Picard-Vessiot extensions for real fields

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1 Introduction

Picard-Vessiot theory can be described as Galois theory of linear differential equations. A derivation of a field K is defined as a map $d : K \to K$ satisfying $d(a+b) = d(a) + d(b)$ and $d(ab) = d(a)b + ad(b)$, for all a, b in K. A differential field is a field endowed with a derivation. We shall use the usual notation $a', a'', \dots, a^{(n)}$ for the successive derivations of the element a. We denote by C_K the field of constants of a differential field K. We shall consider homogeneous linear differential equations over a differential field K of the form

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1 Y' + a_0 Y = 0,$$

where $a_i \in K$ for $i \in \{0, 1, \dots, n-1\}$. If L is a differential field extension of K, i.e. a differential field containing K with derivation extending derivation in K, the set of solutions of $\mathcal{L}(Y) = 0$ in L is a C_L-vector space of dimension $\leq n$. A fundamental system of solutions of $\mathcal{L}(Y) = 0$ is a set of n solutions of the equation in some differential extension L of K, linearly independent over C_L.

A Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over K is a differential field extension of K differentially generated by a fundamental system of solutions of $\mathcal{L}(Y) = 0$, i.e. generated by the elements in the fundamental system and their derivatives, and not adding constants. Picard-Vessiot theory is due to E. Picard and E. Vessiot and in rigorous form to E. Kolchin, who built on the work of J.F. Ritt in differential algebra. It was made more accessible by the book of I. Kaplansky [5]. We refer the reader also to [4], [6] and [8] for the results of Picard-Vessiot theory used in this paper.

Picard-Vessiot theory has been built under the hypothesis that the field of constants C_K of the differential field K is algebraically closed. In this case, one obtains existence and uniqueness, up to K-differential isomorphisms, of the Picard-Vessiot extension of the differential equation and that the differential Galois group of the differential equation, defined as the group of K-differential automorphisms of its Picard-Vessiot extension, is a linear algebraic group of rank n over C_K. It is worth considering whether the condition C_K algebraically closed can be weakened. In particular, the case of real fields is interesting due to the application of Picard-Vessiot theory to the integrability of hamiltonian systems (see [7] or [4]).

In this paper we consider homogeneous linear differential equations $\mathcal{L}(Y) =$

0 defined over a real field K with real closed field of constants. We prove that, in the generic case, there exists a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over K which moreover is a real field. For the results of the theory of real fields used in this paper, we refer the reader to [3].

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2 Auxiliary results

We consider a homogeneous linear differential equation of order n over a real differential field K, with real closed field of constants C_K

$$\mathcal{L}(Y) := Y^{(n)} + a_{n-1}Y^{(n-1)} + \ldots + a_1Y' + a_0Y = 0, \tag{1}$$

where $a_i \in K$ for $i \in \{0, 1, \ldots, n-1\}$.

We recall the definition of Picard-Vessiot extension.

Definition 2.1. *A differential field extension $K \subset L$ is a Picard - Vessiot extension for $\mathcal{L}$ if*

1. *L is differentially generated over K by the set of solutions of $\mathcal{L}(Y) = 0$ in L,*
2. *$\mathcal{L}(Y) = 0$ has in L exactly n solutions lineary independent over C_K,*
3. *every constant of L lies in K,i.e. $C_K = C_L$.*

In the case in which C_K is an algebraically closed field, a Picard-Vessiot extension for $\mathcal{L}$ over K is obtained by constructing the full universal solution algebra R (see below) and considering a maximal differential ideal M of R, i.e. a maximal element in the set of proper differential ideals of R, which is proved to be prime. Then one can prove that the field of fractions of the integer domain R/M fulfills the conditions to be a Picard-Vessiot extension for $\mathcal{L}$ over K.

In this paper we consider the case in which K is a real field and C_K is real closed. We are interested in real differential ideals of the full universal solution algebra R and a crucial point for our construction will be to prove that a maximal real differential ideal of R, i.e. a maximal element in the set of proper real differential ideals of R, is prime. To this end we shall use a theorem of Ritt of which we give the statement for the convenience of the reader (see [9], chapter I.16, or [2] 1.3).

Theorem 2.1. *(Ritt) Let K be a differential field of characteristic zero. Let R be a differential K-algebra finitely differentially generated, I a proper radical differential ideal of R. Then there exists finitely many prime differential ideals $P_1, \ldots, P_s$ of R, such that*

$$I = P_1 \cap \ldots \cap P_s.$$

Moreover, when $P_i \nsubseteq P_j$ for all $i \neq j$, $\quad i, j \in \{1, \ldots, s\}$, then $\{P_1, \ldots, P_s\}$ is unique.

Proposition 2.1. *Let K be a differential field of characteristic zero, R a noetherian differential K-algebra finitely differentially generated. Let I be a maximal real differential ideal of R. Then I is prime.*

Proof. I is radical, because it is real (see [3], lemma 4.1.5). Then, by theorem 2.1, I is an intersection of finite number of prime differential ideals, i.e.

$$I = P_1 \cap \ldots \cap P_s. \tag{2}$$

Moreover, we can assume that $P_i \nsubseteq P_j$ for all $i \neq j$. Indeed, if some $P_i \subset P_j$ for $i \neq j$, we can omit P_j and reduce the decomposition. Therefore (2) is a primary decomposition of the ideal I with

$$rad(P_i) = P_i \neq rad(P_j) = P_j \quad \forall\, i \neq j.$$

Hence, by unicity in theorem 2.1, it is a reduced primary decomposition (see [1], chap.4). So the $P_i's$ are exactly the minimal prime ideals containing I. Now, minimal prime ideals containing the real ideal I are as well real (see [3], lemma 4.1.5). But I is a maximal real differential ideal, so $s = 1$ and $I = P_1$. Therefore I is prime.

□

3 Main result

We consider the homogeneous linear differential equation (1) defined over the real field K. We construct a differential K-algebra containing a full set of solutions of the equation and prove that it can be ordered.
Let us consider the ring $K[Y_{ij}]$, where $0 \leq i \leq n-1$ and $1 \leq j \leq n$. It is a polynomial ring in n^2 indeterminates. We extend the derivation of K to $K[Y_{ij}]$ by defining

$$Y_{ij}' = Y_{i+1,j} \quad \text{for} \quad 0 \leq i \leq n-2,$$

$$Y_{n-1,j}' = -a_{n-1}Y_{n-1,j} - \ldots - a_1 Y_{1j} - a_0 Y_{0j}.$$

We denote $W = \det(Y_{ij})$.We have

$$W = \det \begin{pmatrix} Y_{01} & \ldots & Y_{0n} \\ Y_{11} & \ldots & Y_{1n} \\ \ldots & \ldots & \ldots \\ Y_{n-1,1} & \ldots & Y_{n-1,n} \end{pmatrix} = \det \begin{pmatrix} Y_{01} & \ldots & Y_{0n} \\ Y_{01}' & \ldots & Y_{0n}' \\ \ldots & \ldots & \ldots \\ Y_{01}^{(n-1)} & \ldots & Y_{0n}^{n-1} \end{pmatrix}.$$

So W is the wronskian (determinant) of $Y_{01}, \ldots, Y_{0n}$.
Let $\mathcal{W} = \{W^n\}_{n>0}$ be the multiplicative system of the powers of W. Let $R := K[Y_{ij}]_{\mathcal{W}}$ be the localization of $K[Y_{ij}]$ in $\mathcal{W}$. The derivation of $K[Y_{ij}]$ extends to R in an unique way (see e.g. [4], remark 2.1).
The K-algebra R is called the *full universal solution algebra*. By construction, it contains n solutions of equation (1) linearly independent over constants. Forgetting the differential structure, R is a subring of the field of rational functions in n^2 variables over K, so it can be ordered (see [3], example 1.1.2) and it is an integral domain. Hence 0 is a real differential ideal of R and the set of real differential ideals of R is not empty.
Let P be a maximal real differential ideal of R. Then, by proposition 2.1, P is prime. So the quotient ring R/P is an integral domain. The field of fractions $L = Fr(R/P)$ is a real field, because P is a real differential ideal (see [3], lemma 4.1.6). By our construction L is a real field differentially generated over K by a fundamental system of solutions of equation (1). Concerning the field of constants C_L of the real differential field L, we obtain the following result.

Lemma 3.1. *Let K and L be as above and assume that the constant field C_K of K is real closed. If $c \in C_L \setminus C_K$, then c is transcendent over K.*

Proof. If the constant c were algebraic over K, it would be algebraic over C_K (see e.g. [4] proof of Prop. 3.5). In this case, it would be a real algebraic element. But C_K is a real closed field, so $c \in C_K$.

Remark 3.1. *If the construction above leads to an algebraic extension $K \subset L$, then L is a Picard-Vessiot extension of K.*

Let $\bar{C}_K$ denote the algebraic closure of C_K. Let $\hat{K} := K \otimes_{C_K} \bar{C}_K$. We have $\bar{C}_K = C_K(i)$ and $\hat{K} = K(i)$ with $i^2 = -1$.
We can extend the derivation from K to its complexification $K(i)$ by defining

$$(a + bi)' = a' + b'i \qquad \forall a, b \in K.$$

$C_K(i)$ is the field of constants of $K(i)$ and it is algebraically closed. So there exists a Picard-Vessiot extension for equation (1) over $K(i)$.
Let $R = K[Y_{ij}]_{\mathcal{W}}$ be the full universal solution algebra of equation (1) over K constructed above. Let $\hat{R} = R \otimes_{C_K} \bar{C}_K$. Then $\hat{R}$ is isomorphic to the full universal solution algebra of equation (1) over $\hat{K}$.
Indeed we have a monomorphism

$$m : \hat{K} \to R \otimes_{C_K} \bar{C}_K$$

induced by the inclusion $K \to K[Y_{ij}]_{\mathcal{W}}$. Let Z_{ij} be n^2 independent variables, $0 \leq i \leq n-1, \quad 1 \leq j \leq n$ and extend derivation of $\hat{K}$ to $\hat{K}[Z_{ij}]$ by

$$Z'_{ij} = Z_{i+1,j} \quad \text{for} \quad 0 \leq 1 \leq n-2,$$

$$Z'_{n-1,j} = -a_{n-1} Z_{n-1,j} - \ldots - a_1 Z_{1j} - a_0 Z_{0j}.$$

Then m can be extended to a differential morphism $\hat{K}[Z_{ij}] \to \hat{R}$ and the image by m of the wronskian determinant $\det(Z_{ij})$ of the Z_{0j} is an invertible element in $\hat{R}$. So by [1], proposition 3.2, we obtain a differential morphism

$$\hat{K}[Z_{ij}]_\Omega \to \hat{R},$$

where Ω is the multiplicative system of the powers of the wronskian determinant $\det(Z_{ij})$.
The inverse of this morphism is induced by the natural bilinear morphism of C_K-algebras

$$R \times \bar{C}_K \to \hat{K}[Z_{ij}]_\Omega.$$

For a maximal real differential ideal P of R, the extended ideal P^e is clearly a proper differential ideal of $\hat{R}$. Let M be a maximal differential ideal of $\hat{R}$ containing P^e. We have then an epimorphism $R/P \to R/P^{ec}$, where c denotes contraction of ideals, inducing an epimorphism

$$R/P \otimes_{C_K} \bar{C}_K \to R/P^{ec} \otimes_{C_K} \bar{C}_K,$$

and also an epimorphism $\hat{R}/P^e \to \hat{R}/M$. Let us prove now

$$R/P^{ec} \otimes_{C_K} \bar{C}_K \simeq \hat{R}/P^e.$$

Indeed, the kernel of the epimorphism

$$\hat{R} = R \otimes_{C_K} \bar{C}_K \to R/P^{ec} \otimes_{C_K} \bar{C}_K$$

is $P^{ece} = P^e$. We have then the following inequalities

$$trdeg(L|K) = trdeg(R/P|K) \geq trdeg(\hat{R}/P^e|\hat{K}) \geq trdeg(\hat{R}/M|\hat{K}). \quad (3)$$

Moreover, by construction, $trdeg(L|K) \leq n^2$. We obtain

Theorem 3.1. *Let K be a real differential field with real closed field of constants C_K. Let $\mathcal{L}(Y) = 0$ be a homogeneous linear differential equation of order n defined over K. Let $\hat{K} = K \otimes_{C_K} \bar{C}_K$ and assume that the differential Galois group of $\mathcal{L}$ over $\hat{K}$ is $Gl_n(\bar{C}_K)$. Then there exists a Picard-Vessiot extension L for $\mathcal{L}(Y) = 0$ over K which moreover is a real field.*

Proof. In the above construction the fraction field of $\hat{R}/M$ is a Picard-Vessiot extension for $\mathcal{L}$ over $\hat{K}$. So $trdeg(\hat{R}/M|\hat{K}) = n^2$ (see e.g. [4], corollary 4.1). Hence inequalities (3) are all equalities. We want to prove that the real field L constructed above is a Picard-Vessiot extension for $\mathcal{L}(Y) = 0$ over K. To this end, it remains to prove $C_L = C_K$. Let us assume the contrary. By lemma 3.1, an element $a \in C_L \setminus C_K$ is transcendent over K. Let $S := R/P$ be the quotient of the full universal solution algebra by a maximal real differential ideal constructed above. Let I_1 (resp. I_2) be the ideal of S of denominators (resp. of numerators) of a, i.e. $I_1 := \{b \in S | ba \in S\}, I_2 := \{b \in S | ba^{-1} \in S\}$. Then both ideals are differential ideals and at least one of them contains elements transcendent over K. Let us denote it by I. We assume first that I_1 and I_2 are both proper, so I as well. The extended ideal P^e will be contained strictly in the extension of the differential ideal $P+I$, which will be contained in a maximal differential ideal M of $\hat{R}$. We would have then $trdeg(\hat{R}/P^e|\hat{K}) > trdeg(\hat{R}/M|\hat{K})$ which gives a contradiction. We assume now that only one of I_1 and I_2 is proper, let say I_2. Then a is a non invertible element in S. The ideal (a) of S is a proper differential ideal containing at least one transcendent element so we arrive to a contradiction as before. If the ideals I_1 and I_2 are both equal to S for all elements $a \in C_L \setminus C_K$, we have $C_L \subset S$. In this case, by applying lemma 3.2 below, for each $a \in C_L \setminus C_K$, there is a constant $c \in C_K$ such that $a - c$ is not invertible in S. Hence, the ideal $(a-c)$ of S is a proper differential ideal containing at least one transcendent element so we arrive again to a contradiction. We have then obtained that the real field L is a Picard-Vessiot extension for $\mathcal{L}$ over K.

Lemma 3.2. *Let K be a real differential field with real closed field of constants C_K. Let A be a finitely generated K-algebra without zero divisors and let a be an element of A. Then either a is algebraic over K or there is a constant $c \in C_K$ such that $a - c$ is not invertible in A.*

Proof. Let $\overline{K}$ be the algebraic closure of K, $\overline{A} := A \otimes_K \overline{K}$. Let us observe that if the element $a \otimes 1 - c \otimes 1 = (a-c) \otimes 1$ is not a unit in $\overline{A}$, then the element $a - c$ will be a nonunit in A. Let $V_{\overline{K}}$ be the affine algebraic variety with coordinate ring $\overline{A}$. Then $a \otimes 1$ defines a $\overline{K}$-valued function $f : V_{\overline{K}} \to \overline{K}$. By Chevalley's theorem, $f(V_{\overline{K}})$ is a constructible subset of $\overline{K}$, hence it is either finite or cofinite in $\overline{K}$. If $f(V_{\overline{K}})$ is a finite set, then it is a point because $V_{\overline{K}}$ is irreducible. In this case, f is a constant function then $a \in \overline{K}$. In the second case, as the real closed field C_K is infinite, there exists $c \in C_K$ such that $f(\omega) = c$, for some $\omega \in V_{\overline{K}}$. Let us observe that $f - (c \otimes 1)$ vanishes at ω and therefore $a \otimes 1 - c \otimes 1$ is not invertible in $\overline{A}$.

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