Addendum to the paper "Decomposition into special cubes and its application to quasi-subanalytic geometry"

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In paper [7], we demonstrate how to achieve the model completeness and o-minimality of the real field with restricted quasianalytic functions (result due to Rolin–Speissegger–Wilkie [13]) by means of a technique of decomposition into special cubes; see [8, 9, 10, 11] for other applications of this method. Therein we asked, inter alia, whether, given a polynomially bounded o-minimal expansion $\mathcal{R}$ of the real field, the structure generated by global smooth $\mathcal{R}$-definable functions is model complete. We should note that this follows immediately from Wilkie's complement theorem [14] (see also [12, 6]). In this Addendum, we also wish to indicate that Gabrielov's proof [5] of the complement theorem can be adopted to the real field with restricted smooth $\mathcal{R}$-definable functions.

Gabrielov's approach relies on certain three preliminary lemmas. Below we state their quasianalytic versions, whose proofs can be repeated mutatis mutandis. Next, we shall outline our proof of the complement theorem based on those lemmas. Denote by Q_n the algebra of those $\mathcal{R}$-definable functions that are smooth in the vicinity of the closed cube $[0,1]^n$. The algebras Q_n give rise to the notions of Q-analytic, Q-semianalytic and Q-subanalytic subsets of the cubes $[0,1]^n$, $n \in \mathbb{N}$.

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Lemma 1. *Consider a Q-semianalytic subset E of $[0,1]^n$ of the form*

$$E := \{x \in [0,1]^n : f_1(x) = \ldots = f_k(x) = 0,\ g_1(x) > 0, \ldots, g_l(x) > 0\}$$

with $f_i, g_j \in Q_n$. Then the closure $\overline{E}$ and frontier ∂E are Q-semianalytic too. Moreover, $\overline{E}$ and ∂E can be described by functions which are polynomials in x, in the functions f_i, g_j, and in their (finitely many) partial derivatives.

Consequently, if F is a Q-subanalytic subset of $[0,1]^m$, then so are its closure $\overline{F}$ and frontier ∂F.

Remark. As an easy generalization, one can formulate the parametric version of the above lemma, in which the $\mathcal{R}$-definable functions involved in the description depend smoothly on parameters.

By a Q-leaf we mean a set of the form

$$L := \{x \in [0,1]^n : f_1(x) = \ldots = f_k(x) = 0,\ g_1(x) > 0, \ldots, g_l(x) > 0\},$$

where $f_i, g_j \in Q_n$ and

$$\frac{\partial(f_1, \ldots, f_k)}{\partial(x_{i_1}, \ldots, x_{i_k})}(x) \neq 0 \quad \text{for some } \ 1 \leq i_1 < \ldots < i_k \leq n \ \text{ and for all } \ x \in L.$$

Lemma 2. *Every Q-semianalytic subset E of $[0,1]^n$ is a finite union of Q-leaves.*

The image of a Q-leaf $L \subset [0,1]^n$ under a projection $\pi : \mathbb{R}^n \longrightarrow \mathbb{R}^m$, $n \geq m$, shall be called an immersed Q-leaf, if the restriction of π to L is an immersion. By combining Lemma 2 with the technique of fiber cutting (see e.g. [4, 5, 2, 3, 1, 7]), one can obtain

Lemma 3. *Every Q-subanalytic subset F of $[0,1]^m$ is a finite union of immersed Q-leaves.*

By a Q-cell we mean a cell given by smooth functions with Q-subanalytic graphs. Now we can readily outline our proof of the following main result wherefrom the complement theorem follows immediately.

Main Theorem. *Consider Q-subanalytic subsets $F_1, \ldots, F_r$ of $[0,1]^m$. Then there exists a Q-cell decomposition $\mathcal{C}$ of $[0,1]^m$ which is compatible with the sets F_i, $i = 1, \ldots, r$.*

We proceed by a double induction with respect to m and

$$d := \max\{\dim F_1, \ldots, \dim F_r\}.$$

The case $m = 0$ is trivial, and so take $m > 0$. Again, the case $d = 0$ is evident, and we may suppose $d > 0$. By virtue of Lemma 3, we can assume that F_i are immersed Q-leaves, i.e.

$$F_i = p(E_i), \quad p : \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad p(x_1, \ldots, x_m) = (x_1, \ldots, x_n),$$

for all $i = 1, \ldots, r$. Denote by $q : \mathbb{R}^n \longrightarrow \mathbb{R}^{m-1}$ and $\pi : \mathbb{R}^m \longrightarrow \mathbb{R}^{m-1}$ the canonical projections onto the first $m-1$ coordinates; obviously, $\pi \circ p = q$. Put $d_i := \dim F_i = \dim E_i$, $d_i \leq d$, $i = 1, \ldots, r$, and

$$E_i' := \{x \in E_i : \operatorname{rank} q|E_i = d_i\}, E_i'' := \{x \in E_i : \operatorname{rank} q|E_i = d_i - 1\}.$$

Then $E_i = E_i' \cup E_i''$. Clearly, the restriction

$$\operatorname{res} q : E_i' \setminus q^{-1}(q(\partial E_i')) \longrightarrow q(E_i') \setminus q(\partial E_i')$$

is proper. Now observe that the set S of self-intersections of the image of $\operatorname{res} q$ is a Q-subanalytic subset of $q(E_i')$ as $S \times \{0\} = \overline{V} \cap (q(E_i') \times \{0\})$, where

$$V := \{ (u_1, \ldots, u_{m-1}, \epsilon) \in (q(E_i') \setminus q(\partial E_i')) \times [0, 1] :$$

$$\exists\, v = (v_1, \ldots, v_{n-m+1}), w = (w_1, \ldots, w_{n-m+1}) \in [0, 1]^{n-m+1} :$$

$$0 < |v - w| < \epsilon, \ (u, v), (u, w) \in E_i' \setminus q^{-1}(q(\partial E_i')) \}.$$

Then $T := S \cup q(\partial E_i')$ is a Q-subanalytic set of dimension $< d$, and the restriction

$$\operatorname{res} q : E_i' \setminus q^{-1}(T) \longrightarrow q(E_i') \setminus T$$

is a topological covering, whence so is the restriction

$$\operatorname{res} \pi : p(E_i') \setminus \pi^{-1}(T) \longrightarrow q(E_i') \setminus T.$$

Therefore, the set $p(E_i')$ is over any connected subset (in the sequel we shall consider a Q-cell) of $q(E_i') \setminus T$ is a finite union of the Q-subanalytic graphs of smooth functions.

Further, notice that, for each $u \in q(E_i'')$, the fiber $(E_i'')_u := q^{-1}(u) \cap E_i''$ is a smooth Q-semianalytic arc, and the restriction of p to $(E_i'')_u$ is an immersion of this fiber into $\{u\} \times \mathbb{R}_{x_m}$ whence the fiber $(F_i)_u$ is a finite union of open intervals. By virtue of the parametric version of Lemma 1, the sets

$$Z_i := \bigcup_{u \in q(E_i'')} (\{u\} \times \partial p(E_i'')_u) \subset [0,1]^m$$

are Q-subanalytic of dimension $< d$. By induction hypothesis, there exists a Q-cell decomposition $\{C_p : p = 1, \dots, s\}$ of $[0,1]^m$ compatible with the sets Z_i, $i = 1, \dots, r$. Clearly, for each cell C_p, the sets

$$W_{i,p} := \{u \in [0,1]^{m-1} : (C_p)_u \subset (E_i)_u\} \subset [0,1]^{m-1}$$

are Q-subanalytic. Again by induction hypothesis, one can find a Q-cell decomposition $\mathcal{C}$ compatible with the sets

$$q(E_i'), \quad q(\partial E_i'), \quad W_{i,p}, \quad p(E_i') \cap \pi^{-1} q(\partial E_i') \ \text{ and } \ Z_i;$$

whereas the first three are subsets of $[0,1]^{m-1}$, the last two are subsets of $[0,1]^m$ of dimension $< d$. Indeed, one must construct a Q-cell decomposition compatible with the subsets of $[0,1]^m$ under study, which are of dimension $< d$, and next refine the induced Q-cell decomposition of $[0,1]^{m-1}$ compatibly with the remaining subsets of $[0,1]^{m-1}$.

What still remains to be done is to modify the Q-cell decomposition $\mathcal{C}$, achieved in this fashion, as follows. As we have already seen, over each Q-cell C from the induced Q-cell decomposition of $[0,1]^{m-1}$ such that $C \subset q(E_i')$ but $C \cap q(\partial E_i') = \emptyset$, $i = 1, \dots, r$, the set $p(E_i')$ is a finite union of the Q-subanalytic graphs of smooth functions. Again, one must modify $\mathcal{C}$ by partitioning its Q-cells by means of those Q-subanalytic graphs; this is, of course, linked with a successive refinement of the cube $[0,1]^{m-1}$, which is possible due to induction hypothesis.

It is not difficult to check that eventually we attain a Q-cell decomposition $\mathcal{C}$ of $[0,1]^m$ compatible with the sets $p(E_i')$ and $p(E_i'')$, and a fortiori with the sets $F_i := p(E_i) = p(E_i') \cup p(E_i'')$. We leave the details for the reader.

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