sqrt{2}c & 0 \\ 0 & 2cu_1 & -2c^2 & 2(c^2 - b_2) & \sqrt{2}c & 3\sqrt{2}c & 0 \\ -2c & -\sqrt{2}u_1 & -3\sqrt{2}c & \sqrt{2}c & 1 + 2(b^2 - z) & 1 - 2b^2 & -2bx_4 \\ -2c & \sqrt{2}u_1 & -\sqrt{2}c & 3\sqrt{2}c & 1 - 2b^2 & 1 + 2(b^2 - z) & -2bx_4 \\ -2u_4 & 0 & 0 & 0 & -2bx_4 & -2bx_4 & 2(x_4^2 - z) \end{pmatrix} \tag{68}$$

and

$$B^T = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -x_1 \\ 0 & 0 & 0 & 0 & 0 & \sqrt{2} & -b \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{2} & -b \\ 2bu_1 & 2(cb + u_1x_1 - u_4x_4) & -2cb & -2bu_4 & 0 & 0 & 0 \\ 2bu_1 & -2cb & 2(cb + u_1x_1 - u_4x_4) & -2bu_4 & 0 & 0 & 0 \\ -2u_1x_4 & -2cx_4 & -2cx_4 & 4(u_4x_4 - u_1x_1/2 - cb) & 0 & 0 & 0 \end{pmatrix}; \tag{69}$$

Notice that in 6-th row of C the only non-zero elements are $c_{6,10} = -c_{6,11} = \sqrt{2}$ and in 8-th row of C the only elements which could be different from 0 are $c_{8,12} = c_{8,13} = -2c$ and $c_{8,14} = -2u_4$. Consequently, applying the symmetry of C, adding 10-th row to 11-th, 10-th column to 11-th column, subtracting 14-th row multiplied by c/u_4 from 12-th and 13-row and subtracting 14-th column multiplied by c/u_4 from 12-th and 13-th column

$$det(C) = 8(u_4)^2 det(D),$$

where D is a 10×10 symmetric matrix defined by

$$D = \begin{pmatrix} D_1 & E \\ E^T & D_2 \end{pmatrix}. \tag{70}$$

Here

$$D_1 = \begin{pmatrix} 2(u_1^2 - b_1) & 2cu_1 & 2cu_1 & -2u_1u_4 & -2x_1 & 0 \\ 2cu_1 & 2(c^2 - b_1) & -2c^2 & -2cu_4 & -2b & 1/\sqrt{2} \\ 2cu_1 & -2c^2 & 2(c^2 - b_1) & -2cu_4 & -2b & -1/\sqrt{2} \\ -2u_1u_4 & -2cu_4 & -2cu_4 & 2(u_4^2 - b_1) & -2x_4 & 0 \\ -2x_1 & -2b & -2b & -2x_4 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} & 0 & 0 & 0 \end{pmatrix}; \tag{71}$$

$$D_2 = \begin{pmatrix} 2(u_1^2 - b_2) & 4cu_1 & -\sqrt{2}u_1 & \sqrt{2}u_1 \\ 4cu_1 & -4b_2 & -2\sqrt{2}c & 2\sqrt{2}c \\ -\sqrt{2}u_1 & -\sqrt{2}c & d_{3,3} - (2 + 2c/(u_4)^2)z & d_{3,4} - 2(c/(u_4)^2)z \\ \sqrt{2}u_1 & \sqrt{2}c & d_{4,3} - 2(c/(u_4)^2)z & d_{4,4} - (2 + 2c/(u_4)^2)z \end{pmatrix}, \tag{72}$$

where $d_{3,4} = d_{4,3}$ and $d_{3,3} = d_{4,4}$ do not depend on b_2 and z. Also observe that the coefficients of B^T do not depend on b_2 and z, hence the same is holds true for E. Now we calculate the coefficient $c_{4,4}(u_1)$ of $Det(D^4_{u_1,A,t})$. Notice that

$$det(C(t)) = Det(D^4_{u_1,A,t}(X, BB, v)) = 8u_4^2 det(D(t)),$$

where $C(t)$ and $D(t)$ denote the above written matrices C and D with z replaced by $z + t/(2u_4)$ and $b_2 = b_1 + t/\sqrt{2}$. By definition of determinant

$$det(D(t)) = \sum_{\sigma \in \Pi_{10}} sgn(\sigma)(\sum_{i=1}^{15} d_{i,\sigma(i)}),$$

where Π_{10} denotes the set of all permutations of $\{1, ..., 10\}$. Notice that by the above given formulas the variable t appears only in D_1 and D_2. Consequently to calculate $c_{4,4}(u_1)$ it is enough to consider

$$\sum_{\sigma \in P_1} sgn(\sigma)(\sum_{i=1}^{10} d_{i,\sigma(i)}),$$

where

$$P_1 = \{\sigma \in \Pi : \sigma(\{9, 10\}) = \{9, 10\}, \sigma(j) = j, j = 7, 8\}.$$

Consequently, applying the formula on D_1 and D_2 we can deduce that

$$c_{4,4}(u_1) = 32 det(F) det(D_1)$$

where

$$F = \begin{pmatrix} -(1 + c/(u_4)^2) & -c/u_4^2 \\ -c/(u_4^2) & -(1 + c/(u_4^2)) \end{pmatrix}. \tag{73}$$

Note that $det(F) = 1 + 2c/(u_4^2) > 0$. (In the case of Theorem(4.1) applied to $N = 4$, $u_1 \in [0, 1/\sqrt{3})$, $c = 1/\sqrt{3}$ and $u_4 = \sqrt{1/3 - u_1^2} > 0$.) Hence

to end the proof we should demonstrate that $Det(D_1) \neq 0$ for (X, BB, v) defined in Theorem(4.1). But this can be done as in Theorem(3.4) (see also Remark(4.2)).
Now assume that $N = 4$ and let A, (x^1, x^2, u, d) be as in Theorem(4.2). In this case we have that $u_4 = 0$ and $x_4^1 = 0$. Reasoning in a similar way as above we get that

$$Det(C) = 8c^2 det(D),$$

where D is a 10×10 symmetric matrix defined by

$$D = \begin{pmatrix} D_1 & E \\ E^T & D_2 \end{pmatrix}, \tag{74}$$

such that D_1 is as in the previous case and

$$D_2 = \begin{pmatrix} 2(u_1^2 - b_2) & 4cu_1 & \sqrt{2}u_1 & 0 \\ 4cu_1 & -4b_2 & 4\sqrt{2}c & 0 \\ 2\sqrt{2}u_1 & 4\sqrt{2}c & d_{3,3} - 4z & 0 \\ 0 & 0 & 0 & -2z \end{pmatrix}, \tag{75}$$

Also, as in the previous case, the coefficients of E do not depend on z and b_2. Moreover, the coefficients of D_1 and D_2 do not depend on $a_{N,j}$ for $j = N-3, N-2, N-1$, which are not fixed, for A given by (62), as in Theorem(4.1). Hence, reasoning as above we can show that

$$c_{4,4}(u_1) = 2^6(c^2)/(u_3)^2 Det(D_1).$$

Since $u_1 < 1$, $u_3 = \sqrt{(1-u_1^2)/2} > 0$ (compare with Theorem(4.2)). Hence to end the proof we should demonstrate that $Det(D_1) \neq 0$ for (X, BB, v) defined in Theorem(4.2). But this can be done as in Theorem(3.4) (see also Remark(4.2)).
Now take any $N > 4$. We show that the proof of this case practically reduces to the proof given for $N = 4$. First assume that A, $(x^1, x^2, u, d(t))$ are such as in Theorem(4.1). We will differentiate in the following way:

$$(w_1, ..., w_{3(N-4)}) = (x_1^1, x_1^2, u_1, ..., x_{N-4}^1, x_{N-4}^2, u_{N-4}),$$

$$(w_{3(N-4)+1}, ..., w_{3N+2})$$
$$= (x_{N-3}^1, x_{N-2}^1, x_{N-1}^1, x_N^1, x_{N-3}^2, x_{N-2}^2, x_{N-1}^2, b_1, b_2, b_{12}, b_4, u_{N-2}, u_{N-1}, u_N).$$

(We do not differentiate with respect to x_N^2 and u_{N-3}.) Now we show that (since $u_j = x_j^1 = x_j^2 = 0$ for $j = 1, ..., N-4$) the matrix C_N corresponding to our case has a form

$$C_N = \begin{pmatrix} W_1 & 0 & ... & 0 & 0 \\ 0 & W_2 & ... & 0 & 0 \\ ... & ... & ... & ... & ... \\ 0 & 0 & ... & W_{N-4} & 0 \\ 0 & 0 & ... & 0 & C_4 \end{pmatrix}, \tag{76}$$

where C_4 denotes the matrix obtained for

$$X = (x^1_{N-3}, x^1_{N-2}, x^1_{N-1}, x^1_N, x^2_{N-3}, x^2_{N-2}, x^2_{N-1}, x^2_N)$$

$$u = (u_{N-3}, u_{N-2}, u_{N-1}, u_N), b = (d_1(t), d_2(t), d_{12}(t), d_4(t))$$

in the case $N = 4$. Here, for $i = 1, ..., N-4$, W_i is a 3×3 matrix given by

$$W_i = \begin{pmatrix} -2b_1 & 0 & w_{i,1} \\ 0 & -2b_2 & w_{i,2} \\ w_{i,1} & w_{i,2} & -2z \end{pmatrix}, \tag{77}$$

where

$$w_{i,k} = \sum_{j=N-3}^{N} a_{ij} u_j x^k_j$$

for $k = 1, 2$. Indeed for any $j = 1, ..., N$

$$\frac{\partial h^N_{u_1,A,t}}{x^1_j}(x^1, x^2, u, d(t)) = 2(\sum_{k=1}^{N} a_{jk} x^1_k u_j u_k - d_{12}(t) x^2_j - d_1(t) x^1_j).$$

and

$$\frac{\partial h^N_{u_1,A,t}}{u_j}(x^1, x^2, u, d(t)) = 2(\sum_{k=1}^{N} a_{jk} u_k < x_j, x_k >_2 - d_4(t) u_j).$$

Hence for any $j = 1, ..., N-4$

$$\frac{\partial h^N_{u,A,t}}{x^1_j, w_l}(x^1, x^2, u, d(t)) = 0$$

for $w_l \neq x^1_j$ and $w_l \neq u_j$. The same reasoning applies if we differentiate with respect to x^2_j, $j = 1, ..., N-4$. Analogously, for $j = 1, ..., N-4$,

$$\frac{\partial h^N_{u_1,A,t}}{u_j, w_l}(x^1, x^2, u, d(t)) = 0$$

for $w_l \neq x^i_j$, $i = 1, 2$ and $w_l \neq u_j$. Also for

$w_k, w_j \in \{(x^1_{N-3}, x^1_{N-2}, x^1_{N-1}, x^1_N, x^2_{N-3}, x^2_{N-2}, x^2_{N-1}, u_{N-2}, u_{N-1}, u_N, b_1, b_2, b_{12}, b_4\}$

$$\frac{\partial h^N_{u_1,A,t}}{w_j, w_k}(x^1, x^2, u, d) = \frac{\partial h^4_{u_1,A,t}}{w_j, w_k}(z^1, z^2, v, d),$$

where $h^4_{u_1,A,t}$ is the function from Theorem(4.1) corresponding to $N = 4$ and

$z^1 = (x^1_{N-3}, x^1_{N-2}, x^1_{N-1}, x^1_N), z^2 = (x^2_{N-3}, x^2_{N-2}, x^2_{N-1}), v = (u_{N-2}, u_{N-1}, u_N)$.

This shows our claim concerning the matrix C_N.
Since $w_{i,k}$ for $k = 1, 2$ and $i = 1, ..., N-4$ do not depend on b_2 and z, $b_1 = 2/3$, $b_2 = 2/3 + t/\sqrt{2}$ $z = 4/3 + t/2u_n$

$$c_{4+2(N-4),N}(u_{N-3}) \neq 0$$

for any $u_{N-3} \in [0, 1/\sqrt{3})$, which completes the proof for $N > 4$ in the case of A from Theorem(4.1). The case of A from Theorem(4.2) and $N > 4$ is exactly the same, so we omit it.

REMARK 4.2 *If $x^1, x^2, u, d(t)$ are as in Theorem(4.1) for $N = 4$, applying a symbolic Mathematica program we can show that*

$$Det(D_1) = 64/27(2 + 6\sqrt{2}u_1 x^1_2 + 3(x^1_2)^2) > 0.$$

If $x^1, x^2, u, d(t)$ are as in Theorem(4.2) for $N = 4$, applying a symbolic Mathematica program we can show that

$$Det(D_1) = 8u_1(4x^1_2 u_1(1 - u_1^2) + 4u_1 x^1_2(u_1^2 + \sqrt{4u_1^2 - 3u_1^4})x^1_1$$

$$+u_1(2 - u_1^2 + \sqrt{4u_1^2 - 3u_1^4})(x^1_1)^2) > 0.$$

Now we will prove the main results of this section.

THEOREM 4.6 *Fix $N \in \mathbb{N}$, $N \geq 4$ and $u_{N-3} \in [0, 1]$. Let*

$$f^N_{u_{N-3}}(u_1, ..., u_{N-4}, u_{N-2}, u_{N-1}, u_N, x^1, x^2) = \sum_{i,j=1}^{N} u_i u_j | < x_i, x_j >_2 |.$$

Let $M_{u,N} = \max(f_u^N)$ *under constrains (2) and (3). Then for any* $u_{N-3} \in [0, 1/\sqrt{3})$

$$M_{u_{N-3},N} = 4/3,$$

and for any $u_{N-3} \in [1/\sqrt{3}, 1]$

$$M_{u_{N-3},N} = 1 + (\sqrt{4u_{N-3}^2 - 3u_{N-3}^4} - u_{N-3}^2)/2.$$

Proof. We will proceed by the induction argument with respect to N. First assume $N = 4$. Define

$$U_4 = \{u_1 \in [0, 1/\sqrt{3}) : M_{u_1,4} = 4/3\}.$$

By Lemma(2.6) and Lemma(2.7), $0 \in U_4$. Now we show that U_4 is an open set. Fix $u_1 \in U$. First we consider the case $u_1 = 0$. We apply Theorem(4.4) and Theorem(4.5). Assume that $u_n \to 0$ and $u_n \notin U$. Let $(Z_{u_n}, M_{u_n}(t))$ be such as in Theorem(4.4) (compare with the proof of Theorem(3.3)). Passing to a subsequence, if necessary, and reasoning as in Theorem(3.3), we can assume that $(Z_{u_n}, M_{u_n}(t)) \to (X_o, L_o)$. Let $Z_{u_n} = (u^n, z^{1n}, z^{2n})$. Since $Z_{u_n} \to X_o$

$$sgn < z_{in}, z_{jn} >_2 = a_{ij}$$

for $i, j = 2, 3, 4$ and $n \geq n_o$, where the matrix $\{a_{ij}\}$ is given by (62) for $N = 4$. Without loss of generality, passing to a subsequence if necessary we can assume that for $n \geq n_o$

$$sgn < z_{1n}, z_{jn} >_2 = z_j$$

for $j = 2, 3, 4$, where $z_j = \pm 1$. By Lemma(2.8) we have to consider two cases:

a) $z_2 = z_3 = z_4 = 1$;

b) $z_2 = z_3 = 1, z_4 = -1$.

If a) holds true, then by Theorem(4.4), Theorem(4.3) (applied to $u_{N-3} = 0$) and Theorem(4.5) we get that

$$M_{u_n,4} = 4(1 - u_n^2)/3 < 2/3 + 2/3 = 4/3$$

for $n \geq n_o$, (compare with Theorem(4.1)), which by Theorem(2.1) leads to a contradiction. (Since $u_1 = 0$, $D_{u_1,A,t}^4$ is the same for the function $h_{u_1,A,t}^4$ form Theorem(4.3) like for the function $h_{u_1,A,t}$ form Theorem(4.1)). If b) holds true, by Theorem(4.5) and Theorem(4.1), we get a contradiction with

Theorem(4.4). Consequently, there exists an interval $[0, v) \subset U_4$.
Now assume that $v = u_1 \in U$ and $v > 0$. Assume $u_n \to v$ and $u_n \notin U_4$ for $n \in \mathbb{N}$. Let (Z_{u_n}, M_{u_n}) be such as in Theorem(4.4). Without loss of generality we can assume that $(Z_{u_n}, M_{u_n}(t)) \to (X_v, L_v(t))$. Let $Z_{u_n} = (u^n, z^{1n}, z^{2n})$. Since $Z_{u_n} \to X_v$

$$sgn < z_{in}, z_{jn} >_2 = a_{ij}$$

for $i, j = 1, 2, 3, 4$ for $n \geq n_o$, where the matrix $\{a_{ij}\}$ is as in Theorem(4.1) for $N = 4$. Applying Theorem(4.4), we get that $u_n \in U$ for $n \geq n_o$; a contradiction. Hence the set U_4 is open. It is easy to see that U_4 is also closed. Since $0 \in U_4$ and $[0, 1/\sqrt{3})$ is connected, $U_4 = [0, 1/\sqrt{3})$. Observe that by the continuity of the function $u_{N-3} \to f^N_{u_{N-3}}$, $M(1/\sqrt{3}, 4) = 4/3$. Now define

$$W_4 = \{u_1 \in [1/\sqrt{3}, 1) : M_{u_1,4} = 1 + (\sqrt{4u_1^2 - 3u_1^4} - u_1^2)/2\}.$$

By the above reasoning $1/\sqrt{3} \in W_4$. Let $v = u_1 \in W_4$. Assume that $u_n \to v$ and $u_n \notin W_4$. Applying Theorem(4.2) and proceeding as above we get that $(Z_{u_n}, M_{u_n}(t)) \to (X_v, L_v(t))$. Also reasoning as above, passing to a subsequence if necessary, we can assume that

$$f^4_{u_n} = f^4_{u_n,A},$$

where A is a fixed matrix satisfying (62). By Theorem(4.2), Theorem(4.4) and Theorem(4.5) $u_n \in W_4$ for $n \geq n_o$; a contradiction. Hence W_4 is an open set. Reasoning as above we get that

$$W_4 = [1/\sqrt{3}, 1),$$

which completes the proof for $N = 4$. (It is easy to see that $M_{1,4} = 1$.)
Now assume that our formula for $M_{u_{N-3},N}$ holds true. We will show that it holds for $M_{u_{N+1-3},N+1}$. We will proceed in the same way as in $N = 4$ case. Define

$$U_{N+1} = \{u_{N-2} \in [0, 1/\sqrt{3}) : M_{u_{N-2},N+1} = 4/3\}.$$

By the induction hypothesis and Lemma(2.7), $0 \in U_{N+1}$. Reasoning as in the $N = 4$ case and applying Theorem(4.1), Theorem(4.3),Theorem(4.4) and Theorem(4.5), we can show that U_{N+1} is an open set. It is clear that that U_{N+1} is closed. Hence $U_{N+1} = [0, 1/\sqrt{3})$. Again by the continuity of $u_{N+1-3} \to f^{N+1}_{u_{N+1-3}}$ we get that

$$M_{1/\sqrt{3},N+1} = 4/3.$$

Define

$W_{N+1} = \{u_{N-2} \in [1/\sqrt{3}, 1) : M_{u_{N-2},N+1} = 1 + (\sqrt{4u_{N-2}^2 - 3u_{N-2}^4} - u_{N-2}^2)/2\}.$

By the above reasoning $1/\sqrt{3} \in W_{N+1}$. Applying Theorem(4.2), Theorem(4.4) and Theorem(4.5) and proceeding as in the case $N = 4$, we get that

$$W_{N+1} = [1/\sqrt{3}, 1).$$

It is easy to see that $M_{1,N+1} = 1$. The proof is complete. ■

THEOREM 4.7

$$\lambda_2 = 4/3.$$

Proof. By Theorem(4.6), Theorem(2.2), Lemma(2.6), Lemma(2.7) and Lemma(2.10),

$$\lambda_2^N = 4/3$$

for any $N \in \mathbb{N}$, $N \geq 3$. Let $V \subset l_\infty$ be so chosen that $\lambda_2 = \lambda(V)$. For any $\epsilon > 0$ we can find $N \in \mathbb{N}$ and $V_N \subset l_\infty^{(N)}$, such that

$$ln(d(V_N, V)) \leq \epsilon,$$

where d denotes the Banach-Mazur distance. Since

$$|ln(\lambda(V_N)) - ln(\lambda(V))| \leq ln(d(V_n, V)),$$

(see e.g. [15], p. 113)

$$\lambda_2 = \lambda(V) \leq \lambda(V_N)e^\epsilon \leq \lambda_2^N e^\epsilon.$$

Consequently,

$$\lim_N \lambda_2^N = \lambda_2,$$

which shows that

$$\lambda_2 = 4/3.$$

The proof is complete. ■

REMARK 4.3 *Notice that in [5], it has been proven that*

$$\lambda(V) \leq 4/3$$

for any two-dimensional, real, unconditional Banach space. Recall that a two-dimensional, real Banach space V is called unconditional if there exists v^1, v^2 a basis of V such that for any $a_1, a_2 \in \mathbb{R}$ and $\epsilon_1, \epsilon_2 \in \{-1, 1\}$

$$\|a_1 v^1 + a_2 v^2\| = \|\epsilon_1 a_1 v^1 + \epsilon_2 a_2 v^2\|.$$

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