Rectilinearization of quasi-subanalytic functions and its applications

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Abstract

This paper presents several theorems about the rectilinearization of quasi-subanalytic functions and their application to quantifier elimination for the real field with restricted quasianalytic functions.

1. Introduction. This paper is concerned with the rectilinearization of functions definable in the real field with restricted quasianalytic functions. We are particularly interested in quasianalytic counterparts of the theorem on rectilinearization of a continuous subanalytic function due to Bierstone–Milman [1] and Parusiński [15, 16].

In that classical, real analytic case, Bierstone and Milman employ Hironaka's major complex-analytic tool, the local flattening theorem (see e.g. [6]). This method makes it possible to reduce the problem to the case where the subanalytic function f under study is semianalytic. Via transformation to normal crossings applied to the functions defining the graph of f, the Weierstrass preparation theorem can be applied. This leads to the case where the function f is a root of a polynomial with analytic coefficients.

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Parusiński, in turn, indicates that in the foregoing reduction, instead of local flattening, a weaker result with equidimensionality can be used. He applies, however, the technique of complexification too. Transformation to normal crossings, used once again, reduces the problem to the case where the function f is a root of a polynomial with analytic coefficients, whose discriminant is a normal crossing. This allows him to use a real-analytic version of the Abhyankar–Jung theorem.

Unfortunately, the main tools of the classical analytic geometry, as the technique of flattening and complexification, or finally, the Weierstrass preparation theorem, are unavailable in quasianalytic geometry. Nevertheless, we have at our disposal transformation to normal crossings (cf. [2, 3]), which can serve as a basic tool for the study of definable sets and functions (cf. [19, 18, 8, 9]).

Several theorems concerning rectilinearization of quasi-subanalytic functions are provided in Section 2, which is crucial for the whole article. These results will be needed in the last section for the study of quantifier elimination for the real field with restricted quasianalytic functions. They will also be applied in our subsequent paper [12] about arc-quasianalytic functions.

Section 3 gives an affirmative answer to a problem posed in our previous article [8] concerning the decomposition of a definable (i.e. Q-subanalytic) set into special cubes. As an immediate consequence, we achieve that the real field with restricted quasianalytic functions admits quantifier elimination in the language augmented by the name of the reciprocal function. This generalizes to the quasianalytic settings a theorem of Denef–van den Dries [4] from real analytic geometry.

As in our previous papers [8, 9], we shall deal with a family $\mathcal{Q}(U)$ of quasianalytic Q-functions defined on the open subsets U of the affine spaces $\mathbb{R}^m$, which satisfies the following six conditions:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;

2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);

3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \longrightarrow V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^m$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \longrightarrow U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;

4. $\mathcal{Q}$ is closed under differentiation;

5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;

6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f}_a$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

This gives rise to a category Q of Q-manifolds and their Q-mappings, which admits a transformation to normal crossings and a desingularization by blowing up (cf. [2, 3]). Transformation to normal crossings enables one to successfully build the geometry of Q-semianalytic and Q-subanalytic sets (cf. [19, 8, 9]). In particular, Gabrielov's complement theorem holds also for Q-subanalytic sets.

Consider now the expansion $\mathcal{R}_Q$ of the real field $\mathbb{R}$ by restricted Q-functions, i.e. functions of the form:

$$\tilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^m \\ 0, & \text{otherwise} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^m$. We attach to the language of ordered rings (with the symbols $=, <, 0, 1, +, -, \cdot$) the names of all restricted Q-functions.

$\mathcal{R}_Q$ is a polynomially bounded o-minimal structure which admits Q-cell decomposition and, moreover, has a universal axiomatization in the language of restricted Q-functions augmented by the names of rational powers (cf. [8, 9, 18]); the latter result being a generalization to the quasianalytic settings of a classical theorem by [5] from real analytic geometry. Our establishing of this result relies, however, on a certain problem of the decomposition of quasianalytic germs with respect to their Taylor series, posed by us in [9, 10]. Although evident in the classical real analytic case, it is very delicate in the general quasianalytic settings, and seems to be unsolved as yet.

We describe the reciprocal function $1/x$ and roots $\sqrt[n]{x}$ in the ordinary fashion by stipulating that:

$$x \cdot 1/x = 1 \quad \text{if} \quad x \neq 0 \quad \text{and} \quad 1/x = 0 \quad \text{if} \quad x = 0,$$

$$(\sqrt[n]{x})^n = x \quad \text{if} \quad x \geq 0 \quad \text{and} \quad \sqrt[n]{x} = 0 \quad \text{if} \quad x < 0.$$

2. Rectilinearization of quasi-subanalytic functions. We begin with terminology suitable for the rectilinearization of definable functions. By a quadrant in $\mathbb{R}^m$ in we mean a subset of $\mathbb{R}^m$ of the form:

$$\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = 0, x_j > 0, x_k < 0 \quad \text{for} \;\; i \in I_0, j \in I_+, k \in I_-\},$$

where $\{I_0, I_+, I_-\}$ is a disjoint partition of $\{1, \ldots, m\}$; its trace Q on the cube $[-1, 1]^m$ shall be called a bounded quadrant; put

$$Q_+ := \{x \in [0, 1]^m : x_i = 0, x_j > 0 \quad \text{for} \;\; i \in I_0, j \in I_+ \cup I_-\}.$$

The interior $\mathrm{Int}\,(Q)$ of the quadrant Q is its trace on the open cube $(-1, 1)^m$. A bounded closed quadrant is the closure $\overline{Q}$ of a bounded quadrant Q, i.e. a subset of $\mathbb{R}^m$ of the form:

$$\overline{Q} := \{x \in [-1, 1]^m : x_i = 0, x_j \geq 0, x_k \leq 0 \quad \text{for} \;\; i \in I_0, j \in I_+, k \in I_-\}.$$

We say that a function g on a bounded quadrant Q in $\mathbb{R}^m$ is a fractional normal crossing on Q if it is the superposition of a normal crossing f in the vicinity of the closure $\overline{Q_+}$ of Q_+ and a rational power substitution ψ given by the equality:

$$\psi : \mathbb{R}^m \longrightarrow \mathbb{R}^m, \quad \psi(x_1, \ldots, x_m) = (|x_1|^{\alpha_1}, \ldots, |x_m|^{\alpha_m}),$$

where $\alpha_1, \ldots, \alpha_m$ are non-negative rational numbers. In other words, a fractional normal crossing g on Q is a function of the form

$$g(x_1, \ldots, x_m) = |x_1|^{\frac{n_1}{N}} \cdot \ldots \cdot |x_m|^{\frac{n_m}{N}} \cdot u(|x_1|^{\frac{1}{N}}, \ldots, |x_m|^{\frac{1}{N}}),$$

where N is a positive integer, $n_1, \ldots, n_m$ are non-negative integers such that $n_i = 0$ for $i \in I_0$, and u is a Q-function near $\overline{Q_+}$ which vanishes nowhere on $\overline{Q_+}$.

Before proving the main result of this section, we make a key observation. Consider an o-minimal structure $\mathcal{R}$ in a language $\mathcal{L}$. Since every o-minimal structure has definable Skolem (choice) functions, the following two conditions are equivalent:

- every definable function f in $\mathcal{R}$ is piecewise given by a finite number of terms in the language $\mathcal{L}$;
- the structure $\mathcal{R}$ has a universal axiomatization which admits quantifier elimination in the language $\mathcal{L}$.

In the case of the structure $\mathcal{R}_Q$, every definable function is actually given by one term in the language of restricted Q-functions augmented by the names of rational powers. Indeed, every such function is piecewise given by a finite number of terms in this augmented language, and therefore it is sufficient to show that the characteristic function of any definable subset $E \subset \mathbb{R}^m$ is given by one term.

Further, due to quantifier elimination, we are reduced to sets described by atomic formulae, and next, since our language contains only two relation symbols $=$ and $<$, to the sets given by the formulae $t(x) = 0$ or $t(x) > 0$ with any term $t(x)$. Consequently, we have only to know that the characteristic functions f and g of the subsets $A := \{0\}$ and $B := (0, \infty)$ of the real line $\mathbb{R}$ are given by one term. But this follows from the obvious equalities below:

$$f(x) = 1 - x \cdot \frac{1}{x} \quad \text{and} \quad g(x) = x \cdot \frac{1}{(\sqrt{x})^2}.$$

Theorem 1 (on simultaneous rectilinearization of definable functions). *If $f_1, \ldots, f_s : \mathbb{R}^m \longrightarrow \mathbb{R}$ are definable functions and K is a compact subset of $\mathbb{R}^m$, then there exist a finite collection of modifications*

$$\varphi_i : [-1, 1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1, 1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) the union of the images $\varphi_i((-1, 1)^m)$, $i = 1, \ldots, p$, is a neighbourhood of K.

3) for every bounded quadrant Q_j, $j = 1, \ldots, 3^m$, the restriction to Q_j of each function $f_k \circ \varphi_i$, $k = 1, \ldots, s$, $i = 1, \ldots, p$, either vanishes or is a normal crossing or a reciprocal normal crossing on Q_j.

The proof is based on the fact that every definable function $f_k : \mathbb{R}^m \longrightarrow \mathbb{R}$ is piecewise given by one term t_k in the language of restricted Q-functions augmented by the names of rational powers. We shall proceed by induction on the complexity of the terms t_k; obviously, terms of complexity zero are variables and constants. We wish to explain the induction process more precisely.

For any finite collection (ψ_i) of modifications described in Theorem 1, denote by ψ the modification being the disjoint gluing of the mappings ψ_i. We shall, in fact, prove by a double induction, with respect to the maximum $n = 0, 1, 2, \ldots$ of the complexities of the terms t_k and the number $s = 1, 2, \ldots$ of these terms, that the theorem holds for the superpositions t_k of the terms t_k with any modification ψ as above.

When $n = 0$ and s is an arbitrary positive integer, the foregoing theorem can be established directly via a simultaneous transformation to normal crossings of the components of the mapping ψ. We encounter two distinct induction steps, described by the following two schemes:

I. assuming the theorem to hold for n and s, we will prove it for n and $s + 1$;

II. assuming the theorem to hold for n and all s, we will prove it for $n+1$ and 1.

We first outline how to cope with induction scheme I. Suppose we have terms $t_1, \ldots, t_s, t_{s+1}$. By induction hypothesis, we are able to find a finite collection φ of modifications such that the requirements of the theorem are fulfilled for the superpositions $t_1 \circ \psi \circ \varphi, \ldots, t_s \circ \psi \circ \varphi$. We shall have established this induction scheme if we find a collection χ of modifications that improves $t_{s+1} \circ \psi \circ \varphi$ without spoiling the superpositions already achieved in an appropriate form. We shall explain below how to find such an appropriate collection χ.

Take a collection ω of modifications suitable for $t_{s+1} \circ \psi \circ \varphi(x)$; let $x = \omega(x')$. Next, via simultaneous transformation to normal crossings, one can find a collection σ of modifications $x' = \sigma(x'')$ such that each

$$x_j = \omega_j(x') = (\omega_j \circ \sigma)(x'') \quad \text{and} \quad x'_j = \sigma_j(x''), \quad j = 1, \ldots, m$$

is a normal crossing in the variables x''. Then the superposition $\psi \circ \varphi \circ \omega \circ \sigma$ is the desired collection of modifications.

In the proof of induction scheme II, we must analyze a term t of complexity $n + 1$, and thus encounter the following cases:

$$t = \sqrt[p]{t_1}, \quad t = t_1 \cdot t_2, \quad t = \frac{t_1}{t_2}, \quad t = t_1 + t_2 \quad \text{and} \quad t = g(t_1, \ldots, t_r),$$

where $t_1, \ldots, t_r$ are terms of complexity $\leq n$ and g is a restricted Q-function. The verification of these five cases is routine, and needs again the use of simultaneous transformation to normal crossings of a finite number of Q-functions in the following strengthened form: one can require that the exponents of the normal crossings achieved in the process be totally ordered with respect to the induced partial ordering from $\mathbb{N}^m$. This finishes the proof, the details being left to the reader. $\diamondsuit$

Remark. In the above proof we use the obvious fact that normal crossings are preserved under substitution of powers, and thus under substitutions of normal crossings. Similarly, fractional normal crossings are preserved under substitution of fractional normal crossings.

We could repeat mutatis mutandis the above proof in order to get modifications φ_i that are finite composites of local blowings-up and of power substitution only at the last step, as stated below.

Theorem 1* (on simultaneous rectilinearization of definable functions). *If $f_1, \ldots, f_s : \mathbb{R}^m \longrightarrow \mathbb{R}$ are definable functions and K is a compact subset of $\mathbb{R}^m$, then there exist a finite collection of modifications*

$$\varphi_i : [-1, 1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1, 1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) the union of the images $\varphi_i((-1, 1)^m)$, $i = 1, \ldots, p$, is a neighbourhood of K.

3) for every bounded quadrant Q_j, $j = 1, \ldots, 3^m$, the restriction to Q_j of each function $f_k \circ \varphi_i$, $k = 1, \ldots, s$, $i = 1, \ldots, p$, either vanishes or is a fractional normal crossing or a reciprocal fractional normal crossing on Q_j.

$\diamondsuit$

What supervenes in the proof of Theorem **1*** is only an additional difficulty in showing induction scheme II for the terms

$$t = t_1 + t_2 \quad \text{and} \quad t = g(t_1, \ldots, t_r).$$

This could be easily established, once the following conjecture on fractional normal crossings holds true for the general quasianalytic settings.

Conjecture. *Given a Q-function $f : U \longrightarrow \mathbb{R}$ near zero and $n \in \mathbb{N}$, there exist a finite number of modifications $\sigma_i : [0, 1]^m \longrightarrow \mathbb{R}$ such that:*

i) each σ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[0,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

ii) $\sigma_i([0,1]^m) \subset U$ and the union of the images $\sigma_i((0,1)^m)$ is the trace of a neighbourhood of zero on the orthant $(0,\infty)^m$;

iii) each superposition $f(x_1^{1/n},\dots,x_m^{1/n})\circ\sigma_i$ is a fractional normal crossing on the orthant $(0,1)^m$.

In the classical case of real analytic functions, this conjecture follows from the Abhyankar–Jung theorem and the fact that the germ $f(x_1^{1/n},\dots,x_m^{1/n})$ is integral over the ring of germs of analytic functions. The former can be carried over to the quasianalytic settings, as proven in our paper [11]. The latter result, however, is still open. In our paper [14], we link it with a certain delicate problem of decomposition of quasianalytic germs with respect to their Taylor series. It is evident in the classical real analytic case, but — although discussed with numerous specialists in the theory of ultradifferentiable and quasianalytic functions — in the quasianalytic settings, it seems to remain unsolved as yet.

We now recall the generalization of the classical Abhyankar–Jung theorem to certain henselian $k[x]$-algebras from our paper [11]. Here it will be applied to the $\mathbb{C}$-algebra

$$\mathcal{A}^{\mathbb{C}}(\mathbb{R};x) := \mathcal{A}(\mathbb{R};x)\otimes_{\mathbb{R}}\mathbb{C},$$

where $\mathcal{A}(\mathbb{R};x)$ denotes the local $\mathbb{R}$-algebra of germs at $0\in\mathbb{R}^m$ of Q-functions. For any $r\in\mathbb{N}$, $r\neq 0$, put

$$\mathcal{A}(\mathbb{R};x^{1/r}) := \{a(x_1^{1/r_1},\dots,x_m^{1/r_m}) : a(x)\in\mathcal{A}(\mathbb{R};x)\}$$

and

$$\mathcal{A}^{\mathbb{C}}(\mathbb{R};x^{1/r}) := \{a(x_1^{1/r_1},\dots,x_m^{1/r_m}) : a(x)\in\mathcal{A}^{\mathbb{C}}(\mathbb{R};x)\}.$$

Abhyankar–Jung Theorem. *Every quasiordinary polynomial*

$$P(x;v) = v^n + a_{n-1}(x)v^{n-1} + \cdots + a_0(x) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R};x)[v]$$

has all its roots in $\mathcal{A}^{\mathbb{C}}(\mathbb{R};x^{1/r})$, for some $r\in\mathbb{N}$; actually, one can take $r=n!$. $\diamondsuit$

As an immediate consequence, we obtain the aforementioned real version of the Abhyankar–Jung theorem:

Corollary. *Every quasiordinary polynomial*

$$P(x; v) = v^n + a_{n-1}(x)v^{n-1} + \cdots + a_0(x) \in \mathcal{A}(\mathbb{R}; x)[v]$$

is a product of linear factors and irreducible square trinomials with coefficients in $\mathcal{A}(\mathbb{R}; x^{1/r})$, for some $r \in \mathbb{N}$; actually, one can take $r = n!$. Here, the roots $x^{1/r}$ are regarded as germs on the first orthant in $\mathbb{R}^m$. Moreover, if the first non-zero coefficient $a_i(x) \not\equiv 0$ is a normal crossing, then every non-zero root is a fractional normal crossing. $\diamondsuit$

We are now in a position to outline a proof of the conjecture on fractional normal crossings. Once we know that the germ $f(x_1^{1/n}, x_2^{1/n}, \ldots, x_m^{1/n})$ is integral over $\mathcal{A}(\mathbb{R}; x)$, it is a root of a monic polynomial with Q-analytic coefficients. Hence and by Proposition 2 from [11], this germ is, after a suitable transformation σ to normal crossings by blowing up, a root of a quasiordinary polynomial which satisfies the assumption of the above corollary, and thus it is a fractional normal crossing. Furthermore, we must require that the coordinate functions transform to normal crossings so as to ensure condition ii) of the conjecture.

Now let us proceed with some consequences of Theorem 1. Let U be a definable bounded open subset in $\mathbb{R}^m$, ∂U be its frontier and ρ_1, ρ_2 be the distance functions from the sets U, ∂U, respectively. Given a definable function $f : U \longrightarrow \mathbb{R}$, we can deduce directly from Theorem 1 applied to the functions f, ρ_1, ρ_2, the following consequence:

Theorem 2 (on rectilinearization of a definable function). *Let $U \subset \mathbb{R}^m$ be a bounded open subset and $f : U \longrightarrow \mathbb{R}$ be a definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1, 1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1, 1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{Int}\,(Q))$ with Q ranging over the bounded quadrants contained in $\varphi_i^{-1}(U)$, $i = 1, \ldots, p$;

5) for every bounded quadrant Q, the restriction to Q of each function $f \circ \varphi_i$ either vanishes or is a normal crossing or a reciprocal normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$. $\diamondsuit$

Remarks. 1) It follows from points 1) and 2) that every bounded quadrant of dimension $< m$ contained in $\varphi_i^{-1}(U)$ is adjacent to a bounded quadrant of dimension m (a bounded orthant) contained in $\varphi_i^{-1}(U)$. Hence

$$\varphi_i^{-1}(\overline{U}) = \overline{\varphi_i^{-1}(U)},$$

and therefore point 4) implies that $\overline{U}$ is the union of the images $\varphi_i(\overline{Q})$ of the closures of those bounded quadrants of dimension m (bounded orthants) Q for which $\varphi_i(Q) \subset U$, $i = 1, \ldots, p$.

2) One can formulate Theorem 2, similarly as Theorem 1, for several definable functions $f_1, \ldots, f_s$.

Under the above notation, consider a bounded orthant Q contained in $\varphi_i^{-1}(U)$. Denote by $\mathrm{dom}_i\,(Q)$ the union of Q and all those bounded quadrants that are adjacent to Q and disjoint with $\varphi_i^{-1}(\partial U)$; it is, of course, an open subset of the closure $\overline{Q}$. Moreover, the open subset $\varphi_i^{-1}(U)$ of the cube $[-1,1]^m$ coincides with the union of $\mathrm{dom}_i\,(Q)$, where Q range over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, and with the union of those bounded quadrants that are contained in $\varphi_i^{-1}(U)$.

Consequently, the union of the images $\varphi_i(\mathrm{Int}\,(Q))$, where Q range over the bounded quadrants that are contained in $\varphi_i^{-1}(U)$, coincides with the union of the images

$$\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m),$$

where Q range over the bounded orthants Q that are contained in $\varphi_i^{-1}(U)$.

Corollary (on rectilinearization of a continuous definable function). *Let U be a bounded open subset in $\mathbb{R}^m$ and $f : U \longrightarrow \mathbb{R}$ be a continuous definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \ldots, p$;

5) for every bounded orthant Q, the restriction to $\mathrm{dom}_i\,(Q)$ of each function $f \circ \varphi_i$ either vanishes or is a normal crossing or a reciprocal normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$. $\diamondsuit$

Similarly, Theorem 1* implies the results stated below, which are thus valid provided that the foregoing conjecture holds true for the general quasianalytic settings. We return to this conjecture in our paper [14], where it is linked in more detail with a certain delicate problem of the decomposition of quasianalytic germs with respect to their Taylor series. This problem is evident in the case of real analytic germs.

Theorem 2* (on rectilinearization of a definable function). *Let $U \subset \mathbb{R}^m$ be a bounded open subset and $f : U \longrightarrow \mathbb{R}$ be a definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{Int}\,(Q))$ with Q ranging over the bounded quadrants contained in $\varphi_i^{-1}(U)$, $i = 1, \ldots, p$;

5) for every bounded quadrant Q, the restriction to Q of each function $f \circ \varphi_i$ either vanishes or is a fractional normal crossing or a reciprocal fractional normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$. $\diamondsuit$

Corollary* (on rectilinearization of a continuous definable function). *Let U be a bounded open subset in $\mathbb{R}^m$ and $f : U \longrightarrow \mathbb{R}$ be a continuous definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \quad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \ldots, p$;

5) for every bounded orthant Q, the restriction to $\mathrm{dom}_i\,(Q)$ of each function $f \circ \varphi_i$ either vanishes or is a fractional normal crossing or a reciprocal fractional normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$. $\diamondsuit$

Theorem 2* will be used in the next section in connection with quantifier elimination in the language of restricted quasianalytic functions augmented by the name of the reciprocal function. The significance of the foregoing corollary lies in its application to the theory of arc-quasianalytic functions, presented in our subsequent paper [12]. Finally, let us mention that the above theorems on rectilinearization imply the classical results of E. Bierstone, P.D. Milman [1] and A. Parusiński [15, 16].

3. Application to quantifier elimination. We begin this section with an affirmative answer to a question posed in our previous paper [8] (Open Problem 1):

Theorem 3 (on decomposition into special cubes). *Every bounded Q-subanalytic subset F in $\mathbb{R}^m$ is a finite union of special cubes S_i, i.e. subsets in $\mathbb{R}^m$ of the form*

$$S_i = \varphi_i((-1,1)^{d_i}),$$

where $\varphi_i(x)$ is a special modification, i.e. a diffeomorphism from $(-1,1)^{d_i}$ onto S_i that extends to a Q-mapping in the vicinity of into $[-1,1]^{d_i}$.

Moreover, each φ_i is a composite of finitely many local blowings-up with smooth centers, and therefore each special cube S_i and each inverse mapping

$$\psi_i : S_i \longrightarrow (-1,1)^{d_i}$$

to the special modification φ_i is described by terms in the language of restricted Q-functions augmented by the name of the reciprocal function.

Remarks. 1) Each inverse mapping ψ_j is given piecewise by terms in the language of restricted Q-functions augmented by the name of the reciprocal function $1/x$, because — roughly speaking — it has been locally built in the process of blowing up as a successive superposition of restricted Q-functions and of the reciprocal function $1/x$ off the zero argument.

2) One can formulate Theorem 3 for relatively compact Q-subanalytic subsets of a Q-manifold too. Such a version would even be more suitable for our proof which is lead by induction with respect to the dimension of the ambient space. Actually, the ambient spaces involved in the induction process are the smooth centers of the successive blowings-up.

For the proof, apply Theorem 2* to the function $f := 1 - \chi_F$, where χ_E is the characteristic function of the set F, so as to find a finite collection of modifications

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \quad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) F is a finite union of the images under the φ_i's of the interiors of some bounded quadrants.

Now, in order to present F as a finite union of special cubes of the form specified in the theorem, we can just repeat the proof of the theorem on covering with special cubes from our paper [8]. In this manner, we are able to construct special cubes that are compatible with the coordinate functions and the exceptional divisors of the modifications φ_i, and hence the theorem follows. Recall that the aforementioned proof is by induction with respect to the dimension of the ambient space, and is based on transformation to normal crossings by local blowing up such that the final exceptional divisors have only normal crossings. For more details we refer the reader to [8]. $\diamondsuit$

As an immediate consequence, we obtain the result below, which generalizes to the quasianalytic settings a theorem of Denef–van den Dries [4] on quantifier elimination for restricted real analytic function. Their proof works with convergent power series, and makes use, in particular, of the Weierstrass preparation theorem and the fact that the ring of convergent power series is noetherian.

Corollary. *The expansion $\mathcal{R}_Q$ of the real field $\mathbb{R}$ admits quantifier elimination in the language of restricted Q-functions augmented by the name of the reciprocal function.* $\diamondsuit$

Finally, let us emphasize once again that the results presented in this paper rely on a certain delicate issue concerning the decomposition of quasianalytic germs with respect to their Taylor series (cf. [9, 10, 14]). Whereas it is evident in the classical real analytic case, it seems to remain unsolved in the general quasianalytic settings.

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