On arc-quasianalytic functions

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Abstract

In this paper, we present certain characterizations of definable arc-quasianalytic functions through blowing up. Our approach makes use of a description of quasi-subanalytic functions investigated in our previous article, which relies, in the non-analytic case, on a conjecture concerning quasianalytic fractional normal crossings.

1. Introduction. In our previous article [12], we presented several theorems about the rectilinearization of quasi-subanalytic functions and their application to quantifier elimination for the real field with restricted quasianalytic functions. In this paper we shall apply them to the theory of definable arc-quasianalytic functions in such structures.

The notion of a definable arc-quasianalytic function, introduced in Section 3, generalizes that of an arc-analytic function, considered by Kurdyka [6] in relation with arc-symmetric semialgebraic sets. Kurdyka posed also a

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question whether every such function can be, perhaps under additional assumptions, modified by means of blowing up to an analytic function. We should mention that functions of this type (i.e. analytic after composition with certain modifications) were investigated by Kuo [5] as well.

In the classical, real-analytic case, the affirmative answer was first given by Birstone–Milman [1], and next by Parusiński [15]. Also developed in [1] was a method for rectilinearization of a continuous subanalytic function to the effect that every such function becomes analytic after composing it with a locally finite family of modifications, each of which is a composite of finitely many local blowings-up and local power substitutions. Parusiński [15] improved the above result so that it is enough to substitute powers only at the last step after all local blowings-up.

The main objective of this paper is to carry over the foregoing results to the real field with restricted quasianalytic functions. As in our previous papers (cf. [8, 9, 12]), we shall deal with a family $\mathcal{Q}(U)$ of smooth functions that satisfies certain six conditions introduced by Bierstone–Milman [3], which ensure resolution of singularities and transformation to normal crossings by blowing up. In the sequel, by a quasi-subanalytic function we shall mean a function definable in the real field $\mathbb{R}$ with restricted quasianalytic functions determined by the family $\mathcal{Q}(U)$.

What is crucial for our approach to arc-quasianalytic functions is the theorem on rectilinearization of a continuous definable function from our paper [12] (the corollary to Theorem **2***), stated below. We first recall some necessary notation from [12]. Given a bounded orthant Q and a collection of modifications φ_i, $\mathrm{dom}_i\,(Q)$ denotes the union of Q and all those bounded quadrants that are adjacent to Q and disjoint with $\varphi_i^{-1}(\partial U)$; it is, of course, an open subset of the closure $\overline{Q}$. Moreover, the open subset $\varphi_i^{-1}(U)$ of the cube $[-1,1]^m$ coincides with the union of $\mathrm{dom}_i\,(Q)$, where Q range over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, and with the union of those bounded quadrants that are contained in $\varphi_i^{-1}(U)$.

Consequently, the union of the images $\varphi_i(\mathrm{Int}\,(Q))$, where Q range over the bounded quadrants that are contained in $\varphi_i^{-1}(U)$, coincides with the union of the images

$$\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m),$$

where Q range over the bounded orthants Q that are contained in $\varphi_i^{-1}(U)$.

Theorem 1 (on rectilinearization of a continuous definable function). *Let U be a bounded open subset in $\mathbb{R}^m$ and $f : U \longrightarrow \mathbb{R}$ be a continuous definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \dots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \dots, p$;

5) for every bounded orthant Q, the restriction to $\mathrm{dom}_i\,(Q)$ of each function $f \circ \varphi_i$ either vanishes or is a fractional normal crossing or a reciprocal fractional normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$. $\diamondsuit$

Our proof of Theorem 1 was based on the following conjecture on fractional normal crossings:

Conjecture. *Given a Q-function $f : U \longrightarrow \mathbb{R}$ near zero and $n \in \mathbb{N}$, there exist a finite number of modifications $\sigma_i : [0,1]^m \longrightarrow \mathbb{R}$ such that:*

i) each σ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[0,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

ii) $\sigma_i([0,1]^m) \subset U$ and the union of the images $\sigma_i((0,1)^m)$ is the trace of a neighbourhood of zero on the orthant $(0,\infty)^m$;

iii) each superposition $f(x_1^{1/n}, \dots, x_m^{1/n}) \circ \sigma_i$ is a fractional normal crossing on the orthant $(0,1)^m$.

In the classical case of real analytic functions, this conjecture follows from the real version of the Abhyankar–Jung theorem and the fact that the germ $f(x_1^{1/n}, \dots, x_m^{1/n})$ is integral over the ring of germs of analytic functions (cf. [12] for details). The former can be carried over to the quasianalytic settings, as proven in our paper [11]. The latter is still an open problem. In our paper [14], we link it with a certain delicate problem of decomposition of quasianalytic germs with respect to their Taylor series (also see [9, 10] for its general formulation). It is evident in the classical real analytic case, but in the quasianalytic settings, it seems to remain unsolved as yet. Therefore the results about arc-quasianalytic functions presented in this paper are still relative in the non-analytic case.

2. Arc-quasianalytic functions. A function $f : U \longrightarrow \mathbb{R}$ on an open subset $U \subset \mathbb{R}^m$ is called arc-quasianalytic if it is quasi-subanalytic and for every smooth (of class $\mathcal{C}^\infty$) arc $\gamma : (-1, 1) \longrightarrow U$ the superposition $f \circ \gamma$ is smooth. In the above definition one may confine oneself to considering only Q-analytic arcs.

The proof of the fact that every arc-quasianalytic function is continuous is similar to that for the classical analytic case (see e.g. [6, 1]). For the reader's convenience, we give here a short proof of this fact.

Proposition 2. *Given an open subset U in $\mathbb{R}^m$, every arc-quasianalytic function $f : U \longrightarrow \mathbb{R}$ is continuous.*

Suppose, on the contrary, that the function f is not continuous at a point $a \in U$. Then there are two real numbers $\alpha, \beta \in \mathbb{R}$, $\alpha < \beta$ such that

$$a \in \overline{E_1} \cap \overline{E_2} \quad \text{with} \quad E_1 := \{x \in U : f(x) \leq \alpha\},\ E_2 := \{x \in U : f(x) \geq \beta\}.$$

One can partition the set U into finitely many definable Q-submanifolds $M_1, \ldots, M_p$ such that the function f is smooth on each of them (cf. [8, 9, 18]). Take a smooth definable stratification $\Gamma_1, \ldots, \Gamma_s$ of $\mathbb{R}^m$ compatible with the sets E_1, E_2 and $M_1, \ldots, M_p$. The structure $\mathcal{R}$ admits finite Q-stratifications (i.e. whose strata are Q-submanifolds) of definable subsets, because it admits Q-cell decompositions of definable subsets.

Due to the curve selection lemma and the arc-quasianalyticity of the function f, we get the following implication considered by Bierstone–Milman [1]:

$$\Gamma_j \subset \overline{\Gamma_i} \quad \Rightarrow \quad f(\Gamma_j) \subset \overline{f(\Gamma_i)}\,.$$

It yields that the sets E_1 and E_2 are closed, whence $a \in E_1 \cap E_2$. This contradiction proves Proposition 1. $\diamondsuit$

We can readily pass to the main purpose of this section.

Theorem 2 (on rectilinearization of an arc-quasianalytic function). *Keeping the notation of theorem 1, assume that a function $f \not\equiv 0$ is arc-quasianalytic on a connected open subset U in $\mathbb{R}^m$. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \dots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1,1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \dots, p$;

5) each function $f \circ \varphi_i$ is a smooth (of class $\mathcal{C}^\infty$) quasi-subanalytic function on the union

$$\bigcup_Q \mathrm{dom}_i\,(Q) \cap (-1,1)^m$$

with Q ranging over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, which is an open rectangular subset of the open cube $(-1,1)^m$.

Remark. The question whether every smooth quasi-subanalytic function is Q-analytic is an open problem posed by us in [8]. It seems to remain unsolved as yet.

Theorem 2 follows directly from Theorem 1 and the proposition below.

Proposition 3. *Let F_1 and F_2 be two Q-functions in the vicinity of the sets $\Omega_1 := \{x \in \mathbb{R}^m : x_m \geq 0\}$ and $\Omega_2 := \{x \in \mathbb{R}^m : x_m \leq 0\}$, respectively, and suppose that the functions F_1 and F_2 coincide on $\Omega_1 \cap \Omega_2$. For a positive integer $k \in \mathbb{N}$, consider the functions*

$$f_i : \Omega_i \longrightarrow \mathbb{R}, \quad f_i(x) := F(x_1, \dots, x_{m-1}, |x_m|^{\frac{1}{k}}), \quad i = 1, 2,$$

and denote by $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ their gluing. If for all $x_1, \dots, x_{m-1} \in \mathbb{R}$ the functions $f(x_1, \dots, x_{m-1}, \cdot)$ of one variable x_m are smooth, then so is the function f.

For the proof, fix a positive integer $N \in \mathbb{N}$. Then

$$F_i(x) = g_{i,0}(x_1, \dots, x_{m-1}) + g_{i,1}(x_1, \dots, x_{m-1}) \cdot x_m + g_{i,2}(x_1, \dots, x_{m-1}) \cdot x_m^2 +$$
$$+ \cdots + g_{i,(kN-1)}(x_1, \dots, x_{m-1}) \cdot x_m^{kN-1} + h_i(x) \cdot x_m^{kN} \quad \text{for} \quad i = 1, 2$$

in the vicinity of Ω_i, where $g_{i,j}$ are certain Q-functions on $\mathbb{R}^{m-1}$ and h_i are certain Q-functions in the vicinity of Ω_i, $i = 1, 2$, $j = 0, 1, \dots, kN-1$. Hence we get for $i = 1, 2$:

$$f_i(x) = g_{i,0}(x_1, \dots, x_{m-1}) + g_{i,1}(x_1, \dots, x_{m-1}) \cdot |x_m|^{\frac{1}{k}} +$$
$$+ g_{i,2}(x_1, \dots, x_{m-1}) \cdot |x_m|^{\frac{2}{k}} + \cdots + g_{i,(kN-1)}(x_1, \dots, x_{m-1}) \cdot |x_m|^{\frac{kN-1}{k}} +$$
$$+ h_i(x_1, \dots, x_{m-1}, |x_m|^{\frac{1}{k}}) \cdot |x_m|^N.$$

Our assumption about the functions $f(x_1, \dots, x_{m-1}, \cdot)$ implies that

$$g_{i,j} \equiv 0 \quad \text{if} \quad i = 1, 2, \; j = 1, \dots, kN-1 \quad \text{and} \quad k \not| j,$$

and that the functions

$$\sum_{j=0,\, k|j}^{kN-1} g_{i,j}(x_1, \dots, x_{m-1}) \cdot |x_m|^{\frac{j}{k}}, \quad i = 1, 2,$$

glue to a smooth function on $\mathbb{R}^m$. Since the functions

$$h_i(x_1, \dots, x_{m-1}, |x_m|^{\frac{1}{k}}) \cdot |x_m|^N \quad \text{on} \quad \Omega_i, \quad i = 1, 2,$$

glue to a function of class $\mathcal{C}^N$ on $\mathbb{R}^m$, so do the functions f_1 and f_2. As the fixed integer N is arbitrarily large, the proof is complete. $\diamondsuit$

Corollary 1 (a characterization of arc-quasianalytic functions). *Let U be an open subset in $\mathbb{R}^m$. Then a function $f : U \longrightarrow \mathbb{R}$ is arc-quasianalytic iff there exists a finite collection of definable modifications*

$$\varphi_i : (-1, 1)^m \longrightarrow \mathbb{R}^m, \quad i = 1, \dots, p,$$

such that

1) $\bigcup_{i=1}^p \varphi_i((-1,1)^m) = U$*;*

2) each φ_i is a definable mapping which is a composite of finitely many local blowings-up with smooth centers;

3) each $f \circ \varphi_i$ is a smooth quasi-subanalytic function.

Indeed, whereas the "if direction" is obvious, the "only if" is a special case of Theorem 4. $\diamondsuit$

Remark. The above criterion for a function to be definable and arc-quasianalytic is an o-minimal counterpart of the classical analytic one due to Birstone–Milman [1] and Parusi nski [15].

Corollary 2. *If $f : U \longrightarrow \mathbb{R}$ is an arc-quasianalytic function, then f is a smooth quasi-subanalytic function outside a closed definable subset $Z \subset U$ of codimension ≥ 2.* $\diamondsuit$

Finally, let us emphasize once again that the results presented in this paper rely on a certain delicate issue concerning the decomposition of quasi-analytic germs with respect to their Taylor series (cf. [9, 10, 12, 14]). It is evident in the classical real analytic case, but — although discussed with numerous specialists in the theory of ultradifferentiable and quasianalytic functions — seems to remain unsolved in the general quasianalytic settings.

References

[1] E. Bierstone, P.D Milman, *Arc-analytic functions*, Invent. Math. **101** (1990), 411–424.

[2] —, —, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, Invent. Math. **128** (1997), 207–302.

[3] —, —, *Resolution of singularities in Denjoy–Carleman classes*, Selecta Math., New Ser. **10** (2004), 1–28.

[4] E. Bierstone, P.D. Milman, *Geometric and differential properties of subanalytic sets*, Ann. Math. **147** (1998), 731–785.

[5] T.-C. Kuo, *On classification of real singularities*, Invent. Math. **82** (1985), 257–262.

[6] K. Kurdyka, *Ensembles semi-algébriques symétriques par arcs*, Math. Ann. **282** (1988), 445–462.

[7] K.J. Nowak, *A proof of the valuation property and preparation theorem*, Ann. Polon. Math. **92** (1) (2007), 75–85.

[8] —, *Decomposition into special cubes and its application to quasi-subanalytic geometry* — to appear in Ann. Polon. Math. **96** (2009).

[9] —, *Quantifier elimination, valuation property and preparation theorem in quasianalytic geometry via transformation to normal crossings* — to appear in Ann. Polon. Math. **96** (2009).

[10] —, *On two problems concerning quasianalytic Denjoy–Carleman classes*, RAAG Preprint **241**, 2007, and IMUJ Preprint **19**, 2007.

[11] —, *On the Abhyankar–Jung theorem for henselian $k[x]$-algebras of formal power series*, IMUJ Preprint **2**, 2009.

[12] —, *Rectilinearization of quasi-subanalytic functions and its applications*, IMUJ Preprint **3**, 2009.

[13] —, *Hyperbolic polynomials and quasianalytic perturbation theory*, IMUJ Preprint **5**, 2009.

[14] —, *A note on quasianalytic functions with roots substituted* — in preparation.

[15] A. Parusiński, *Subanalytic functions*, Trans. Amer. Math. Soc. **344** (2) (1994), 583–595.

[16] —, *Lipschitz stratification of subanalytic sets*, Ann. Scient. Ecole Norm. Sup. **27** (1994), 661–696.

[17] A. Rambaud, *Quasi-analyticité, o-minimalité et élimination des quantificateurs*, PhD Thesis, Université Paris 7, Paris, 2005.

[18] J.-P. Rolin, P. Speissegger, A.J. Wilkie, *Quasianalytic Denjoy–Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (4) (2003), 751–777.

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