MINIMAL MULTI-CONVEX PROJECTIONS ONTO SUBSPACES OF POLYNOMIALS

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Abstract. Let $X = (C^N[0,1], \|\cdot\|)$, where $N \geq 3$ and let V be a linear subspace of Π_N, where Π_N denotes the space of algebraic polynomials of degree less than or equal to N.

Denote by $\mathcal{P}_S = \mathcal{P}_S(X,V) = \{P : X \to V \mid P\text{–linear and bounded } P|_V = \mathrm{id}_V,\ PS \subset S\}$, where S denotes a cone of multi-convex functions.

In [25, 26] the multi-convex projections were defined and it was shown the explicite formula for projection with minimal norm in $\mathcal{P}_S$ for $V = \Pi_N$.

In this paper we present a generalization of these results in the case of V being certain, proper subspaces of Π_N.

1. Introduction

Let X be a real Banach space and let $V \subset X$ be a finite-dimensional subspace. A linear and continuous mapping $P : X \to V$ is called *a projection* if $P|_V = \mathrm{id}_V$. Denote by $\mathcal{P}(X,V)$ the set of all projections from X onto V. Moreover let S denote a cone, that is a convex set closed under nonnegative multiplication. We say that projection P preserve S-shape if $PS \subset S$ (notation $P \in \mathcal{P}_S$). Particulary, if S denotes a cone of functions which certain derivatives are nonnegative, then projection preserving S-shape will be called *multi-convex.*

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The main problem consider in this paper is to find a *minimal multi-convex projection*, that is a projection in $\mathcal{P}_S$ of minimal norm. Note the problem of finding projection of minimal norm (so called minimal projection) or minimal shape preserving projection has been widely studied by many authors (see e.g. [1]–[38]). Recent papers (ex. [26, 25]) gave answer to this problem in case, where subspace V is entire space of polynomials of degree less than or equal to N (see [26]) under assumption that the norm of minimal projection in $\mathcal{P}_s$ is not greater than 2. In this paper we replace the space of polynomials by spaces of incomplete ones. Also the above mentioned assumption is not needed.

The article is divided into four sections. The first one is an introduction. The second one consists of notation and important results that are used in the paper. In third section we describe a basic projection and its properties. The main results concerning recurrence formula are enclosed in fourth section.

2. Notation

As it was mentioned before, the main goal of this paper is to define a minimal multi-convex projection from $C^N[0,1]$ (where $N \geq 3$) onto certain subspaces of space of polynomials of degree less than or equal to N. To describe these subspaces and norms in $C^N[0,1]$ we use a special sequence.

For $n \leq N$ ($N \geq 3$), let

$$\{k_i\}_{i=0}^n \subset \{0, 1, \dots, N\} \tag{1}$$

be such that

(K.1) $\forall i \in \{0, \dots n\} \ : \ k_i < k_{i+1}$,
(K.2) $k_0 = 0$,
(K.3) $k_{n-1} = N - 1$ and $k_n = N$.

Now, set a norm on $C^N[0,1]$ as

$$\|f\| = \max\{\|f^{(k_i)}\|_\infty : i \in \{0, \dots, n\}\}. \tag{2}$$

By (K.2) such expression defines a norm and the couple $(C^N[0,1], \|\cdot\|)$ is a Banach space, (for brevity we write X as a $(C^N[0,1], \|\cdot\|)$).

Also using our sequence $\{k_i\}$ we define

$$v_i(t) = \frac{t^{k_i}}{k_i!}, \ t \in [0,1], \ i = 0, \dots, n, \tag{3}$$

and

$$V = \operatorname{span}\{v_i, i = 0, \dots n\} \subset X. \tag{4}$$

Furthermore, we denote

$$\mathcal{P}(X, V) = \{P : X \to V \ : \ P - \text{linear and bounded } P_{|V} = \mathrm{id}_V\}.$$

Now define

$$S = \{f \in X \ : \ \forall t \in [0;1], i \in \{0,1,\ldots,n\} \ f^{(k_i)}(t) \geq 0\}. \tag{5}$$

Then, $\mathcal{P}_S$ denotes set of all projections preserving S, that is

$$\mathcal{P}_S = \{P \in \mathcal{P}(X,V) \ : \ PS \subset S\}.$$

Now we present some result which will be use later.

First two are related to the geometry of considered spaces.

Lemma 2.1 (Lemma 4.4 [26]). *Let $g \in X^*$ be given by*

$$g = \sum_{i=0}^{N-1} \delta_0^{(i)} + \delta_1^{(N-1)},$$

where $\delta_t^{(i)}(f) = f^{(i)}(t)$. Set

$$W_1 = \{F \in X^{**} \mid F(g) = N+1 \text{ and } \|F\| = 1\}.$$

Then $W_1 \neq \emptyset$.

Lemma 2.2 (Theorem 4.4 [26]). *Let $W = \{F \in X^{**} : F(u_i) = 1, \ i = 0,\ldots,n-1, \ \|F\| = 1\}$. Assume μ is a probabilistic Borel measure such that*

$$u(f) = \int_0^1 f^{(N)}(t)d\mu(t).$$

Then for any $F \in W$ and for any Borel measure μ,

$$F(u) \geq 0.$$

Next two lemmas concern the form of any projection in $\mathcal{P}_S$.

Lemma 2.3 (Lemma 5.1 [26]). *Let $Q \in \mathcal{P}_S$. Then there exists $u \in X^*$ such that*

$$Qf = \sum_{i=0}^{n-1} u_i(f)v_i + u(f)v_n, \tag{6}$$

where

$$u(f) = \int_0^1 f^{(n)}(t)d\mu(t) \tag{7}$$

and μ is a probabilistic Borel measure.

Lemma 2.4 (Corollary 5.1 [26]). *Let $Q_l \in \mathcal{P}_{S_l}$. Then there exist $u_i^l, u \in X^*$, for $i = 0,\ldots,l$ such that*

$$Q_l f = \sum_{i=0}^{l-1} u_i^l(f)v_i^{[l]} + \sum_{i=l}^{n+l-1} u_{i-l}(f^{lk_1})v_i^{[l]} + u(f)v_{n+l}^{[l]},$$

where u_i and $v_i^{[l]}$ are defined in (8), (12). *(see page 4 and 8)*

3. Minimal multi-convex projection – basic case

We start with

Theorem 3.1 (Explicite formula for P)**.** *Let $X = (C^N[0,1], \|\cdot\|)$ (for $N \leq 3$) and let $\{k_i\}_{i=0}^n$ (for $n \geq N$) satisfy (K.1)-(K.3). For $f \in X$ and $t \in [0,1]$ define*

$$Pf(t) = \sum_{i=0}^{n} u_i(f) v_i(t),$$

where

$$\begin{aligned} u_i(f) &= f^{(k_i)}(0), \qquad i = 0, \ldots, n-1, \\ u_n(f) &= \left(f^{(k_{n-1})}(1) - f^{(k_{n-1})}(0)\right) \\ v_i(t) &= \frac{t^{k_i}}{k_i!}, \ t \in [0,1], \ i = 0, \ldots, n. \end{aligned} \tag{8}$$

Then P is multi-convex projection from X onto V, where V is given by (4).

To prove Theorem 3.1 we need

Observation 3.2. *The projection defined above can be written as*

$$\begin{aligned} Pf(t) &= f(0) + f^{(k_1)}(0)\frac{t^{k_1}}{k_1!} + f^{(k_2)}(0)\frac{t^{k_2}}{k_2!} + \ldots + f^{(k_{n-2})}(0)\frac{t^{k_{n-2}}}{k_{n-2}!} + \\ &\quad + f^{(N-1)}(0)\frac{t^{N-1}}{(N-1)!} + \left(f^{(N-1)}(1) - f^{(N-1)}(0)\right)\frac{t^N}{N!} \end{aligned}$$

Note that due to properties of differentiation, P is linear and bounded. Next observation and two corollaries concern the subspace V.

Observation 3.3. *For $i = 0, \ldots, n$ and $j = 0, \ldots, n-1$, k_i-th derivative of v_j is expressed by formula*

$$v_j^{(k_i)}(t) = \begin{cases} \frac{t^{k_j - k_i}}{(k_j - k_i)!}, & i < j \\ 1, & i = j \\ 0, & i > j. \end{cases}$$

Corollary 3.4. *For $i = 0, \ldots, n$ and $j = 0, \ldots, n$*

$$\begin{aligned} v_j^{(k_i)}(0) &= \delta_{ij} \\ v_j^{(k_{n-1})}(1) &= 1 \text{ if and only if } j = n-1 \text{ or } j = n. \end{aligned}$$

Corollary 3.5. *For $i = 0, \dots, n-1$ and $j = 0, \dots, n-1$*

$$\begin{aligned} u_i(v_j) &= \delta_{ij} \\ u_i(v_n) &= 0 \\ u_n(v_j) &= 0 - 0 = 0, \\ u_n(v_n) &= 1 - 0 = 1, \end{aligned}$$

where u_i, v_i are as in Theorem 3.1.

Proof of Theorem 3.1. We divide this proof into two parts.

(1) By Corollary 3.5, we obtain

$$Pv_j(t) = \sum_{i=0}^{n} \delta_{ij} v_i(t) = v_j(t).$$

Since P is linear and bounded, $P \in \mathcal{P}_S$.

(2) Now we show that P is multi-convex.

It is obvious, that for $t \in [0,1]$, $v_i(t) \geq 0$. Moreover,

$$(v_{i-1} - v_i)(t) \geq 0.$$

Let $f \in S$. Thus $u_i(f) \geq 0$, $f^{(N-1)}(0) \geq 0$ and $f^{(N-1)}(1) \geq 0$. Hence

$$\begin{aligned} Pf(t) &= \sum_{i=0}^{n} u_i(f) v_i(t) \\ &= \sum_{i=0}^{n-2} u_i(f) v_i(t) + f^{(N-1)}(0) v_{n-1}(t) \\ &\quad + \left(f^{(N-1)}(1) - f^{(N-1)}(0)\right) v_n(t) \\ &= \sum_{i=0}^{n-2} u_i(f) v_i(t) + f^{(N-1)}(0) \left(v_{n-1}(t) - v_n(t)\right) \\ &\quad + f^{(N-1)}(1) v_n(t) \geq 0 \end{aligned}$$

By Observation 3.3 for $j < n-1$, we get:

$$\begin{aligned} (Pf)^{(k_j)}(t) &= \sum_{i=j}^{n} u_i(f) v_{i-j}(t) \\ &= \sum_{i=j}^{n-2} u_i(f) v_{i-j}(t) + f^{(N-1)}(0)(v_{n-1-j}(t) \\ &\quad - v_{n-j}(t)) + f^{(N-1)}(1) v_{n-j}(t) \geq 0. \end{aligned}$$

Also

$$(Pf)^{(k_{n-1})}(t) \;=\; f^{(N-1)}(0)\left(v_0(t)-v_1(t)\right)+f^{(N-1)}(1)v_0(t)\geq 0$$

and

$$\begin{aligned}(Pf)^{(k_n)}(t) &= \left(f^{(N-1)}(1)-f^{(N-1)}(0)\right)v_0(t)\\ &= \left(f^{(N-1)}(1)-f^{(N-1)}(0)\right)=\int_0^1 f^{(N)}(s)ds\geq 0.\end{aligned}$$

This shows that $Pf\in S$ and consequently $P\in\mathcal{P}_S$.

□

Theorem 3.6. *Let $X=(C^N[0,1],\|\cdot\|)$ (for $N\leq 3$) and $\{k_i\}_{i=0}^n$ (for $n\geq N$) satisfies (K.1)-(K.2). Let P be as in Theorem 3.1.*
Then P has minimal norm in $\mathcal{P}_S$ and

$$\|P\|=\sum_{i=0}^{n-1}\frac{1}{k_i!}.$$

Proof. Note that

$$\begin{aligned}\|Pf\| &\leq \|Pf\|_\infty\\ &= \sup_{t\in[0,1]}\left|\sum_{i=0}^{n-2}u_i(f)v_i(t)+f^{(n-1)}(0)\left(v_{n-1}(t)-v_n(t)\right)+f^{(n-1)}(1)v_n(t)\right|\\ &\leq \sup_{t\in[0,1]}\left\{\sum_{i=0}^{n-2}|u_i(f)v_i(t)|+\left|f^{(n-1)}(0)\left(v_{n-1}(t)-v_n(t)\right)\right|+\left|f^{(n-1)}(1)v_n(t)\right|\right\}\\ &\leq \sup_{t\in[0,1]}\left\{\sum_{i=0}^{n-2}\|f\|\,|v_i(t)|+\|f\|\,|v_{n-1}(t)-v_n(t)|+\|f\|\,|v_n(t)|\right\}\\ &= \|f\|\sup_{t\in[0,1]}\left\{\sum_{i=0}^{n-2}|v_i(t)|+|v_{n-1}(t)-v_n(t)|+|v_n(t)|\right\}\\ &= \|f\|\sup_{t\in[0,1]}\left\{\sum_{i=0}^{n-2}v_i(t)+v_{n-1}(t)-v_n(t)+v_n(t)\right\}=\|f\|\sum_{i=0}^{n-1}\frac{1}{k_i!},\end{aligned}$$

which shows that

$$\|P\|\leq\sum_{i=0}^{n-1}\frac{1}{k_i!}$$

Before proving next inequality we need

Lemma 3.7. $X=C^N[0,1]$. *There exists $\{f_k\}_{k=1}^\infty\subset X$ such that:*

(1) $\forall i = 0, \dots, N-1 \ : \ \lim_{k\to\infty} f_k^{(i)}(0) = 1$,
(2) $\lim_{k\to\infty} f_k^{(N-1)}(1) = 1$,
(3) $\forall i = 0, \dots, N \ : \ \|f_k^{(i)}\|_\infty = 1$.

Proof. It follows from Lemma 4.4 in [26] (see Lemma 2.1) and the Goldstine Theorem. □

Now applying sequence $\{f_k\}_{k=1}^{\infty} \subset X$ we obtain

$$\lim_{k\to\infty} Pf_k(t) = \sum_{i=0}^{n-1} v_i(t) = \sum_{i=0}^{n-1} \frac{t^{k_i}}{k_i!}.$$

Hence

$$\sup_{t\in[0,1]} |\lim_{k\to\infty} Pf_k(t)| = \sup_{t\in[0,1]} \left|\sum_{i=0}^{n-1} \frac{t^{k_i}}{k_i!}\right| = \sum_{i=0}^{n-1} \frac{1}{k_i!}.$$

As a result

$$\sum_{i=0}^{n-1} \frac{1}{k_i!} \leq \|P\|.$$

The last part concerns minimality of P in $\mathcal{P}_S$.
Let $Q \in \mathcal{P}_s$. By Lemma 2.3

$$Qf = \sum_{i=0}^{n-1} u_i(f)v_i + u(f)v_n,$$

where $u(f)$ is given by 7. By Lemma 2.2 for every $F \in W$

$$\|Q\| \geq \sum_{i=0}^{n-1} \frac{1}{k_i!} + F(u) \geq \sum_{i=0}^{n-1} \frac{1}{k_i!},$$

Hence

$$\|Q\| \geq \|P\|$$

and P is minimal.

□

4. The construction of projections for other multi-convex shape

Due to a special construction, we can obtain by recursive formula projections preserving others shapes from our projection P defined in previous section.

Define

$$T_m : C[0,1] \to C[0,1], \quad m = k_1 - 1,$$

by

$$T_m f(s) = \int_0^s \int_0^{s_{m-1}} \dots \int_0^{s_1} f(s_0) ds_0 ds_1 \dots ds_{m-1}. \tag{9}$$

Observation 4.1. *For any $f \in C[0,1]$ and $m \in \mathbb{N}$*

$$(T_m f)^{(m)} = f.$$

The operator T_m will be used in our recurrence formula.

Now we need some additional notation.
In $C^{N+lk_1}[0,1]$ we consider a norm:

$$\|f\|_l = \max_{i\in\{0,\ldots,n\}, j\in\{0,\ldots,l-1\}} \{\|f^{jk_1}\|_\infty, \|f^{(k_i+lk_1)}\|_\infty\}. \tag{10}$$

It is clear that, the norm $\|\cdot\|$ defined in section 2 is our $\|\cdot\|_0$ norm. Let $X_l = (C^{N+lk_1}[0,1], \|\cdot\|_l)$, so $X = X_0$.
Moreover, put

$$V_l = \text{span}\{v_0^{[l]}, v_1^{[l]}, \ldots, v_{n+l}^{[l]}\}, \tag{11}$$

where

$$\begin{aligned} v_0^{[l]}(t) &= 1, \\ v_i^{[l]}(t) &= \int_0^t T_m v_{i-1}^{[l-1]}(s)ds, \quad i = 1, \ldots, n+l. \end{aligned} \tag{12}$$

$$\tag{13}$$

Let

$$\begin{aligned} \mathcal{P}_l &= \mathcal{P}_l(X_l, V_l) \\ &= \{P : X_l \to V_l \ : \ P - \text{linear and bounded}, \ P_{|V_l} = \text{id}_{V_l}\}. \end{aligned} \tag{14}$$

Now we wille define a cone S_l.

$$S_l = \{f \in X_l \ : \ \forall t \in [0;1], i \in \{0,1,\ldots,n\} \ f^{(k_i+lk_1)}(t) \geq 0\}. \tag{15}$$

We denote shape-preserving projection with respect to S_l as

$$\mathcal{P}_{S_l} = \{P \in \mathcal{P}_l \ : \ PS_l \subset S_l\}.$$

Theorem 4.2 (Recurrence formula)**.** *For fixed $N \geq 3$ and $n \leq N$ suppose $\{k_i\}_{i=0}^n$ satisfies (K.1)–(K.3). For given $l \in \mathbb{N}$, let X_l, V_l and S_l be as above. Suppose $P_l \in \mathcal{P}_{S_l}$.*

Then an operator defined by

$$P_{l+1}f(t) = \frac{f(0)+f(1)}{2} + \int_0^t T_m P_l f^{(k_1)}(s)ds - \frac{1}{2}\int_0^1 T_m P_l f^{(k_1)}(s)ds \tag{16}$$

belongs to $P_{l+1} \in \mathcal{P}_{S_{l+1}}$.

Proof. First we show that P_{l+1} is a projection.
Note that

$$\left(v_0^{[l+1]}\right)^{(k_1)}(t) = 0$$

and for $j = 1, \dots, n+l+1$,

$$\begin{aligned}
(v_j^{[l+1]})^{(k_1)}(t) &= (v_j^{[l+1]})^{(1+m)}(t) \\
&= \left(\left(v_j^{[l+1]}\right)'\right)^{(m)}(t) \\
&= \left(\left(\int_0^t T_m v_{j-1}^{[l]}(s)ds\right)'\right)^{(m)} \\
&= \left(T_m v_{j-1}^{[l]}\right)^{(m)}(t) = v_{j-1}^{[l]}(t).
\end{aligned}$$

Consequently

$$\begin{aligned}
P_{l+1}v_0^{[l+1]}(t) &= \frac{v_0^{[l+1]}(0) + v_0^{[l+1]}(1)}{2} + \int_0^t T_m P_l \left(v_0^{[l+1]}\right)^{(k_1)}(s)ds \\
&\quad -\frac{1}{2}\int_0^1 T_m P_l \left(v_0^{[l+1]}\right)^{(k_1)}(s)ds \\
&= \frac{1+1}{2} + \int_0^t T_m P_l 0 ds - \frac{1}{2}\int_0^1 T_m P_l 0 ds \\
&= 1 + 0 - \frac{1}{2}\cdot 0 \\
&= 1 = v_0^{[l+1]}(t)
\end{aligned}$$

Also for $i = 1, \dots, n+l$

$$\begin{aligned}
P_{l+1}v_i^{[l+1]}(t) &= \frac{v_i^{[l+1]}(0) + v_i^{[l+1]}(1)}{2} + \int_0^t T_m P_l \left(v_i^{[l+1]}\right)^{(k_1)}(s)ds \\
&\quad -\frac{1}{2}\int_0^1 T_m P_l \left(v_i^{[l+1]}\right)^{(k_1)}(s)ds \\
&= \frac{v_i^{[l+1]}(0) + v_i^{[l+1]}(1)}{2} + \int_0^t T_m P_l \left(v_{i-1}^{[l]}\right)(s)ds \\
&\quad -\frac{1}{2}\int_0^1 T_m P_l \left(v_{i-1}^{[l]}\right)(s)ds
\end{aligned}$$

$$\begin{aligned}
&= \frac{v_i^{[l+1]}(0)+v_i^{[l+1]}(1)}{2} + \int_0^t T_m\left(v_{i-1}^{[l]}\right)(s)ds \\
&\quad -\frac{1}{2}\int_0^1 T_m\left(v_{i-1}^{[l]}\right)(s)ds \\
&= \frac{1}{2}\left(v_i^{[l+1]}(0)+v_i^{[l+1]}(1)\right) + v_i^{[l+1]}(t) \\
&\quad -\frac{1}{2}\left(v_i^{[l+1]}(1)-v_i^{[l+1]}(0)\right) \\
&= v_i^{[l+1]}(t)+v_i^{[l+1]}(0) = v_i^{[l+1]}(t),
\end{aligned}$$

what ends this part of the proof.

Now assume that we have proved

$$\forall j \in \{0,\dots,l-1\} \ : \ (P_{l+1}f)^{(jk_1)} = \left(P_l f^{(k_1)}\right)^{((j-1)k_1)} \tag{17}$$

and

$$\forall i \in \{0,\dots,n\} \ : \ (P_{l+1}f)^{(k_i+lk_1)} = \left(P_l f^{(k_1)}\right)^{(k_i+(l-1)k_1)}. \tag{18}$$

Let $f \in S_{l+1}$. Hence $f^{(k_j+(l+1)k_1)}(t) \geq 0$ for $t \in [0,1]$ and $j = 0,1,\dots,n$.

By (17–18) and definition of P_l

$$\begin{aligned}
(P_{l+1}f)^{(k_j+lk_1)} &= \left(P_l f^{(k_1)}\right)^{(k_j+(l-1)k_1)} \\
&= \left(P_{l-1} f^{(2k_1)}\right)^{(k_j+(l-2)k_1)} \\
&= \dots \\
&= \left(P f^{(lk_1)}\right)^{(k_j)}.
\end{aligned}$$

Hence for $j = 0,\dots,n-2$

$$\begin{aligned}
\left(P f^{(lk_1)}\right)^{(k_j)}(t) &= \sum_{i=j}^{n} u_i(f^{(lk_1)})v_{i-j}(t) \\
&= \sum_{i=j}^{n-2} u_i(f^{(lk_1)})v_{i-j}(t) + \left(f^{(lk_1)}\right)^{(N-1)}(1)v_{n-j}(t) \\
&\quad + \left(f^{(lk_1)}\right)^{(N-1)}(0)\left(v_{n-1-j}(t)-v_{n-j}(t)\right) \\
&= \sum_{i=j}^{n-2} f^{(k_i+lk_1)}(0)v_{i-j}(t) + f^{(lk_1+N-1)}(1)v_{n-j}(t) \\
&\quad + f^{(lk_1+N-1)}(0)\left(v_{n-1-j}(t)-v_{n-j}(t)\right) \geq 0.
\end{aligned}$$

Also

$$\left(P f^{(lk_1)}\right)^{(k_{n-1})}(t) = f^{(lk_1+N-1)}(0)(1-t) + f^{(lk_1+N-1)}(1)t \geq 0$$

and

$$\left(Pf^{(lk_1)}\right)^{(k_n)}(t) = \left(f^{(lk_1+N-1)}(1) - f^{(lk_1+N-1)}(0)\right) \geq 0.$$

To end the proof, we need to show (17) and (18)

Note that

$$\begin{aligned}
(P_{l+1}f)^{(jk_1)}(t) &= \left(\frac{f(0)+f(1)}{2} + \int_0^t T_m P_l f^{(k_1)}(s)ds \right.\\
&\quad \left. - \frac{1}{2}\int_0^1 T_m P_l f^{(k_1)}(s)ds\right)^{(jk_1)} \\
&= \left(\frac{f(0)+f(1)}{2} - \frac{1}{2}\int_0^1 T_m P_l f^{(k_1)}(s)ds\right)^{(jk_1)} \\
&\quad + \left(\int_0^t T_m P_l f^{(k_1)}(s)ds\right)^{(jk_1)} \\
&= \left(\int_0^t T_m P_l f^{(k_1)}(s)ds\right)^{(k_1+(j-1)k_1)} \\
&= \left(\int_0^t T_m P_l f^{(k_1)}(s)ds\right)^{(1+m+(j-1)k_1)} \\
&= \left(T_m P_l f^{(k_1)}(t)\right)^{(m+(j-1)k_1)} \\
&= \left(P_l f^{(k_1)}\right)^{((j-1)k_1)}(t),
\end{aligned}$$

which shows (17).

Also

$$\begin{aligned}
(P_{l+1}f)^{(k_i+lk_1)}(t) &= \left(\frac{f(0)+f(1)}{2} + \int_0^t T_m P_l f^{(k_1)}(s)ds \right.\\
&\quad \left. - \frac{1}{2}\int_0^1 T_m P_l f^{(k_1)}(s)ds\right)^{(k_i+lk_1)} \\
&= \left(\int_0^t T_m P_l f^{(k_1)}(s)ds\right)^{(k_i+lk_1)}
\end{aligned}$$

$$\begin{aligned}
&= \left(\int_0^t T_m P_l f^{(k_1)}(s)ds\right)^{(k_1+k_i+(l-1)k_1)} \\
&= \left(T_m P_l f^{(k_1)}(t)\right)^{(m+k_i+(l-1)k_1)} \\
&= \left(P_l f^{(k_1)}\right)^{(k_i+(l-1)k_1)}(t),
\end{aligned}$$

which shows (18)

The proof is complete $\square$

As we know, the recurrence formula given in Theorem 4.2 builds a multiconvex projection. Now we show that if P_l is minimal in $\mathcal{P}_{S_l}$, then P_{l+1} is minimal in $\mathcal{P}_{S_{l+1}}$, which is the main result of this paper.

First we need two lemmas.

Lemma 4.3. *Let $p \in \Pi_N$, $p(s) = \sum_{i=0}^{N} a_i \frac{s^i}{i!}$ and let C_p be a constant such that*

$$\sup_{s\in[0,1]} \left\{ \sum_{i=0}^{N} |a_i| \frac{s^i}{i!} \right\} \leq C_p.$$

Then

$$\left| \int_0^t T_m p(s)ds - \frac{1}{2}\int_0^1 T_m p(s)ds \right| \leq \frac{1}{2k_1!}C_p.$$

Proof. Set

$$p(s) = \sum_{i=0}^{n} a_i \frac{s^i}{i!}, \qquad T_m p(s) = \sum_{i=0}^{N} a_i \frac{s^{i+m}}{(i+m)!}, \qquad \widetilde{p}(t) = \int_0^t T_m p(s)ds.$$

Then

$$\int_0^t T_m p(s)ds - \frac{1}{2}\int_0^1 T_m p(s)ds = \widetilde{p}(t) - \frac{1}{2}\widetilde{p}(1).$$

Note that

$$\begin{aligned} \widetilde{p}(t) &= \sum_{i=0}^{N} a_i \frac{t^{i+m+1}}{(i+m+1)!} \\ &= \sum_{i=0}^{N} a_i \frac{t^{i+k_1}}{(i+k_1)!} \\ &= \frac{1}{k_1!}\sum_{i=0}^{N} \frac{a_i k_1! i!}{(i+k_1)!}\frac{t^{i+k_1}}{i!} \\ &= \frac{1}{k_1!}\sum_{i=0}^{N} \frac{a_i}{\binom{i+k_1}{k_1}}\frac{t^{i+k_1}}{i!}. \end{aligned}$$

Hence

$$\begin{aligned}
\sup_{t\in[0,1]}\left|\widetilde{p}(t)-\frac{1}{2}\widetilde{p}(1)\right| &= \sup_{t\in[0,1]}\left|\frac{1}{k_1!}\sum_{i=0}^{N}\frac{a_i}{\binom{i+k_1}{k_1}}\frac{t^{i+k_1}-\frac{1}{2}}{i!}\right| \\
&= \frac{1}{k_1!}\sup_{t\in[0,1]}\left|\sum_{i=0}^{N}\frac{a_i}{\binom{i+k_1}{k_1}}\frac{t^{i+k_1}-\frac{1}{2}}{i!}\right| \\
&\le \frac{1}{k_1!}\sup_{t\in[0,1]}\left\{\sum_{i=0}^{N}\left|\frac{a_i}{\binom{i+k_1}{k_1}}\right|\left|\frac{t^{i+k_1}-\frac{1}{2}}{i!}\right|\right\} \\
&\le \frac{1}{k_1!}\sup_{t\in[0,1]}\left\{\sum_{i=0}^{N}|a_i|\left|\frac{t^{i+k_1}-\frac{1}{2}}{i!}\right|\right\} \\
&\le \frac{1}{k_1!}\sum_{i=0}^{N}|a_i|\frac{\frac{1}{2}}{i!} = \frac{1}{2k_1!}\sum_{i=0}^{n}|a_i|\frac{1}{i!} \le \frac{1}{2k_1!}C_p,
\end{aligned}$$

as required. □

It is worth to notice that above Lemma plays crucial role in this paper. It permits to generalize [[26], Theorem 2.4] without assumption that $\|P_l\| \ge 2$.

Lemma 4.4. *If* $\|P_l\| \ge 1 + \frac{1}{k_1!}$, *then*

$$\|P_{l+1}f\|_\infty \le \|P_l\|.$$

Proof. Let $\|f\|_{l+1} = 1$. By definition of $\|\cdot\|_l$ and $\|\cdot\|_{l+1}$, we obtain that $\|f^{(k_1)}\|_l = 1$. Hence

$$\begin{aligned}
\|P_{l+1}f\|_\infty &= \sup_{t\in[0,1]}\left|\frac{f(0)+f(1)}{2}+\int_0^t T_mP_lf^{(k_1)}(s)ds-\frac{1}{2}\int_0^1 T_mP_lf^{(k_1)}(s)ds\right| \\
&\le \sup_{t\in[0,1]}\left|\frac{\|f\|+\|f\|}{2}+\int_0^t T_mP_lf^{(k_1)}(s)ds-\frac{1}{2}\int_0^1 T_mP_lf^{(k_1)}(s)ds\right| \\
&= \|f\| + \sup_{t\in[0,1]}\left|\int_0^t T_mP_lf^{(k_1)}(s)ds-\frac{1}{2}\int_0^1 T_mP_lf^{(k_1)}(s)ds\right| \\
&= 1 + \sup_{t\in[0,1]}\left|\int_0^t T_mP_lf^{(k_1)}(s)ds-\frac{1}{2}\int_0^1 T_mP_lf^{(k_1)}(s)ds\right| \\
&= \frac{1+\frac{1}{k_1!}}{1+\frac{1}{k_1!}} + \sup_{t\in[0,1]}\left|\int_0^t T_mP_lf^{(k_1)}(s)ds-\frac{1}{2}\int_0^1 T_mP_lf^{(k_1)}(s)ds\right|
\end{aligned}$$

By Lemma 4.3, applied to $p(s) = P_l f^{(k_1)}(s) \in \Pi_{N+lk_1}$ and $C_p = \|P_l\|$, we obtain:

$$\begin{aligned}\|P_{l+1}f\|_\infty &\le \frac{1+\frac{1}{k_1!}}{1+\frac{1}{k_1!}} + \frac{1}{2k_1!}\|P_l\| \\ &\le \frac{1}{1+\frac{1}{k_1!}}\|P_l\| + \frac{1}{2k_1!}\|P_l\| \\ &\le \frac{2k_1!+1+\frac{1}{k_1!}}{2k_1!+2}\|P_l\| \\ &\le \|P_l\| \quad \text{(cause } k_1 \ge 1).\end{aligned}$$

□

First, we show that our recurrence formula keeps the norm of projection P_l constant.

Theorem 4.5. *For fixed P_l such that $P_l \in \mathcal{P}_l$ with $\|P_l\| \ge 1 + \frac{1}{k_1!}$ and P_{l+1} received by the recurrence formula 4.2 we have*

$$\|P_{l+1}\| = \|P_l\|.$$

Proof. Note that

$$\begin{aligned}\|P_{l+1}f\|_{l+1} &= \max\{\|(P_{l+1}f)^{jk_1}\|_\infty, \|(P_{l+1}f)^{(k_i+lk_1)}\|_\infty : \\ & \quad i \in \{0,\dots,n\}, j \in \{0,\dots,l-1\}\}\end{aligned}$$

Assume that $\|f\|_{l+1} = 1$.
Then by Lemma 4.4

(1) $\|P_{l+1}f\|_\infty \le \|P_l\|$.

By (17)

(2) $\|\left(P_{l+1}\right)^{(jk_1)} f\|_\infty \le \|\left(P_l\right)^{(j-1)k_1)}\left(f^{(k_1)}\right)\|_\infty \le \|P_l\|$.

By (18)

(3) $\|\left(P_{l+1}\right)^{(k_i+lk_1)} f\|_\infty \le \|\left(P_l\right)^{(k_i+(l-1)k_1)}\left(f^{(k_1)})\right)\|_\infty \le \|P_l\|$.

Consequently,

$$\|P_{l+1}\| \le \|P_l\|.$$

□

By to Lemma 2.4, any $Q_l \in \mathcal{P}_{S_l}$ may be represented as

$$Q_l f = \sum_{i=0}^{l-1} u_i^l(f) v_i^{[l]} + \sum_{i=l}^{n+l-1} u_{i-l}(f^{lk_1}) v_i^{[l]} + u(f) v_{n+l}^{[l]}.$$

It is obvious that

$$\|Q_l f\|_l \ge \|(Q_l f)^{(lk_1)}\|_\infty.$$

Lemma 4.6. *If $Q_l \in \mathcal{P}_{S_l}$ and $\|f\|_l = 1$, $f \in X_l$, then*

$$\|(Q_l f)^{(lk_1)}\|_\infty \geq \|Q\|,$$

where Q is a corresponding projection in $\mathcal{P}_S$.

Proof. By direct calculation (see page 9), we obtain

$$\begin{aligned}(Q_l f)^{(lk_1)} &= \left(\sum_{i=0}^{l-1} u_i^l(f) v_i^{[l]} + \sum_{i=l}^{n+l-1} u_{i-l}(f^{lk_1}) v_i^{[l]} + u(f) v_{n+l}^{[l]}\right)^{(lk_1)} \\ &= \left(\sum_{i=l}^{n+l-1} u_{i-l}(f^{lk_1}) v_i^{[l]} + u(f) v_{n+l}^{[l]}\right)^{(lk_1)} \\ &= \sum_{i=0}^{n-1} u_i(f^{lk_1}) v_i + u(f) v_n.\end{aligned}$$

What implies

$$\|(Q_l f)^{(lk_1)}\|_\infty \geq \|Q\|.$$

□

Now we show the main result of this paper.

Theorem 4.7 (Minimality of P_l)**.** *Let P be a minimal multiconvex projection ($P \in \mathcal{P}_S$) and let P_l be projection created by applying l-times recurrence formula (16) (see Theorem 4.2).*

Then for any projection $Q_l \in \mathcal{P}_{S_l}$

$$\|Q_l\|_l \geq \|P_l\|_l.$$

Proof. By Lemma 4.6, for $f \in X$, $\|f\|_l = 1$,

$$\|Q\|_l \geq \|Q_l f\|_l \geq \|(Q_l f)^{(lk_1)}\|_\infty \geq \|Q\| \geq \|P\| \geq \|P_l\|_l,$$

which proofs the minimality of P_l in $\mathcal{P}_{S_l}$. □

Remark. If $V = \Pi_N$ then Theorem 4.7 reduces to Theorem 2.4 in [26].

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