On the Abhyankar–Jung theorem for henselian $k[x]$-algebras of formal power series

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Abstract

Given an algebraically closed field k of characteristic zero, we present the Abhyankar–Jung theorem for henselian $k[x]$-algebras of formal power series, which are closed under reciprocal, power substitution and division by a coordinate. Examples of such algebras come, for instance, from polynomially bounded o-minimal structures.

The classical proofs of the Newton–Puiseux theorem (for formal or convergent power series with complex coefficients) applied Newton's algorithm to compute term by term the fractional series (called Puiseux series) arising as t-roots of an algebraic equation $f(x, t) = 0$. This algorithm had been described in [6], "Methodus fluxionum et serierum infinitorum" (see also [12]). It consists in computation using the so-called Newton polygon, determined by the exponents of a given polynomial.

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The Abhyankar–Jung theorem can be regarded as a higher dimensional generalization of the Newton–Puiseux theorem. It asserts that the roots of a Weierstrass (formal or convergent) polynomial with discriminant being a normal crossing are fractional (formal or convergent) series. The first proof for the case of two variables is due to H.W. Jung [3]. He used topological and complex analytical methods, the point being that the complex plane with two lines cut out is topologically the product of two punctured discs, whence its fundamental group is $\mathbb{Z} \times \mathbb{Z}$. In the further reasoning, one needs to apply the Riemann removable singularity theorem. S.S. Abhyankar achieved the general result by means of purely algebraic methods, namely, some properties of the Galois group of the polynomial under study. The methods of Jung and Abhyankar are described in e.g. [4].

I. Luengo [5] observed that a quasiordinary formal Weierstrass polynomial $f(x;t) \in k[[x]][t]$ over an algebraically closed field k of characteristic zero, with vanishing coefficient in t^{n-1}, $n = \deg f$, is ν-quasiordinary in the sense of Hironaka [2]; the latter is a certain property of the Newton polyhedron. This allowed him to give a proof of the Abhyankar–Jung theorem which extends that of the Newton–Puiseux theorem.

In this paper, we give a proof of the Abhyankar–Jung theorem for henselian $k[x]$-algebras $\mathcal{A}(k;x)$ of formal power series, which are closed under reciprocal, power substitution and division by a coordinate. It is based on Luengo's observation and Hensel's lemma. Important examples of such algebras are $\mathcal{Q}_m \otimes_{\mathbb{R}} \mathbb{C}$, where $\mathcal{Q}_m$ is the ring of germs of quasianalytic functions at $0 \in \mathbb{R}^m$, considered in our papers [7, 8]; or, where $\mathcal{Q}_m$ is the ring of germs at $0 \in \mathbb{R}^m$ of smooth functions definable in the real field with restricted quasianalytic functions.

In our subsequent papers [9, 10, 11], we shall apply the version of the Abhyankar–Jung theorem presented here to o-minimal geometry. Now we begin with precise definitions and statements of the results just described.

We call a polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in k[[x]][t], \quad x = (x_1, \ldots, x_m),$$

quasiordinary, if its discriminant $D(x)$ is a normal crossing:

$$D(x) = x^{\gamma} \cdot u(x) \quad \text{with} \quad \gamma \in \mathbb{N}^m,\ u(x) \in k[[x]],\ u(0) \neq 0.$$

We say that $f(x;t)$ is a Weierstrass polynomial, if its coefficients $a_i(x)$ belong to the maximal ideal of $k[[x]]$, i.e. $a_k(0) \neq 0$.

Let us write

$$f(x;t) = \sum_{\alpha \in \mathbb{N}^m} \sum_{k=0}^{n} a_{\alpha,k} \cdot x^\alpha t^k$$

and put

$$E(f) := \{(\alpha_1, \dots, \alpha_m, k) \in \mathbb{N}^{m+1} : a_{\alpha,k} \neq 0\}.$$

By the Newton polyhedron $N(f)$ of the polynomial $f(x,v)$ we mean the convex hull of $E(f) + \mathbb{N}^{m+1}$. We say, after H. Hironaka [2], that the polynomial $f(x;v)$ is ν-quasiordinary with an exponent $\delta = (\delta_1 \dots, \delta_m) \in \mathbb{Q}^m$, if

1) $N(f) \subset S + [0,\infty)^{n+1}$ and $S \cap E(f) \neq \emptyset$, where S is the segment joining the points $(0, \dots, 0, n)$ and $(\delta_1, \dots, \delta_m, 0)$;

2) the polynomial

$$P(x,t) := \sum_{(\alpha,k) \in S} a_{\alpha,k} x^\alpha t^k$$

is not a power of a linear form.

The first condition means that the projection of the set $N(f) \cap \{t < n\}$ from the point $(0, \dots, 0, n)$ onto the hyperplane $t = 0$ is exactly $\delta + [0,\infty)^m$.

Now let us recall a result due to I. Luengo [5].

Proposition 1. *Every quasiordinary Weierstrass polynomial*

$$f(x;t) = t^n + a_{n-2}(x) t^{n-2} + \cdots + a_0(x) \in k[[x]][t], \quad x = (x_1, \dots, x_m),$$

with vanishing coefficient in t^{n-1}, is ν-quasiordinary. $\diamondsuit$

Since $a_{n-1}(x) \equiv 0$, only condition 1) from the above definition needs a verification in the proof of the propositon. By means of the Tschirnhausen transformation

$$t' = t + 1/n \cdot a_{n-1}(x),$$

one can always come to the case of a polynomial with vanishing coefficient in t^{n-1} and without changing the discriminant. The converse is not true as shown in the following example from [5].

Example. The polynomial

$$g(x_1, x_2; t) := t^4 - 2x_1 x_2^2 \cdot t^2 + x_1^4 x_2^4 + x_1^2 x_2^7$$

is ν-quasiordinary but not quasiordinary since its discriminant $D(x_1, x_2)$ is divisible by $x_1 x_2 (x_1^2 + x_2^3)$.

Remark 1. Since the discriminant of a monic polynomial is a weighted polynomial in its coefficients, the discriminant $D(x)$ of the foregoing ν-quasiordinary polynomial $f(x; t)$ with exponent δ is divisible by $x^{(n-1)\delta}$. Therefore, if the discriminant $D(x)$ is a normal crossing $D(x) = x^\gamma \cdot u(x)$, then $\gamma \geq (n-1)\delta$, i.e. $\gamma_i \geq (n-1)\delta_i$ for all $i = 1, \ldots, m$. In particular, $\delta_i = 0$ whenever $\gamma_i = 0$.

Let k be an algebraically closed field of characteristic zero. Consider a henselian $k[x]$-subalgebra $\mathcal{A}(k; x)$ of $k[[x]]$, $x = (x_1, \ldots, x_m)$, which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For any $r_1, \ldots, r_m \in \mathbb{N}$, $r_1, \ldots, r_m \neq 0$, put

$$\mathcal{A}(k; x_1^{1/r_1}, \ldots, x_m^{1/r_m}) := \{a(x_1^{1/r_1}, \ldots, x_m^{1/r_m}) : a(x) \in \mathcal{A}(k; x)\};$$

when $r_1 = \ldots = r_m = r$, we shall denote the above algebra by $\mathcal{A}(k; x^{1/r})$.

Abhyankar–Jung Theorem. *With the above notation, every quasiordinary polynomial*

$$f(x; t) = t^n + a_{n-1}(x) t^{n-1} + \cdots + a_0(x) \in \mathcal{A}(k; x)[t]$$

has all its roots in $\mathcal{A}(k; x^{1/r})$, for some $r \in \mathbb{N}$; actually one can take $r = n!$.

The proof is by induction with respect to the degree of the polynomial $f(x; t)$. Performing the Tschirnhausen transformation, we may assume that $a_{n-1}(x) \equiv 0$. If $f(x; t)$ is not a Weierstrass polynomial, then $f(0; t)$ is not a power of a linear form. Since the ring $\mathcal{A}(k; x)$ of coefficients is henselian, the polynomial $f(x; t)$ is reducible: $f(x; t) = f_1(x; t) f_2(x; t)$. The theorem thus follows from the induction hypothesis.

Otherwise $f(x; t)$ is a Weierstrass polynomial, and then, by Proposition 1, $f(x; t)$ is a ν-quasiordinary polynomial with an exponent $\delta \in \mathbb{Q}^m$. Take any multi-index $(\beta_1, \ldots, \beta_m, l) \in E(f)$ that lies on the segment S from the definition of ν-quasiordinarity. This property of the polynomial $f(x; t)$ implies immediately the inequalities:

$$(n - l)\alpha \geq (n - k)\delta \quad \text{for all} \quad (\alpha, k) \in E(f).$$

Moreover, for at least one multi-index from $E(f) \cap S$, we have equality.

Therefore, in the new coordinates

$$x_1 = y_1^{n-l}, \ \dots, \ x_m = y_m^{n-l}, \ t = w \cdot y_1^{\delta_1} \cdots y_m^{\delta_m},$$

each $a_k(x) = a_k(y_1^{n-l}, \ \dots, y_m^{n-l})$, $k = 0, 1, \dots, n-2$, is divisible by

$$y_1^{(n-k)\delta_1} \cdots y_m^{(n-k)\delta_m}.$$

Hence

$$f(x;t) = f(y^{n-l}, \ \dots, y_m^{n-l}; t) =$$
$$= t^n + a_{n-2}(y_1^{n-l}, \dots, y_m^{n-l}) \cdot t^{n-2} + \dots + a_0(y_1^{n-l}, \dots, y_m^{n-l}) =$$
$$= t^n + y_1^{2\delta_1} \cdots y_m^{2\delta_m} \cdot b_{n-2}(y) \cdot t^{n-2} + \dots + y_1^{n\delta_1} \cdots y_m^{n\delta_m} \cdot b_0(y),$$

with $b_k(y) \in \mathcal{A}(k; y)$. Moreover, at least one coefficient from among $b_k(y)$, $k = 0, \dots, n-2$, is a unit: $b_k(0) \neq 0$. We thus get

$$f(x;t) = y_1^{n\delta_1} \cdots y_m^{n\delta_m} \cdot g(y; w),$$

where

$$g(y; w) = w^n + b_{n-2}(y) w^{n-2} + \dots + b_0(y) \in \mathcal{A}(k; y)[w].$$

Consequently, the polynomial $g(0, w)$ is not a power of a linear form. Since the ring $\mathcal{A}(k; y)$ of coefficients is henselian, the polynomial $g(y; w)$ is reducible: $g(y; w) = g_1(y; w) \cdot g_2(y; w)$. Therefore the proof is complete again by the induction hypothesis. $\diamondsuit$

Remark 2. Suppose that the discriminant $D(x)$ of the polynomial $f(x;t)$ is a normal crossing of the form

$$D(x) = x_1^{\gamma_1} \cdots x_p^{\gamma_p} \cdot u(x) \quad \text{with} \quad u(0) \neq 0, \quad 0 \leq p \leq m.$$

Then $\delta_{p+1} = \dots = \delta_m = 0$ (cf. Remark 1), and thus the inequalities $(n-l)\alpha \geq (n-k)\delta$ from the above proof are trivially satisfied for all α and k. It is therefore sufficient to change only the first p from among the variables x. Consequently, all the roots of the polynomial $f(x; v)$ belong to $\mathcal{A}(k; x_1^{1/r}, \dots, x_p^{1/r}, x_{p+1}, \dots, x_m)$.

We conclude this article with the following comment of great importance for applications in quasianalytic geometry(cf. [9, 10, 11]. Let A be a normal domain, K its quotient field and $f(t) \in A[t]$ be a monic polynomial. It is well known from commutative algebra that if $f(t)$ is irreducible over A, then so it is over K. Consequently, if the field K is of characteristic zero, then every irreducible monic polynomial $f(t) \in A[t]$ is square-free. We cannot directly apply this assertion to the domain

$$\mathcal{A}^{\mathbb{C}}(\mathbb{R}, x) := \mathcal{A}(\mathbb{R}, x) \otimes_{\mathbb{R}} \mathbb{C}, \quad \text{where} \quad \mathcal{A}(\mathbb{R}, x) := \mathcal{Q}_m;$$

here $\mathcal{Q}_m$ is the ring of germs of quasianalytic functions at $0 \in \mathbb{R}^m$, considered in our papers [7, 8], or is the ring of germs at $0 \in \mathbb{R}^m$ of smooth functions definable in the real field with restricted quasianalytic functions.

However, each quasi-meromorphic germ (i.e. element $a(x)/b(x)$ of the quotient field of the domain $\mathcal{A}(\mathbb{R}, x)$ or $\mathcal{A}^{\mathbb{C}}(\mathbb{R}, x)$) that is integral over $\mathcal{A}(\mathbb{R}, x)$ or $\mathcal{A}^{\mathbb{C}}(\mathbb{R}, x)$, respectively, can be transformed to a quasianalytic germ by a finite sequence of blowings-up with smooth centers. Indeed, one can transforme by blowing up the germs $a(x)$ and $b(x)$ to such normal crossings

$$y^{\alpha} \cdot u(y) \quad \text{and} \quad y^{\beta} \cdot v(y), \quad \text{respectively,}$$

that $u(y)$, $v(y)$ are units and either $\alpha \leq \beta$ or $\alpha \geq \beta$. In our case, since the fraction $y^{\alpha-\beta} \cdot u(y)v^{-1}(y)$ is integral over the ring of quasianalytic germs, there must be $\alpha \geq \beta$, which is the desired result.

Therefore, we can adapt the foregoing algebraic assertion to the quasianalytic settings in the following form:

Proposition 2 (on square-free factorization). *If $f(x;t)$ is a monic polynomial with coefficients in $\mathcal{A}(\mathbb{R}, x)$ or $\mathcal{A}^{\mathbb{C}}(\mathbb{R}, x)$, then there exists a modification $\sigma : W \longrightarrow \Omega$ of a neighbourhood Ω of zero, which is a finite composite of blowings-up with smooth centers, such that the pull-back polynomial $f^{\sigma}(y;t)$ factorizes near each point $y_0 \in \sigma^{-1}(0) \subset W$ into a product of square-free monic polynomials.* ♢

Combining the above with transformation to normal crossings by blowing up, we immediately obtain

Corollary. *Every monic polynomial $f(x;t) \in \mathcal{A}(\mathbb{R}, x)[t]$ or $f(x;t) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R}, x)[t]$ factorizes into a product of quasiordinary polynomials after a suitable transformation of its quasianalytic coefficients by a finite sequence of blowings-up with smooth centers.* ♢

In this fashion, the version of the Abhyankar–Jung theorem presented herein applies to arbitrary monic polynomials $f(x;t) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R},x)[t]$.

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