# A note on the points realizing the distance to a definable set

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Abstract. We prove a definable/subanalytic version of a useful lemma, presumably due to John Nash, concerning the points realizing the Euclidean distance to an analytic submanifold of $\mathbb{R}^n$.

## 1. Introduction

Among the auxiliary results proved in [N] by J. Nash one encounters the following interesting lemma:

**Lemma 1.1** ([N]). *Let $M$ be an analytic submanifold of an open set $\Omega \subset \mathbb{R}^n$. Then there exists an arbitrarily small neighbourhood $M \subset U \subset \Omega$ such that*

(1) *for every point $x \in U$ there exists a unique point $m = m(x) \in M$ such that the Euclidean distance $\mathrm{dist}(x, M) = ||x - m(x)||$;*
(2) *the function $m\colon U \ni x \mapsto m(x) \in M$ is analytic.*

As this lemma is the starting point of the whole paper, for the convenience of the reader we recall the proof, simplifying somewhat its original version.

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*Proof of the Nash Lemma.* First, recall that for a point $a \in M$, any radius $r > 0$ and any point $x$ in the ball $\mathbb{B}(a, r)$, there is a point $y \in \mathbb{B}(a, 2r) \cap M$ realizing $\mathrm{dist}(x, M)$.

The problem being local we fix a point $a \in M$ and an analytic parametrization $f\colon (V, 0) \to (M, a)$, $V \subset \mathbb{R}^d$ open, $d = \dim M$.

Observe that if $y \in M$ realizes the distance $\mathrm{dist}(x, M)$ then the vector $x - y$ is normal to $M$ at $y$ ([2]). Consider the analytic function

$$F\colon \mathbb{R}^n \times V \ni (x, t) \mapsto \left(\left\langle x - f(t), \frac{\partial f}{\partial t_j}(t)\right\rangle\right)_{j=1}^{d} \in \mathbb{R}^d$$

and observe that

$$\det \frac{\partial F}{\partial t}(a, 0) = (-1)^d \sum_{1 \le i_1 < \ldots < i_d \le n} \left(\det \frac{\partial(f_{i_1}, \ldots, f_{i_d})}{\partial t}(0)\right)^2 \neq 0.$$

Therefore, by the Implicit Function Theorem, there is a neighbourhood $G \times W$ of $(a, 0)$, with $G \cap M \subset f(V)$, and an analytic function $t\colon G \ni x \mapsto t(x) \in W \subset V$ such that $F^{-1}(0) \cap (G \times W) = \Gamma_t$ where $\Gamma_t$ denotes the graph of $t = t(x)$.

Put $m(x) := f(t(x))$ for $x \in G$. Clearly, for an $r > 0$ such that $\mathbb{B}(a, 2r) \subset G$, a point $x \in \mathbb{B}(a, r)$ and any point $y \in \mathbb{B}(a, 2r) \cap M$ realizing $\mathrm{dist}(x, M)$, we obtain $F(x, f^{-1}(y)) = 0$ and so $y = m(x)$ which ends the proof. $\square$

*Remark* 1.2. It is obvious from the proof that this lemma holds true too, when the word 'analytic' is replaced by the words 'of class $\mathscr{C}^\infty$', while in the case of a $\mathscr{C}^k$-submanifold, one obtains eventually a function $m$ which is only of class $\mathscr{C}^{k-1}$ (see also [KP] — I am indebted to Prof. M. Jarnicki for this reference).

Since this lemma is a very useful tool it is natural to ask what happens if we let $M$ have singularities. The example of the analytic curve $M = \{y^2 = x^3\} \subset \mathbb{R}^2$ shows that apart from a semi-analytic curve $F \subset \mathbb{R}^2$ one still has (1), $\overline{F} \cap M = \mathrm{Sng}M$ (where $\mathrm{Sng}M$ denotes the set of singular points), and the function from (2) has a semi-analytic graph in $\mathbb{R}^4$. Moreover, $m$ is analytic in a still smaller set, namely $\mathbb{R}^2 \setminus \overline{E} \setminus (\{0\} \times \mathbb{R})$.

A still simpler example suggests that the result for semi-analytic sets is somewhat more delicate: let $M = [0, +\infty) \times \{0\} \subset \mathbb{R}^2$. Then (1) holds for $U = \mathbb{R}^2$ but the function $m$ from (2) is analytic only in $\mathbb{R}^2 \setminus (\{0\} \times \mathbb{R})$. Still, $m$ is semi-analytic everywhere.

Finally, if we take $M$ to be the analytic cone $\{x^2 + y^2 = z^2\} \subset \mathbb{R}^3$, we are able to define $m$ at all points of $(\mathbb{R}^3 \setminus \{x = y = 0\}) \cup \{0\}$. For the points lying on the $z$-axis the distance is realized in a circle contained

[2]Suppose that $y \in f(V)$. Then the differentiable function $V \ni t \mapsto ||x - f(t)||^2 \in \mathbb{R}$ has a local minimum at $t_0 := f^{-1}(y)$ which means that its differential at $t_0$ is zero: $(\langle x - y, \partial f/\partial t_j(t_0)\rangle)_{j=1}^d = 0$.

in the cone, which means that arbitrarily near the singularity there are points whose distance to $M$ is realized by infinitely many points of $M$.

In this article we propose to explore a definable/subanalytic version of the Nash lemma. We refer the reader to [DS] for a concise presentation of subanalytic geometry, and to [C] for tame geometry.

By $\mathbb{B}(a,r)$ we denote the open Euclidean ball and by $\overline{\mathbb{B}}(a,r)$ the closed one; $[a,b]$ denotes the segment $\{tb+(1-t)a \mid t\in[0,1]\}$ for $a,b\in\mathbb{R}^n$. For a set $M\subset\mathbb{R}^n$ and $k\in\mathbb{N}\cup\{\omega,\infty\}$ let

$$\mathrm{Reg}_k M := \{x\in M \mid M \text{ is a } \mathscr{C}^k\text{– submanifold in a neighbourhood of } x\},$$

where $\mathscr{C}^\omega$ means analycity (in that case we will also write $\mathrm{Reg}M := \mathrm{Reg}_\omega M$ and put $\mathrm{Sng}M := M\setminus \mathrm{Reg}M$ for the singular locus.)

We shall use the following theorem due to J.-B. Poly and G. Raby:

**Theorem 1.3** ([PR])**.** *Let $M\subset\mathbb{R}^n$ be a closed, nonempty set and $\delta(x) := \mathrm{dist}(x,M)^2$. Then for any $k\geq 2$ or $k\in\{\omega,\infty\}$,*

$$\mathrm{Reg}_k M = \{x\in\mathbb{R}^n \mid \delta \textit{ is of class } \mathscr{C}^k \textit{ at } x\}\cap M.$$

Although we have already used the following elementary fact, it seems reasonable — as it is widely used hereafter — to stress that for any closed set $M\subset\mathbb{R}^n$, a point $a\in M$, a radius $r>0$, and any $x\in\mathbb{B}(a,r)$, one has the implication

$$y\in M \text{ is such that } ||x-y|| = \mathrm{dist}(x,M) \Rightarrow y\in\mathbb{B}(a,2r).$$

## 2. Points realizing the distance to a subanalytic set

The general 'singular' counterpart of the Nash Lemma is our following theorem:

**Theorem 2.1.** *Let $M\subset\mathbb{R}^n$ be a subanalytic set. Then there exist a subanalytic neighbourhood $W\supset M$ in which $M$ is closed and two subanalytic nowheredense sets $E,F\subset\mathbb{R}^n$ such that*

1. *$E\subset W$ and $x\in W\setminus E$ iff there exists a unique $m(x)\in M$ such that* $\mathrm{dist}(x,M)=||x-m(x)||$*;*
2. *the function $m\colon W\setminus E\to M$ is subanalytic;*
3. *$E\subset F\subset W$, $F$ is closed in $W$, $F\cap M=\mathrm{Sng}M$ and $x\in W\setminus\overline{E}$ is a point of analycity of $m$ iff $x\in W\setminus F$.*

*Proof.* We may assume that $M$ is closed ([3]). All the properties of subanalytic sets used hereafter can be found in [DS].

The function

$$\phi\colon \mathbb{R}^n\times\mathbb{R}^n\ni(x,y)\mapsto ||x-y|| - \mathrm{dist}(x,M)\in\mathbb{R}$$

[3]The set $M$ being locally closed, it is closed in the subanalytic open set $\Omega_M := \mathbb{R}^n\setminus(\overline{M}\setminus M)$ and so after proving the result for $\overline{M}$, one needs only to take $W\cap\Omega_M$ instead of $W$ and intersect $E,F$ with $\Omega_M$ as well.

is continuous and subanalytic, whence $X = \phi^{-1}(0)$ is subanalytic and closed. Put $Y := X \cap (\mathbb{R}^n \times M)$ and consider the function

$$\mu_Y \colon \mathbb{R}^n \ni x \mapsto \#\pi^{-1}(x) \cap Y \in \mathbb{P}_1,$$

where $\pi(x,y) = x$ and $\mu_Y(x) = \infty$, whenever $\#\pi^{-1}(x) \cap Y = +\infty$. The set $\pi^{-1}(x) \cap Y$ consists of all points of $M$ realizing $\mathrm{dist}(x, M)$. Since $M$ is closed, $\mu_Y(x) \neq 0$.

Obviously, for any $\varrho > 0$ and any point $a \in M$, there is

$$\pi^{-1}(\mathbb{B}(a,\varrho)) \subset \mathbb{B}(a,\varrho) \times (\mathbb{B}(a, 2\varrho) \cap M) =: M^{(a)}.$$

Fix a radius and let $W := M + \mathbb{B}(0,\varrho)$. It is a subanalytic neighbourhood of $M$ ([4]). Now we put $Z := (W \times M) \cap Y$ and consider the function $\mu_Z$ instead of $\mu_Y$, restricted to $W$.

The function $\mu_Z$ is locally bounded on $W$. Indeed, for $x \in W$, there is $x \in \mathbb{B}(a,\varrho)$ with some $a \in M$, but then $\pi^{-1}(\mathbb{B}(a,\varrho))$ is contained in the subanalytic bounded set $\mathbb{B}(a, 2\varrho) \cap M$. This implies that $\pi|_{\mathbb{B}(a,\varrho)}$ admits a local uniform bound on the number of connected components of its fibres, which in turn means that there exists an $N$ such that

$$\mu_Z(\mathbb{B}(a,\varrho)) \subset \{1, 2, \ldots, N, \infty\}.$$

Thanks to that, $\mu_Z$ is locally bounded.

The set of values of $\mu_Z$ is discrete, hence showing that $\mu_Z$ is subanalytic is equivalent to showing all its fibres are. Observe that

$$\{\mu_Z = k\} = \{\mu_Z > k\} \setminus \{\mu_Z > k-1\}$$

and $\{\mu_Z = \infty\} = \bigcap \{\mu_Z > k\}$. It suffices to prove that the sets $\{\mu_Z > k\}$ are subanalytic (then the intersection is locally finite and so it defines a subanalytic set, too).

Fix a point $x \in W$ and take $\mathbb{B}(a,\varrho)$ be a ball containing $x$. We consider $\mu_Z$ on this ball, which mean there is a bound on the values. On this ball $\mu_Z$ coincides with the bounded function $\mu_{\tilde{Z}}$ defined (as earlier was $\mu_Y$) by the bounded subanalytic set $\tilde{Z} := M^{(a)} \cap Z$. Consider the fibred product

$$\tilde{Z}^{\{N\}} := \{z = (z_1, \ldots, z_N) \in \tilde{Z}^N \mid \pi(z_i) = \pi(z_j), i < j\},$$

which is a subanalytic set as the pre-image on $\tilde{Z}^N$ by $\pi \times \ldots \times \pi$ ($N$ times) of the diagonal of $\mathbb{B}(a,\varrho)^N$.

Let $\rho(z) := z_1$ and $f_{ij}(z) := z_i - z_j$, $i < j$, for $z \in \tilde{Z}^{\{N\}}$. Then

$$\{\mu_{\tilde{Z}} > k-1\} = \rho\left(\bigcup_{1 \le i_1 < \ldots < i_k \le N} \bigcap_{1 \le s < t \le k} \{f_{i_s j_t} \neq 0\}\right),$$

whence this set is subanalytic for all $k \in \{1, \ldots, N-1\}$ ([5]).

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[4] Since it is exactly the set $\{x \in \mathbb{R}^n \mid \mathrm{dist}(x, M) < \varrho\}$ and the distance is a continuous subanalytic function in $\mathbb{R}^n$.

[5] One could take for a fixed $k$ the fibred product $Z^{\{k\}}$ to obtain a simpler description (without the intersection).

Put $E := \{x \in W \mid \mu_Z(x) \neq 1\}$. This is a subanalytic set satisfying

$$x \in W \setminus E \Leftrightarrow \exists!\, m(x) \in M \colon \mathrm{dist}(x, M) = ||x - m(x)||.$$

To prove that $\mathrm{int}E = \varnothing$, suppose on the contrary that there is a ball $\mathbb{B}(b, \rho) \subset E$. Let $\rho_b := \mathrm{dist}(b, M)$ ($\rho_b > \rho$ since $\mathbb{B}(b, \rho) \cap M = \varnothing$) and observe that

$$\overline{\mathbb{B}}(b, \rho_b) \cap M = \partial\mathbb{B}(b, \rho_b) \cap M \supset \{b_1, b_2\}$$

with $b_1 \neq b_2$. Consider any point $c \in \mathbb{B}(b, \rho) \cap [b, b_1]$ different from $b$ (obviously $c \neq b_1$) and put $\rho' := ||b - c||$. Then $\mathrm{dist}(c, M) = \rho_b - \rho'$, but clearly $\overline{\mathbb{B}}(c, \rho_b - \rho') \cap M = \{b_1\}$ which contradicts $c \in E$.

We obtained thus a well-defined function $m \colon W \setminus E \to M$. Its graph $\Gamma_m$ is subanalytic as it coincides with

$$Z \cap ((W \setminus E) \times \mathbb{R}^n).$$

Of course $\overline{E}$ is nowheredense too and one can ask about the analycity of $m$ in the open, nonempty set $W \setminus \overline{E}$. Note that $\overline{E} \subset \mathrm{Sng}M$ ([6]). This follows from the fact that for any $a \in \mathrm{Reg}M$ there is (by the Nash Lemma) a ball $\mathbb{B}(a, r)$ in which $m$ is uniquely determined.

Since $m$ is subanalytic and locally bounded, then (it is a classical result due to Tamm and used to prove that the singularities of a subanalytic set form a subanalytic set) the set

$$\mathcal{N}(m) := \{x \in W \setminus \overline{E} \mid m \text{ is not analytic at } x\}$$

is subanalytic (and closed, obviously). Moreover, it is nowheredense ([7]) Now let $F := \overline{E} \cup \mathcal{N}(m)$. It is a closed, nowheredense, subanalytic set apart from which $m$ is analytic. Clearly, by virtue of the Nash Lemma, there must be $F \cap M \subset \mathrm{Sng}M$. In order to prove the converse inclusion fix a point $a \in W \setminus F$. The function $m$ is well-defined at this point and we have the relation $\delta(x) = ||x - m(x)||^2$ ($\delta$ is the square of the distance function as in Theorem 1.3) which means that $\delta(x)$ is analytic in a neighbourhood of $a$. If we picked $a \in M$, we obtain $a \in \mathrm{Reg}M$, thanks to Theorem [PR]. This achieves the proof. $\square$

Observe that there is no direct relation at all between $\dim F$, $\dim E$ and $\dim M$ (the dimension of $E$ depends rather on $n$ than on $\dim M$ cf. $M = \{y^2 = x^3, z = 0\} \subset \mathbb{R}^3$)). However, one can conjecture the following:
For any $x \in W$ let $M(x) \subset M$ denote the section of $Z$ at $x$ (we keep

[6] The set $E$ could be empty as shown in the examples in the introduction.

[7] This follows from two observations: first, by a result of Tamm (cf. [DS]), there is a $k \in \mathbb{N}$ such that $W \setminus \mathcal{N}(m)$ is exactly the set of points at which $m$ is of class $\mathscr{C}^k$, and then one proves that the complement of the latter is nowheredense — e.g. as in the definable setting from [C]. Note that $\mathcal{N}(m)$ may not correspond to $\pi(\mathrm{Sng}\Gamma_m)$ (just think of the function $t^{1/3}$ on $\mathbb{R}$.)

the notations from the proof above). It is a subanalytic compact set. Let $k_x := \dim M(x)$. Put

$$H_x := \{x' \in W \mid \dim M(x') = k_x\}.$$

It is a subanalytic set (it follows from Lemma ?? in [DDP], for instance; in the definable setting see [C] Theorem 3.18). Note that for $x \in W \setminus E$, $k_x = 0$ and $H_x \supset W \setminus E$, i.e. $\dim H_x = n$ in this case.

*Conjecture.* If $x \in E$, then $k_x + \dim H_x = n - 1$.

## 3. The semialgebraic and the definable cases

If we change the word *subanalytic* to *semialgebraic*, respectively *definable*, in the previous theorem, the theorem still holds true with some due changes. In particular, the neighbourhood $W$ can be taken to be $\Omega_M = \mathbb{R}^n \setminus (\overline{M} \setminus M)$.

**Theorem 3.1.** *Let $M \subset \mathbb{R}^n$ be a semialgebraic set. Then there exist a semialgebraic neighbourhood $W \supset M$ in which $M$ is closed and two semialgebraic nowheredense sets $E, F \subset \mathbb{R}^n$ such that*

1. *$E \subset W$ and $x \in W \setminus E$ iff there exists a unique $m(x) \in M$ such that* $\mathrm{dist}(x, M) = ||x - m(x)||$*;*
2. *the function $m \colon W \setminus E \to M$ is semialgebraic;*
3. *$E \subset F \subset W$, $F$ is closed in $W$, $F \cap M = \mathrm{Sng} M$ and $x \in W \setminus \overline{E}$ is a point of analycity of $m$ iff $x \in W \setminus F$. In particular $m$ is Nash-analytic in $W \setminus F$.*

Recall that 'Nash-analytic' in the real setting means exactly the same as 'semialgebraic and $\mathscr{C}^\infty$'.

*Proof.* The proof follows the same lines as the proof of the main theorem 2.1. Actually, since semialgebraic sets form an o-minimal structure, the first two points follow from the next theorem. We only need to justify the third point. The set $F = \overline{E} \cup \mathcal{N}(m)$ is constructed as in the proof of Theorem 2.1 (semialgebraic sets are subanalytic) but now it is semialgebraic, since $\mathcal{N}(m)$ is semialgebraic ([8]). Then $m$ is clearly Nash-analytic in $W \setminus F$. Note that one can take $W = \Omega_M$ (see the next proof). □

The main difference in the general definable setting is that there may be no possibility of considering neither analycity, nor $\mathscr{C}^\infty$-class.

**Theorem 3.2.** *Let $M \subset \mathbb{R}^n$ be a definable (in some o-minimal structure) set. Then there exist a definable neighbourhood $W \supset M$ in which $M$ is closed, a definable nowheredense set $E \subset \mathbb{R}^n$ and for any $k \geq 2$ another definable, nowheredense set $F_k \subset \mathbb{R}^n$ such that*

[8]Indeed, since as in the subanalytic case the complement of $\mathcal{N}(m)$ coincides with the set of points at which $m$ is of class $\mathscr{C}^k$ for some appropriate $k$, and the partial derivatives of a semialgebraic function are semialgebraic, the latter set is semialgebraic. Therefore, so is $\mathcal{N}(m)$.

(1) *$E \subset W$ and $x \in W \setminus E$ iff there exists a unique $m(x) \in M$ such that* $\mathrm{dist}(x, M) = ||x - m(x)||$*;*
(2) *the function $m\colon W \setminus E \to M$ is definable;*
(3) *$E \subset F_k \subset W$, $F_k$ is closed in $W$, $M \setminus F_k = \mathrm{Reg}_k M$ and the function $m$ is of class $\mathscr{C}^{k-1}$ at $x \in W \setminus \overline{E}$ iff $x \in W \setminus F_k$.*

*Proof.* We give here only an outline. The set $\Omega_M$ is definable, whence one can assume that $M$ is closed. The function $\phi$ is definable and so are the sets $X, Y, W, Z$ and $\tilde{Z}$. Actually the whole thing now is sort of 'global',

It is a classical fact (see [C]) that the set

$$\mathcal{N}_{k-1}(m) := \{x \in W \setminus \overline{E} \mid m \text{ is not of class } \mathscr{C}^{k-1} \text{ at } x\}$$

is definable and nowheredense.

Let $F_k := \overline{E} \cup \mathcal{N}_{k-1}(m)$. It is definable, closed and $\mathrm{int} F_k = \varnothing$.

By the Nash Lemma (cf. Remark 1.2), if $a \in \mathrm{Reg}_k M$, then $a \in W \setminus F$. On the other hand, if $a \in M \setminus F$, then $m$ is defined and of class $\mathscr{C}^{k-1}$ (Remark 1.2) at $a$. Therefore, $\delta(x) = ||x - m(x)||^2$ is of class $\mathscr{C}^{k-1}$ at $a$, too, and so by Theorem 1.3, $a \in \mathrm{Reg}_{k-1} M$. But if we differentiate $\delta(x)$ for $x$ in a neighbourhood $\mathbb{B}(a, r)$ such that $\mathbb{B}(a, 2r) \cap M$ is a $\mathscr{C}^{k-1}$-submanifold, then (we write $m(x) = (m_1(x), \ldots, m_n(x))$)

$$\frac{\partial \delta}{\partial x_j}(x) = 2(x_j - m_j(x)) - 2\sum_{i=1}^{n}(x_j - m_i(x))\frac{\partial m_i(x)}{\partial x_j} =$$
$$= 2(x_j - m_j(x)) - 2\left\langle x - m(x), \frac{\partial m}{\partial x_j}(x)\right\rangle.$$

Since $\partial m / \partial x_j(x) \in T_{m(x)} M$, by the very definition of $m(x)$, the vector $x - m(x)$ is normal to $M$ at $m(x)$ (cf. the first footnote). Therefore,

$$\mathrm{grad}\delta(x) = 2(x - m(x))$$

which means that $\delta$ is of class $\mathscr{C}^k$ in $\mathbb{B}(a, r)$ and so $a \in \mathrm{Reg}_k M$ by Theorem 1.3, as wanted. □

*Remark* 3.3. In a forthcoming paper we shall prove a parameter version of the main theorem.

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