Hyperbolic polynomials and quasianalytic perturbation theory

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Abstract

This paper investigates hyperbolic polynomials with quasianalytic coefficients. Our main purpose is to prove a factorization theorem for such polynomials. As an application, we generalize the results of K. Kurdyka and L. Paunescu about analytic families of symmetric matrices to the quasianalytic settings.

Hyperbolic polynomials with analytic coefficients in one variable were studied by Rellich [20, 21]. This was linked with his investigation into the behaviour of eigenvalues of symmetric matrices under one-parameter analytic perturbation. This theory, initiated by Rellich, culminated in the work of Kato [7]. One-parameter families of hyperbolic polynomials were contemporarily studied in [2, 8], as well. Very recently, Kurdyka–Paunescu [11] developed multi-parameter analytic perturbation theory. Our purpose is to carry over this theory to the quasianalytic settings.

As in our previous papers (cf. [14, 15, 17]), we shall deal with a family $\mathcal{Q}(U)$ of smooth functions that satisfies certain six conditions introduced

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by Bierstone–Milman [4], which ensure resolution of singularities and transformation to normal crossings by blowing up. In the sequel, by a quasi-subanalytic function we shall mean a function definable in the real field $\mathbb{R}$ with restricted quasianalytic functions determined by the family $\mathcal{Q}(U)$.

In this paper, we are actually interested in the the family $\widetilde{\mathcal{Q}}(U)$ of smooth quasi-subanalytic functions rather than in the family $\mathcal{Q}(U)$ itself. For abbreviation, such smooth quasi-subanalytic functions shall be called definable quasianalytic, or even quasianalytic. Clearly, the new family $\widetilde{\mathcal{Q}}(U)$ satisfies the above-mentioned six conditions too.

Denote by $\mathcal{Q}_m = \mathcal{A}(\mathbb{R}; x)$ the ring of germs at $0 \in \mathbb{R}^m$ of smooth quasi-subanalytic functions, and put

$$\mathcal{A}^{\mathbb{C}}(\mathbb{R}; x) := \mathcal{A}(\mathbb{R}; x) \otimes_{\mathbb{R}} \mathbb{C};$$

here $m \in \mathbb{N}$ and $x = (x_1, \ldots, x_m)$. $\mathcal{A}^{\mathbb{C}}(\mathbb{R}; x)$ may be regarded, of course, as a henselian $\mathbb{C}[x]$-subalgebra of the formal power series algebra $\mathbb{C}[[x]]$, which is closed under reciprocal, power substitution and division by a coordinate.

For any $r_1, \ldots, r_m \in \mathbb{N}$, $r_1, \ldots, r_m \neq 0$, put

$$\mathcal{A}(\mathbb{R}; x_1^{1/r_1}, \ldots, x_m^{1/r_m}) := \{a(x_1^{1/r_1}, \ldots, x_m^{1/r_m}) : a(x) \in \mathcal{A}(\mathbb{R}; x)\}$$

and

$$\mathcal{A}^{\mathbb{C}}(\mathbb{R}; x_1^{1/r_1}, \ldots, x_m^{1/r_m}) := \{a(x_1^{1/r_1}, \ldots, x_m^{1/r_m}) : a(x) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R}; x)\};$$

when $r_1 = \ldots = r_m = r$, we shall denote the above algebras by $\mathcal{A}(\mathbb{R}; x^{1/r})$ and $\mathcal{A}^{\mathbb{C}}(\mathbb{R}; x^{1/r})$, respectively. A special case of the Abhyankar–Jung theorem from our paper [16], is the following quasianalytic one (for the classical versions for formal or convergent series, we refer the reader to e.g. [6, 22, 1, 12, 13, 19]):

Abhyankar–Jung Theorem. *Every quasiordinary polynomial*

$$h(x; t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R}; x)[t]$$

has all its roots in $\mathcal{A}^{\mathbb{C}}(\mathbb{R}; x^{1/r})$*, for some* $r \in \mathbb{N}$*; actually one can take* $r = n!$.

Corollary 1. *Consider a quasiordinary polynomial*

$$h(x; t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}^{\mathbb{C}}(\mathbb{R}; x)[t].$$

Then there exists an $r \in \mathbb{N}$ *such that for each closed orthant* Q_k *in* $\mathbb{R}^m$, $k = 1, \dots, 2^m$, *we have in the vicinity of* $0 \in \mathbb{R}^m$ *a factorization of the form*

$$h(x;t) = \prod_{i=1}^{n} (t - \varphi_{ik}(|x_1|^{1/r}, \dots, |x_m|^{1/r})) \quad \textit{for} \quad x \in Q_k,$$

where $\varphi_{ik} \in \mathcal{A}^{\mathbb{C}}(\mathbb{R}; x)$; *actually one can take* $r = n!$. ◇

Below stated is a real version of the Abhyankar–Jung theorem.

Corollary 2. *Let*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}(\mathbb{R}; x)[t]$$

be a quasiordinary polynomial. Then there exists an $r \in \mathbb{N}$ *such that for each closed orthant* Q_k *in* $\mathbb{R}^m$, $k = 1, \dots, 2^m$, *we have in the vicinity of* $0 \in \mathbb{R}^m$ *a factorization of the form*

$$f(x;t) = \prod_{i=1}^{p} (t - \varphi_{ik}(|x|^{1/r})) \prod_{j=1}^{q} (t^2 - \alpha_{jk}(|x|^{1/r})t + \beta_{jk}^2(|x|^{1/r})) \quad \textit{for} \quad x \in Q_k,$$

where $p + 2q = n$, $\varphi_{ik}, \alpha_{jk}, \beta_{jk} \in \mathcal{A}(\mathbb{R}; x)$ *and* $|x|^{1/r} = (|x_1|^{1/r}, \dots, |x_m|^{1/r})$; *actually one can take* $r = n!$. ◇

Before turning to hyperbolic polynomials, we still need to look more carefully to Corollary 1. For any closed subset $A \subset \mathbb{R}^m$, let $\mathcal{C}(A)$ and $\mathcal{D}(A)$ be the $\mathbb{R}$-algebras of those quasi-subanalytic functions on A which are continuous and smooth, respectively; put

$$\mathcal{C}(A, \mathbb{C}) := \mathcal{C}(A) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad \mathcal{D}(A, \mathbb{C}) := \mathcal{D}(A) \otimes_{\mathbb{R}} \mathbb{C}.$$

By symmetry, we may confine our considerations to the first closed orthant $Q = Q_1 = [0, \infty)^m$. The functions

$$\varphi_i(x^{1/r}) := \varphi_{i1}(x_1^{1/r}, \dots, x_m^{1/r})$$

have representatives which belong to $\mathcal{C}([0, \delta]^m, \mathbb{C})$ with $\delta > 0$ small enough; denote by $\widehat{\varphi}_i(x^{1/r})$ their Puiseux series. Let ϵ be a primitive r-th root of unity. It is easy to check that each algebraic conjugate

$$\widehat{\varphi}_i(\epsilon^{\alpha_1} x_1^{1/r}, \dots, \epsilon^{\alpha_m} x_m^{1/r}), \quad \alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{N}^m,$$

of any $\widehat{\varphi_i}(x^{1/r})$ is the Puiseux series $\widehat{\varphi_j}(x^{1/r})$ of some $\varphi_j(|x|^{1/r})$. In other words, the Puiseux series $\widehat{\varphi_i}(x^{1/r})$, $i = 1, \ldots, n$, are preserved under algebraic conjugacy.

We call a monic polynomial $f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}(\mathbb{R}; x)[t]$ hyperbolic if, for each value of the parameters x, all its roots are real. This is a shortened name for "a quasianalytic family of hyperbolic polynomials". Keeping the foregoing notation, it is clear that the Puiseux series $\widehat{\varphi_i}(x^{1/r})$, $i = 1, \ldots, n$, of the roots $\varphi_i(x^{1/r})$ of the hyperbolic quasiordinary polynomial $f(x;t)$ are real series. Since they are preserved under algebraic conjugacy, we get $\widehat{\varphi_i}(x^{1/r}) \in \mathbb{R}[[x]]$.

The above reasoning about Puiseux series may be repeated at each point from $[0,\delta]^m$. Therefore it follows from Glaeser's composite function theorem (see e.g. [5, 3]) that the functions $\psi_i(x) := \varphi_i(x^{1/r})$ are smooth:

$$\psi_i(x) = \varphi_i(x^{1/r}) \in \mathcal{D}([0,\delta]^m), \quad i = 1, \ldots, n.$$

Note that we applied, in fact, a very special case of Glaeser's theorem. Denote by $T_a\psi_i(x)$ the Taylor series at a point $a \in Q_1$ of the smooth function ψ_i, $i = 1, \ldots, n$.

Summing up, for each closed orthant Q_k in $\mathbb{R}^m$, $k = 1, \ldots, 2^m$, we have in the vicinity of $0 \in \mathbb{R}^m$ a factorization of the form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_{ik}(x)) \quad \text{for} \quad x \in Q_k \cap [-\delta,\delta]^m,$$

where $\psi_{ik} \in \mathcal{D}(Q_k \cap [-\delta,\delta]^m)$ with $\delta > 0$ small enough. But for every $k = 1, \ldots, 2^m$, the roots $\psi_{ik}(x)$ of the polynomial $f(x;v)$ determine common Taylor series $\widehat{\varphi_i}(x^{1/r}) \in \mathbb{R}[[x]]$, $i = 1, \ldots, n$. Consequently, those roots can be glued together to n smooth functions definable in a cube $[-\delta,\delta]^m$. Indeed, consider two adjacent orthants Q_k, Q_l with common face F, and next fix $i = 1, \ldots, n$ and put

$$F_j := \{a \in F : T_a\psi_{i,k}(x) = T_a\psi_{j,l}(x)\}, \quad j = 1, \ldots, n.$$

It is clear that $F_1, \ldots, F_n$ are closed, pairwise disjoint subsets of F such that $F = F_1 \cup \ldots \cup F_n$. Since F is a connected set, we get $F = F_{j(i)}$ for a unique $j(i) = 1, \ldots, n$. This means that the functions $\psi_{i,k}(x)$ and $\psi_{j(i),l}$ can be glued together, as asserted.

In this manner, we have thus proved the main result of this paper, which concerns the factorization of quasiordinary hyperbolic polynomials with quasianalytic coefficients in several variables.

Factorization Theorem for Hyperbolic Polynomials. *Every hyperbolic quasiordinary polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}(\mathbb{R};x)[t]$$

can be factorized in the following form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)) \quad \textit{for all} \quad x \in [-\delta, \delta]^m,$$

where $\psi_i(x) \in \mathcal{D}([-\delta, \delta]^m)$ *and* $\delta > 0$ *is small enough.* $\diamondsuit$

By virtue of Proposition 2 from our previous paper [16], every monic polynomial with quasianalytic coefficients factorizes into a product of quasiordinary polynomials after a suitable transformation of its coefficients by a finite sequence of blowings-up with smooth centers. Hence and via transformation of the discriminant of a given hyperbolic polynomial with quasianalytic coefficients to normal crossings by blowing up, we immediately obtain the following

Corollary 1. *Consider a hyperbolic polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x), \quad a_i(x) \in \widetilde{\mathcal{Q}}(\Omega),$$

with quasianalytic coefficients on an open subset $\Omega \subset \mathbb{R}^m$*. Then one can find a modification* $\sigma : W \longrightarrow \Omega$*, which is a locally finite composite of blowings-up with smooth centers, such that at each point* $y_0 \in W$ *the pull-back polynomial*

$$f^\sigma(y;t) = f(\sigma(y);t) = t^n + a_{n-1}(\sigma(y))t^{n-1} + \cdots + a_0(\sigma(y))$$

has a factorization of the form

$$f^\sigma(y;t) = \prod_{i=1}^{n} (t - \psi_i(y)),$$

where $\psi_i(y)$*,* $i = 0, \ldots, n-1$*, are the germs of some definable quasianalytic functions at* y_0*.* $\diamondsuit$

Corollary 2. *Given a hyperbolic polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x), \quad a_i(x) \in \widetilde{\mathcal{Q}}(\Omega),$$

as above, there exists a quasi-subanalytic subset $\Sigma \subset \Omega$ *of codimension at least* 2 *such that at each point* $x_0 \in \Omega \setminus \Sigma$ *we have a factorization of the form*

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)),$$

where $\psi_i(x)$, $i = 0, \ldots, n-1$, *are the germs of some definable quasianalytic functions at* x_0. $\diamondsuit$

Before deducing next results, we introduce some terminology. Denote by $\mathcal{M}(\mathbb{R};x)$ the field of quasi-meromorphic germs at $0 \in \mathbb{R}^m$, i.e. the quotient field of the domain $\mathcal{A}(x)$. Similarly, $\mathcal{M}_{x_0}(\mathbb{R};x)$ stands for the field of quasi-meromorphic germs at a point $x_0 \in \mathbb{R}^m$. We consider in the vector spaces $\mathbb{R}^n$ and $\mathcal{M}(\mathbb{R};x)^n$ over the fields $\mathbb{R}$ and $\mathcal{M}(\mathbb{R};x)$, respectively, the standard inner products given by the formula

$$v \bullet w = v_1 w_1 + \cdots + v_n w_n \quad \text{and} \quad f(x) \bullet g(x) = f_1(x) g_1(x) + \cdots + f_n(x) g_n(x).$$

The spectral theorem for symmetric matrices is valid for any real closed field. The assumption of real closedness is necessary to ensure that the characteristic polynomial of a given symmetric matrix, which always is hyperbolic, factorizes into linear factors. We are able to dispense with it, but instead we must use the foregoing Corollary 1, i.e. transform the discriminant of that characteristic polynomial, and next factorize it into linear factors. Therefore, repeating mutatis mutandis the proof of the spectral theorem from linear algebra, we obtain the following counterpart over the field of quasi-meromorphic functions.

Spectral Theorem with Quasianalytic Parameters. *Let* M *be a symmetric* $n \times n$ *matrix with quasianalytic entries from* $\mathcal{A}(\mathbb{R};x)$. *Then we can find a modification* $\sigma : W \longrightarrow \Omega$ *of a neighbourhood* Ω *of zero, which is a finite composite of blowings-up with smooth centers, such that for each point* $y_0 \in \sigma^{-1}(0) \subset W$ *the vector space* $\mathcal{M}_{y_0}(\mathbb{R};y)^n$ *over the field* $\mathcal{M}_{y_0}(\mathbb{R};y)$ *has an orthogonal basis*

$$w^1(y), \ldots, w^n(y) \in \mathcal{A}_{y_0}(\mathbb{R};y)^n$$

that consists of eigenvectors of the pull-back matrix M^σ. $\diamondsuit$

A matrix with entries from the $\mathbb{R}$-algebra $\mathcal{A}(\mathbb{R}; x)$ may be regarded as a quasianalytic family of real matrices parametrized by x. Our next objective is to achieve a simultaneous quasianalytic diagonalization of the pull-back matrix M^σ after performing a suitable modification σ which is a finite composite of blowings-up with smooth centers. Let

$$\lambda_1(y), \ldots, \lambda_n(y) \in \mathcal{A}_{y_0}(\mathbb{R}; y)$$

be the eigenvalues of M^σ, which may not be pairwise distinct. The above theorem yields a quasianalytic family

$$w^1(y), \ldots, w^n(y) \in \mathbb{R}^n$$

of orthogonal eigenvectors which form a basis of $\mathbb{R}^n$ generically near $y_0 \in W$; say over a set $W_0 = W \setminus \Sigma$ where $\Sigma \subset W$ is a closed definable subset of codimension at least one. Fix a vector $w(x) = w^j(x)$, $j = 1, \ldots, n$, and its eigenvalue $\lambda(x) = \lambda_l(x)$. Take any sequence $(y_k) \subset W_0$ which tends to y_0: $y_k \longrightarrow y_0$, and such that the limit

$$v(y_0) = \lim_{k\to\infty} v(y_k), \quad \text{where} \;\; v(y_k) = \frac{w(y_k)}{\|w(y_k)\|},$$

exists. We obviously have

$$(M^\sigma(y_0) - \lambda(y_0)) \cdot v(y_0) = \lim_{k\to\infty} (M^\sigma(y_k) - \lambda(y_k)) \cdot v(y_k) = 0.$$

Since the vectors $w^j(y)$ are pairwise orthogonal, we obtain in this fashion an orthonormal basis

$$v^1(y_0), \ldots, v^n(y_0) \in \mathbb{R}^n$$

that consists of eigenvectors of the matrix $M^\sigma(y_0)$.

We wish to construct a quasianalytic family

$$v^1(y), \ldots, v^n(y) \in \mathbb{R}^n$$

of orthonormal bases that consist of eigenvectors of the matrices $M^\sigma(y)$. Clearly, this will be possible once we know that all the components of each vector $w^j(y)$, $j = 1, \ldots, n$, are divisible in $\mathcal{A}_{y_0}(\mathbb{R}; y)$ by one of them. It is well known that the last condition can be ensured by means of a successive transformation to normal crossings by blowing up. Consequently, we achieved the theorem stated below, which generalizes to the quasianalytic settings the result of Kurdyka–Paunescu [11] from real analytic perturbation theory.

Theorem on Quasianalytic Diagonalization. *Consider a symmetric* $n \times n$ *matrix* M *with quasianalytic entries from* $\mathcal{A}(\mathbb{R};x)$. *Then there exists a modification* $\sigma : W \longrightarrow \Omega$ *of a neighbourhood* Ω *of zero, which is a finite composite of blowings-up with smooth centers, such that the pull-back matrix* M^{σ} *admits a simultaneous quasianalytic diagonalization near each point* $y_0 \in \sigma^{-1}(0) \subset W$. *This diagonalization can be performed through a quasianalytic choice of orthonormal bases that consist of eigenvectors of* M^{σ}. $\diamondsuit$

Remark 1. Both the spectral theorem with quasianalytic parameters and Proposition 1 remain valid, with the same proof, in the case of quasianalytic families of hermitian matrices.

Remark 2. All the above results can be, as shown by Kurdyka–Paunescu, carried over to the case of polynomials with purely imaginary roots, and thence to that of antisymmetric matrices. Indeed, a polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{A}(\mathbb{R};x)[t]$$

has purely imaginary roots iff the polynomial $i^{-n}f(x;it)$ is hyperbolic (cf. [11] for details).

Remark 3. It is well known that, in general, one cannot find bases of eigenvectors even in a continuous way. This is caused by that the angle between linearly independent eigenvectors, which correspond to distinct eigenvalues, may tend to zero when approaching a given point y_0.

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