Weighted shifts on directed trees

Zenon J. Jabłoński, Il Bong Jung, and Jan Stochel

Dedicated to Professor Franciszek H. Szafraniec on the occasion of his 70th birthday

IM UJ Preprint 2009/10

Abstract. A new class of (not necessarily bounded) operators related to (mainly infinite) directed trees is introduced and investigated. Operators in question are to be considered as a generalization of classical weighted shifts, on the one hand, and of weighted adjacency operators, on the other; they are called weighted shifts on directed trees. The basic properties of such operators, including closedness, adjoints, polar decomposition and moduli are studied. Circularity and the Fredholmness of weighted shifts on directed trees are discussed. The relationships between domains of a weighted shift on a directed tree and its adjoint are described. Hyponormality, cohyponormality, subnormality and complete hyperexpansivity of such operators are entirely characterized in terms of their weights. Related questions that arose during the study of the topic are solved as well. Particular trees with one branching vertex are intensively studied mostly in the context of subnormality and complete hyperexpansivity of weighted shifts on them. A strict connection of the latter with k-step backward extendibility of subnormal as well as completely hyperexpansive unilateral classical weighted shifts is established. Models of subnormal and completely hyperexpansive weighted shifts on these particular trees are constructed. Various illustrative examples of weighted shifts on directed trees with the prescribed properties are furnished. Many of them are simpler than those previously found on occasion of investigating analogical properties of other classes of operators.

2000 *Mathematics Subject Classification.* Primary 47B37, 47B20; Secondary 47A05, 44A60.

Key words and phrases. Directed tree, weighted shift, adjoint operator, polar decomposition, circular operator, inclusion of domains, Fredholm operator, semi-Fredholm operator, hyponormal operator, cohyponormal operator, subnormal operator, completely hyperexpansive operator.

The first and the third authors were supported by the MNiSzW grant N201 026 32/1350. The second author was supported by Basic Science Research Program through the National Research Foundation of Korea (NRF) grant funded by the Korea government (MEST) (2009-0093125).

Contents

Chapter 1. Introduction

The main goal of this paper is to implement some methods of graph theory into operator theory. We do it by introducing a new class of operators, which we propose to call *weighted shifts on directed trees.* This considerably generalizes the notion of a weighted shift, the classical object of operator theory (see e.g., [**70**] for a beautiful survey article on bounded weighted shifts, and [**58**] for basic facts on unbounded ones). As opposed to the standard graph theory which concerns mostly finite graphs (see e.g., [**65, 25**]), we mainly deal with infinite graphs, in fact infinite directed trees. Much part of (non-selfadjoint) operator theory trivializes when one considers weighted shifts on finite directed trees. This is the reason why we have decided to include assorted facts on infinite graphs. The specificity of operator theory forces peculiarity of problems to be solved in graph theory. This is yet another reason for studying infinite graphs.

Matrix theory is always behind graph theory: finite undirected graphs induce adjacency matrices which are always symmetric. However, if undirected graphs are infinite, then we have to replace adjacency matrices by symmetric operators (cf. [**62, 63**]). It turns out that adjacency operators may not be selfadjoint (cf. [**64**], see also [**5**]). If we want to study non-selfadjoint operators, we have to turn our interest to directed graphs, and replace the adjacency matrix by an (in general, unbounded) operator, called the adjacency operator of the graph. This was done for the first time in [**32**]. It turns out that the adjacency operators ("which form a small fantastic world", cf. [**32**]) can be expressed as infinite matrices whose entries are 0 or 1. If we look at the definition of the adjacency operator of a directed tree $\mathscr{T}$ (with bounded valency), we find that it coincides with that of the weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ with weights $\lambda_v \equiv 1$ (see Definition 3.1.1 and Proposition 3.1.3). The questions of when the adjacency operator is positive, selfadjoint, unitary, normal and (co-) hyponormal have been answered in [**32**] (characterizations of some algebraic properties of adjacency operators have been given there as well). Spectral and numerical radii of adjacency operators have been studied in [**13**] (the case of undirected graphs) and in [**33, 79**] (the case of directed graphs).

The notion of adjacency operator has been generalized in [**31**, Section 6] to the case of infinite directed fuzzy graphs G (i.e., graphs whose arrows have stochastic values); such operator, denoted by $A(G)$ in [**31**] (and sometimes called a weighted adjacency operator of G), is assumed to be bounded. In view of Proposition 3.1.3, it is a simple matter to verify that if G is a directed tree, then the weighted adjacency operator $A(G)$ coincides with our weighted shift operator on G. It was proved in [**31**, Theorem 6.1] that the spectral radius of the weighted adjacency operator $A(G)$ of an infinite directed fuzzy graph G belongs the approximate point spectrum of $A(G)$. Our approach to this question is quite different. Namely, we first prove that a weighted shift on a directed tree is circular (cf. Theorem 3.3.1), and then deduce the Perron-Frobenius type theorem (cf. Corollary 3.3.2). As an immediate consequence of circularity, we obtain the symmetricity of the spectrum of a weighted shift on a directed tree with respect to the real axis.

We now explain why in the case of directed trees we prefer to call a weighted adjacency operator a *weighted shift on a directed tree.* In the present paper, by a *classical weighted shift* we mean either a unilateral weighted shift S in ℓ^2 or a bilateral weighted shift S in $\ell^2(\mathbb{Z})$ ($\mathbb{Z}$ stands for the set of all integers). To be more precise, S is understood as the product VD, where, in the unilateral case, V is the unilateral isometric shift on ℓ^2 of multiplicity 1 and D is a diagonal operator in ℓ^2 with diagonal elements $\{\lambda_n\}_{n=0}^{\infty}$; in the bilateral case, V is the bilateral unitary shift on $\ell^2(\mathbb{Z})$ of multiplicity 1 and D is a diagonal operator in $\ell^2(\mathbb{Z})$ with diagonal elements $\{\lambda_n\}_{n=-\infty}^{\infty}$ (diagonal operators are assumed to be closed, cf. Lemma 2.2.1). In fact, S is a unique closed linear operator in ℓ^2 (respectively, $\ell^2(\mathbb{Z})$) such that the linear span of the standard orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of ℓ^2 (respectively, $\{e_n\}_{n=-\infty}^{\infty}$ of $\ell^2(\mathbb{Z})$) is a core[1] of S and

$$Se_n = \lambda_n e_{n+1} \quad \text{for } n \in \mathbb{Z}_+ \text{ (respectively, for } n \in \mathbb{Z}), \tag{1.1}$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers. Roughly speaking, the operators $S_{\boldsymbol\lambda}$ which are subject of our investigations in the present paper can be described as follows (cf. (3.1.4)):

$$S_{\boldsymbol\lambda} e_u = \sum_{v \in \mathsf{Chi}(u)} \lambda_v e_v, \quad u \in V, \tag{1.2}$$

where $\{e_v\}_{v\in V}$ is the standard orthonormal basis of $\ell^2(V)$ indexed by a set V of vertexes of a directed tree $\mathscr{T}$, $\mathsf{Chi}(u)$ is the set of all children of u and $\{\lambda_v\}_{v \in V^\circ}$ is a system of complex numbers called the weights of $S_{\boldsymbol\lambda}$. If we apply this definition to the directed trees $\mathbb{Z}_+$ and $\mathbb{Z}$ (see Remark 3.1.4 for a detailed explanation), we will see that in this particular situation the equality (1.2) takes the form

$$S_{\boldsymbol\lambda} e_n = \lambda_{n+1} e_{n+1} \quad \text{for } n \in \mathbb{Z}_+ \text{ (respectively, } n \in \mathbb{Z}). \tag{1.3}$$

Comparing (1.1) with (1.3), one can convince himself that the operator $S_{\boldsymbol\lambda}$ can be viewed as generalization of a classical weighted shift operator. This is the main reason why operators $S_{\boldsymbol\lambda}$ are called here weighted shifts on directed trees.

The reader should be aware of the difference between notation (1.1) and (1.3). *In the present paper, we adhere to the new convention* (1.3).

It is well known that the adjoint of an injective unilateral classical weighted shift is not a classical weighted shift. It is somewhat surprising that the adjoint of a unilateral classical weighted shift is a weighted shift in our more general sense (cf. Remark 3.4.2).

Hereafter, we study weighted shifts on directed trees imposing no restrictions on their cardinality. However, if one wants to investigate densely defined weighted shifts with nonzero weights, then one ought to consider them on directed trees which are at most countable (cf. Proposition 3.1.10).

Less than half of our paper, namely chapters 3 and 4, deals with unbounded weighted shifts on directed trees. In Chapter 3, we investigate the question of when assorted properties of classical weighted shifts remain valid for weighted shifts on directed trees. The first basic property of classical weighted shifts stating that each of them is unitarily equivalent to another one with nonnegative weights has a natural counterpart in the context of directed trees (cf. Theorem 3.2.1). Circularity is another significant property of classical weighted shifts which turns out to be valid for their generalizations on directed trees (cf. Theorem 3.3.1). The adjoint and the modulus of a weighted shift on a directed tree are explicitly exhibited in Propositions 3.4.1 and 3.4.3, respectively. As a consequence, a clearly expressed description of the polar decomposition of a weighted shift on a directed tree is

[1] See Section 2.2 for appropriate definitions.

derived in Proposition 3.5.1. It enables us to characterize Fredholm and semi-Fredholm weighted shifts on a directed tree (cf. Propositions 3.6.2 and 3.6.9). It turns out that the property of being Fredholm, when considered in the class of weighted shifts on a directed tree with nonzero weights, can be stated entirely in terms of the underlying tree. Such a tree is called here Fredholm (cf. Definition 3.6.3). In general, if a directed tree admits a Fredholm weight shift (with not necessarily nonzero weights), then it is automatically Fredholm, but not conversely (cf. Propositions 3.6.2 and 3.6.4). Proposition 3.6.2 provides an explicit formula for the index of a Fredholm weighted shift on a directed tree (cf. the formula (3.6.1)). Owing to this formula, the index depends on both the underlying tree and the weights of the weighted shift in question (in fact, it depends on the geometry of the set of vertexes corresponding to vanishing weights). However, if all the weights are nonzero, then the index depends only on the underlying tree, and as such is called the index of the Fredholm tree (cf. Definition 3.6.3). The index of a Fredholm weighted shift on a directed tree can take all integer values from $-\infty$ to 1 (cf. Lemma 3.6.6 and Theorem 3.6.8).

The question of when the domain of a classical weighted shift is included in the domain of its adjoint has a simple answer. A related question concerning the reverse inclusion has an equally simple answer. However, the same problems, when formulated for weighted shifts on a directed tree, become much more elaborate. This is especially visible in the case of the reverse inclusion in which we require that a family of rank one perturbations of positive diagonal operators be uniformly bounded; these operators are tided up to the vertexes possessing children (cf. Theorem 4.2.2). Some examples of unbounded weighted shifts on a directed tree illustrating possible relationships between the domain of the operator in question and that of its adjoint are stated in Example 4.3.1.

Starting from Chapter 5, we concentrate mainly on the study of bounded operators. We begin by considering the question of hyponormality. We first show that a hyponormal weighted shift on a directed tree with nonzero weights is injective, and consequently that the underlying tree is leafless (this no longer true if we admit zero weights, cf. Remark 5.1.4). A complete characterization of the hyponormality of weighted shifts on directed trees is given in Theorem 5.1.2. It turns out that the property of being hyponormal is not too restrictive with respect to the underlying tree (even in the class of weighted shifts with nonzero weights). The situation changes drastically when we pass to cohyponormal weighted shifts on a directed tree. If the tree has a root, then there is no nonzero cohyponormal weighted shift on it. On the other hand, if the tree is rootless and admits a nonzero cohyponormal weighted shift, then the set of vertexes corresponding to nonzero weights is a subtree of the underlying tree which can be geometrically interpreted as either a broom with infinite handle or a straight line. This property is an essential constituent of the characterization of cohyponormality of nonzero weighted shifts on a directed tree that is given in Theorem 5.2.2. As a consequence, any injective cohyponormal weighted shift on a directed tree is a bilateral classical weighted shift (cf. Corollary 5.2.3).

The last section of Chapter 5 is devoted to showing how to separate hyponormality and paranormality classes with weighted shifts on directed trees. It is well known that the class of paranormal operators is essentially larger than that of hyponormal ones (see Section 5.1 for the appropriate definition). This was deduced by Furuta [**34**] from the fact that there are non-hyponormal squares of hyponormal operators. The first rather complicated example of a hyponormal operator whose square is not hyponormal was given by Halmos in [**39**] (however it is not injective). Probably the simplest example of such operator which is additionally injective is to be found in [**45**, page 158] (see [**41**, Problem 209] for details). One more example of this kind (with the injectivity property), but still complicated, can be found in [**26**, Example]. In the present article, we offer two examples of injective hyponormal weighted shifts on directed trees whose squares are not hyponormal. The first one, parameterized by three independent real parameters, is built on a relatively simple directed three that has only one branching vertex (cf. Example 5.3.2). The other one, parameterized by two real parameters, is built on a directed tree which is a "small perturbation" of a directed binary tree (cf. Example 5.3.3). Let us point out that there are no tedious computations behind our examples. According to our knowledge, the first direct example (making no appeal to non-hyponormal squares) of an injective paranormal operator which is not hyponormal appeared in [**16**, Example 3.1] (see also [**55, 56**] for non-injective examples of this kind). In Example 5.3.1 we construct an injective paranormal weighted shift on a directed tree which is not hyponormal; the underlying directed tree is the simplest possible directed tree admitting such an operator (because there is no distinction between hyponormality and paranormality in the class of classical weighted shifts).

Chapter 6 is devoted to the study of (bounded) subnormal weighted shifts on directed trees. The main characterization of such operators given in Theorem 6.1.3 asserts that a weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T} = (V, E)$ is subnormal if and only if each vertex $u \in V$ induces a Stieltjes moment sequence, i.e., $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence. Hence, it is natural to examine the set of all vertexes which induce Stieltjes moment sequences. Since the operator in question is bounded, the Stieltjes moment sequence induced by $u \in V$ turns out to be determinate; its unique representing measure is denoted by μ_u (cf. Notation 6.1.9). The first question we analyze is whether the property of inducing a Stieltjes moment sequence is inherited by the children of a fixed vertex. In general, the answer to the question is in the negative. The situation in which the answer is in the affirmative happens extremely rarely, actually, only when the vertex has exactly one child (cf. Lemma 6.1.5 and Example 6.1.6). This fact, when applied to the leafless directed trees without branching vertexes, leads to the well known Berger-Gellar-Wallen criterion for subnormality of injective classical weighted shifts (cf. Corollaries 6.1.7 and 6.1.8). Though the answer to the reverse question is in the negative, we can find a necessary and sufficient condition for a fixed vertex (read: a parent) to induce a Stieltjes moment sequence whenever its children do so (cf. Lemma 6.1.10); this condition is called the consistency condition. Lemma 6.1.10 also gives a formula linking measures induced by the parent and its children. The key ingredient of its proof consists of Lemma 6.1.2 which answers a variant of the question of backward extendibility of Stieltjes moment sequences.

The usefulness of the consistency condition (as well as the strong consistency condition) is undoubted. This is particularly illustrated in the case of directed trees $\mathscr{T}_{\eta,\kappa}$ that have only one branching vertex (cf. (6.2.10)). Such trees are one step more complicated than those involved in the definitions of classical weighted shifts (see Remark 3.1.4). Parameter η counts the number of children of the branching vertex of $\mathscr{T}_{\eta,\kappa}$, while κ counts the number of possible backward steps along the tree when starting from its branching vertex. Employing Lemma 6.1.10, we first characterize the subnormality of weighted shifts on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights (cf. Theorem 6.2.1 and Corollary 6.2.2) and then build models for such operators (cf. Section 6.3). According to Procedure 6.3.1, to construct the model weighted shift on $\mathscr{T}_{\eta,\kappa}$, we first take a sequence $\{\mu_i\}_{i=1}^{\eta}$ of Borel probability measures on a finite interval $[0, M]$, each of which possessing finite negative moments up to order $\kappa + 1$ (cf. (6.3.1)). The next step of the procedure depends on whether $\kappa = 0$ or $\kappa \geqslant 1$. In the first case, we choose any sequence $\{\lambda_{i,1}\}_{i=1}^{\eta}$ of positive real numbers satisfying the consistency condition (6.2.12) and define the weights of the model weighted shift by the formula (6.3.2). In the other case, we choose any sequence $\{\lambda_{i,1}\}_{i=1}^{\eta}$ of positive real numbers satisfying the strong consistency condition (6.2.13) and the estimate (6.3.3), and define the weights of the model weighted shift by the formulas (6.3.2), (6.3.4) and (6.3.5). The question of the existence of a sequence $\{\lambda_{i,1}\}_{i=1}^{\eta}$ which meets our requirements is answered in Lemma 6.3.2. Note that if $1 \leqslant \kappa < \infty$, then the weight $\lambda_{-\kappa+1}$ corresponding to the child of the root of $\mathscr{T}_{\eta,\kappa}$ is not uniquely determined by the sequences $\{\mu_i\}_{i=1}^{\eta}$ and $\{\lambda_{i,1}\}_{i=1}^{\eta}$; it is parameterized by a positive real number ϑ ranging over an interval in which one endpoint is 0 and the other is uniquely determined by $\{\mu_i\}_{i=1}^{\eta}$ and $\{\lambda_{i,1}\}_{i=1}^{\eta}$. If the parameter ϑ coincides with the nonzero endpoint, the corresponding subnormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ is called extremal. The extremality can be expressed entirely in terms of the weighted shift in question (cf. Remark 6.2.3). Procedure 6.3.1 enables us to link the issue of subnormality of weighted shifts (with nonzero weights) on the directed tree $\mathscr{T}_{\eta,\kappa}$ with the problem of k-step backward extendibility of subnormal unilateral classical weighted shifts which was originated by Curto in [**20**] and continued in [**22**] (see also [**42**] and referenced cited in the paragraph surrounding (6.3.10)). Roughly speaking, the subnormality of a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights is completely determined by the $(\kappa + 1)$-step backward extendibility of unilateral classical weighted shifts which are tied up to the children of the branching vertex via the formula (6.3.11) (cf. Proposition 6.3.4).

The class of completely hyperexpansive operators was introduced by Aleman in [**4**] on occasion of his study of multiplication operators on Hilbert spaces of analytic functions, and independently by Athavale in [**7**] on account of his investigation of operators which are antithetical to contractive subnormal operators. We also point out the trilogy by Stankus and Agler [**1, 2, 3**] concerning m-isometric transformations of a Hilbert space which are always completely hyperexpansive whenever $m \leqslant 2$. Again, as in the case of subnormality, the complete hyperexpansivity of a weighted shift $S_{\boldsymbol{\lambda}}$ on a directed tree $\mathscr{T} = (V, E)$ with nonzero weights can be characterized by requiring that each vertex $u \in V$ induces a completely alternating sequence, i.e., $\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^{\infty}$ is a completely alternating sequence (cf. Theorem 7.1.4). The structure of the set of all vertexes of $\mathscr{T}$ inducing completely alternating sequences is studied in two consecutive lemmas (cf. Lemmata 7.1.6 and 7.1.8). The first of them deals with the question of whether the property of inducing a completely alternating sequence is inherited by the children of a fixed vertex. The answer is exactly the same as in the case of subnormality. In the latter lemma we formulate a necessary and sufficient condition for a fixed vertex $u \in V$ to induce a completely alternating sequence whenever its children do so; this condition is again called the consistency condition, but now it is written in terms of representing measures of completely alternating sequences $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^{\infty}$, where v ranges over the set $\mathsf{Chi}(u)$ of all children of u. The proof of Lemma 7.1.8 rely on Lemma 7.1.2 which solves the question of backward extendibility of completely alternating sequences and provides the formula for representing measures of backward extensions of a given completely alternating sequence. As a consequence, we obtain a formula binding representing measures of completely alternating sequences induced by the parent and its children (cf. Lemma 7.1.8).

As in the case of subnormality, the directed tree $\mathscr{T}_{\eta,\kappa}$ serves as a good test for the applicability of Lemmata 7.1.6 and 7.1.8. What we get are the characterizations of complete hyperexpansivity of weighted shifts on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights (cf. Theorem 7.2.1 and Corollary 7.2.3). In opposition to subnormality, the only completely hyperexpansive weighted shifts on $\mathscr{T}_{\eta,\infty}$ with nonzero weights are isometries. This is the reason why in further parts of the paper we consider only the case when κ is finite. Modelling of complete hyperexpansivity of weighted shifts on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights, though still possible, is much more elaborate. The procedure leading to a model weighted shifts on $\mathscr{T}_{\eta,\kappa}$ starts with a sequence $\boldsymbol{\tau} = \{\tau_i\}_{i=1}^{\eta}$ of positive Borel measures on $[0,1]$ whose total masses are uniformly bounded (these measures eventually represents completely alternating sequences induced by the children of the branching vertex). The next step of the procedure requires much more delicate reasoning. It depends on the behaviour of weights of a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ corresponding to the children of the branching vertex. They must satisfy the conditions (7.3.1), (7.3.2) and (7.3.3) which are rather complicated and somewhat difficult to deal with (cf. Lemma 7.3.2). This means that if we want $\{t_i\}_{i=1}^{\eta}$ to be a sequence of weights of some completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ that correspond to the children of the branching vertex, it must verify the conditions (7.3.1), (7.3.2) and (7.3.3). Theorem 7.3.4 asserts that the above necessary conditions turn out to be sufficient as well. However, what remains quite unclear is under what circumstances a sequence $\{t_i\}_{i=1}^{\eta}$ satisfying these three conditions exits. The solution of this problem is given in Proposition 7.3.5. It is unexpectedly simple: each measure τ_i must have a finite negative moment of order $\kappa + 1$ and at least one of them must generate a unilateral classical weighted shift possessing a completely hyperexpansive $(\kappa+1)$-step backward extension (see (7.2.1) for an explanation). This is another significant difference between complete hyperexpansivity and subnormality because in the latter case each unilateral classical weighted shift generated by the child of the branching vertex must possess subnormal $(\kappa + 1)$-step backward extension (compare Propositions 7.4.4 and 6.3.4). The problem of k-step backward extendibility of completely hyperexpansive unilateral classical weighted shifts was investigated in [**48**]. The whole process of modelling complete hyperexpansivity on $\mathscr{T}_{\eta,\kappa}$ is summarized in Procedure 7.3.6.

Section 7.4 deals with the question of when for a given sequence $\{t_i\}_{i=1}^{\eta}$ of positive real numbers there exists a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ whose weights corresponding to the children of the branching vertex form the sequence $\{t_i\}_{i=1}^{\eta}$. The necessary and sufficient conditions for that are given in Propositions 7.4.1 and 7.4.2, respectively.

Chapter 7 ends with Section 7.5 which concerns the issue of extendibility of a system of weights of a completely hyperexpansive weighted shift on a subtree $\mathscr{T}$ of a directed tree $\hat{\mathscr{T}}$ to a system of weights of some completely hyperexpansive weighted shift on $\hat{\mathscr{T}}$ (both weighted shifts are assumed to have nonzero weights). In many cases such a possibility does not exist. Similar effect appears in the case of subnormal weighted shifts, however the assumptions imposed on the pair $(\mathscr{T}, \hat{\mathscr{T}})$ in the former case are much more restrictive than those in the latter (compare Propositions 6.1.12 and 7.5.1). Example 7.5.2 illustrates the validity of the phrase "much more restrictive" as well as shows that none of the assumptions (i) and (ii) of Proposition 7.5.1 can be removed.

In the last chapter of the paper (i.e., Chapter 8) we discuss the question of when a directed tree admits a weighted shift with a prescribed property (dense range, hyponormality, subnormality, normality, etc.) and characterize p-hyponormality of weighted shifts on directed trees (cf. Theorem 8.2.1). In Example 8.2.4, we single out a family of weighted shifts on $\mathscr{T}_{2,1}$ (with nonzero weights) in which ∞-hyponormality and subnormality are proved to be independent (modulo an operator). The same family is used to show how to separate p-hyponormality classes. Note also that p-hyponormal unilateral or bilateral classical weighted shifts are always hyponormal (cf. Corollary 8.2.2).

We now make three concluding remarks. First, we note that a weighted shift on a rootless directed tree $\mathscr{T}$ is a weighted composition operator on L^2 space with respect to the counting measure on the set of vertexes of $\mathscr{T}$ (cf. Definition 3.1.1). The next observation is that a weighted shift on a directed tree can be viewed as a weighted shift with operator weights on one of the following simple directed trees

$$\mathbb{Z}_+,\ \mathbb{Z},\ \mathbb{Z}_- \text{ and } \{1, \ldots, \kappa\}\ (\kappa < \infty).$$

This can be inferred from a decomposition of a directed tree described in the conditions (vi) and (viii) of Proposition 2.1.12. In general, the nth operator weight is an unbounded operator acting between different Hilbert spaces whose dimensions vary in n. The third remark is that the theory of weighted shifts on directed trees can easily be implemented in the context of directed forests. The reason for this is that a weighted shift on a directed forest is equal to the orthogonal sum of weighted shifts on directed trees which form the underlying directed forest.

In this paper we use the following notation. The fields of real and complex numbers are denoted by $\mathbb{R}$ and $\mathbb{C}$, respectively. The symbols $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$ stand for the sets of integers, nonnegative integers and positive integers, respectively. Given a topological space X, we write $\mathfrak{B}(X)$ for the σ-algebra of all Borel subsets of X. If $\zeta \in X$, then δ_ζ stands for the Borel probability measure on X concentrated on $\{\zeta\}$. We denote by χ_Y and $\operatorname{card}(Y)$ the characteristic function and the cardinal number of a set Y, respectively (it is clear from the context on which set the characteristic function χ_Y is defined).

Chapter 2. Prerequisites

2.1. Directed trees. Since the graph theory is mainly devoted to the study of finite graphs and our paper deals mostly with infinite graphs, we have decided to include in this section some basic notions and facts on the subject which are essential for the rest of the paper. For the basic concepts of the theory of graphs, we refer the reader to [**65**]. We say that a pair $\mathcal{G} = (V, E)$ is a *directed graph* if V is a nonempty set and E is a subset of $V \times V \setminus \{(v, v) \colon v \in V\}$. Put

$$\widetilde{E} = \{\{u, v\} \subseteq V \colon (u, v) \in E \text{ or } (v, u) \in E\}.$$

For simplicity, we suppress the explicit dependence of V, E and $\widetilde{E}$ on $\mathcal{G}$ in the notation. An element of V is called a *vertex* of $\mathcal{G}$, a member of E is called an *edge* of $\mathcal{G}$, and finally a member of $\widetilde{E}$ is called an *undirected edge*. If W is a nonempty subset V, then obviously the pair

$$\mathcal{G}_W := (W, (W \times W) \cap E) \tag{2.1.1}$$

is a directed graph which will be called a (directed) *subgraph* of $\mathcal{G}$. A directed graph $\mathcal{G}$ is said to be *connected* if for any two distinct vertexes u and v of $\mathcal{G}$ there exists a finite sequence $v_1, \ldots, v_n$ of vertexes of $\mathcal{G}$ ($n \geqslant 2$) such that $u = v_1$, $\{v_j, v_{j+1}\} \in \widetilde{E}$ for all $j = 1, \ldots, n-1$, and $v_n = v$; such a sequence will be called an *undirected path* joining u and v. Set

$$\mathsf{Chi}(u) = \{v \in V \colon (u, v) \in E\}, \quad u \in V.$$

A member of $\mathsf{Chi}(u)$ is called a *child* of u. If for a given vertex $u \in V$, there exists a unique vertex $v \in V$ such that $(v, u) \in E$, then we say that u has a parent v and write $\mathsf{par}(u)$ for v. Since the correspondence $u \mapsto \mathsf{par}(u)$ is a partial function (read: a relation) in V, we can compose it with itself k-times ($k \geqslant 1$); the result is denoted by par^k. We adhere to the convention that par^0 is the identity mapping on V. We will write $\mathsf{par}^k(u)$ only when u is in the domain of par^k. A finite sequence $\{u_j\}_{j=1}^n$ of distinct vertexes is said to be a *circuit* of $\mathcal{G}$ if $n \geq 2$, $(u_j, u_{j+1}) \in E$ for all $j = 1, \ldots, n-1$, and $(u_n, u_1) \in E$. A vertex v of $\mathcal{G}$ is called a *root* of $\mathcal{G}$, or briefly $v \in \mathsf{Root}(\mathcal{G})$, if there is no vertex u of $\mathcal{G}$ such that (u, v) is an edge of $\mathcal{G}$. Clearly, the cardinality of the set $\mathsf{Root}(\mathcal{G})$ may be arbitrary. If $\mathsf{Root}(\mathcal{G})$ is a one-element set, then its unique element is denoted by $\mathsf{root}(\mathcal{G})$, or simply by root if this causes no ambiguity. We write $V^\circ = V \setminus \mathsf{Root}(\mathcal{G})$.

The proof of the following fact is left to the reader[2].

PROPOSITION 2.1.1. *Let $\mathcal{G}$ be a directed graph satisfying the following conditions*

(i) *$\mathcal{G}$ is connected,*
(ii) *each vertex $v \in V^\circ$ has a parent.*

Then the set $\mathsf{Root}(\mathcal{G})$ contains at most one element.

We say that a directed graph $\mathscr{T}$ is a *directed tree* if it has no circuits and satisfies the conditions (i) and (ii) of Proposition 2.1.1. Note that none of these three properties defining the directed tree follows from the others. A subgraph of a directed tree $\mathscr{T}$ which itself is a directed tree will be called a *subtree* of $\mathscr{T}$. A directed tree may or may not possess a root, however, in the other case, a root must be unique. Note also that each finite directed tree always has a root. The reader

[2] All facts stated in this section without proofs can be justified by induction or methods employed in the proof of Proposition 2.1.4.

should be aware of the fact that we impose no restriction on the cardinality of the set V. A directed tree $\mathscr{T}$ such that $\operatorname{card}(\mathsf{Chi}(u)) = 2$ for all $u \in V$ will be called a *directed binary tree*.

Given a directed tree $\mathscr{T}$, we put $V' = \{u \in V \colon \mathsf{Chi}(u) \neq \varnothing\}$ and

$$V_{\prec} = \{u \in V \colon \operatorname{card}(\mathsf{Chi}(u)) \geqslant 2\}. \tag{2.1.2}$$

A member of the set $V \setminus V'$ is called a *leaf* of $\mathscr{T}$, while a member of the set $V_{\prec}$ is called a *branching vertex* of $\mathscr{T}$. A directed tree $\mathscr{T}$ is said to be *leafless* if $V = V'$. Every leafless directed tree is infinite, and every directed binary tree is leafless.

The following decomposition of the set V° plays an important role in our further investigations. Its proof is left to the reader.

Proposition 2.1.2. *If $\mathscr{T}$ is a directed tree, then $\mathsf{Chi}(u) \cap \mathsf{Chi}(v) = \varnothing$ for all $u, v \in V$ such that $u \neq v$, and*[3]

$$V^\circ = \bigsqcup_{u \in V} \mathsf{Chi}(u). \tag{2.1.3}$$

Let $\mathscr{T}$ be a directed tree. Given a set $W \subseteq V$, we put $\mathsf{Chi}(W) = \bigsqcup_{v \in W} \mathsf{Chi}(v)$ (in view of Proposition 2.1.2, $\mathsf{Chi}(W)$ is well-defined). Define

$$\mathsf{Chi}^{\langle 0 \rangle}(W) = W, \quad \mathsf{Chi}^{\langle n+1 \rangle}(W) = \mathsf{Chi}(\mathsf{Chi}^{\langle n \rangle}(W)), \quad n = 0, 1, 2, \ldots,$$

$$\mathsf{Des}(W) = \bigcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n \rangle}(W). \tag{2.1.4}$$

The members of $\mathsf{Des}(W)$ are called *descendants* of W. Since $\mathsf{Chi}(\cdot)$ is a monotonically increasing set-function, so is $\mathsf{Chi}^{\langle n \rangle}(\cdot)$. As a consequence, we have

$$W_1 \subseteq W_2 \subseteq V \implies \mathsf{Des}(W_1) \subseteq \mathsf{Des}(W_2). \tag{2.1.5}$$

An induction argument shows that

$$\mathsf{Chi}^{\langle n+1 \rangle}(W) = \bigcup_{v \in \mathsf{Chi}(W)} \mathsf{Chi}^{\langle n \rangle}(\{v\}). \tag{2.1.6}$$

In general, the sets $\mathsf{Chi}^{\langle n \rangle}(W)$, $n = 0, 1, 2, \ldots$, are not pairwise disjoint. For $u \in V$, we shall abbreviate $\mathsf{Chi}^{\langle n \rangle}(\{u\})$ and $\mathsf{Des}(\{u\})$ to $\mathsf{Chi}^{\langle n \rangle}(u)$ and $\mathsf{Des}(u)$, respectively. It is clear that (use an induction argument)

$$\mathsf{Des}(u) \subseteq W \text{ whenever } \mathsf{Chi}(W) \subseteq W \text{ and } u \in W, \tag{2.1.7}$$

which means that $\mathsf{Des}(u)$ is the smallest subset of V which is "invariant" for $\mathsf{Chi}(\cdot)$ and which contains u (cf. (2.1.8) below). Since, by (2.1.4),

$$\mathsf{Chi}(\mathsf{Des}(u)) = \bigcup_{n=1}^{\infty} \mathsf{Chi}^{\langle n \rangle}(u) \subseteq \mathsf{Des}(u), \quad u \in V, \tag{2.1.8}$$

we get

$$\mathsf{Des}(\mathsf{Des}(u)) = \mathsf{Des}(u), \quad u \in V. \tag{2.1.9}$$

[3] The notation "$\bigsqcup$" is reserved to denote pairwise disjoint union of sets.

It follows from the definition of the partial function par and the fact that $\mathscr{T}$ has no circuits that the sets $\mathsf{Chi}^{\langle n\rangle}(u)$, $n = 0, 1, 2, \ldots$, are pairwise disjoint, and hence

$$\mathsf{Des}(u) = \bigsqcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(u), \quad u \in V. \tag{2.1.10}$$

It may happen that $\mathsf{Chi}^{\langle n\rangle}(u) = \varnothing$ for some $u \in V$ and $n \geqslant 1$. In what follows, we also use the modified notation $\mathsf{Chi}_{\mathscr{T}}(u)$, $\mathsf{Chi}^{\langle n\rangle}_{\mathscr{T}}(u)$ and $\mathsf{Des}_{\mathscr{T}}(u)$ in order to make clear the dependence of $\mathsf{Chi}(u)$, $\mathsf{Chi}^{\langle n\rangle}(u)$ and $\mathsf{Des}(u)$ on the underlying directed tree $\mathscr{T}$.

The following simple observation turns out to be useful (its proof is left to the reader).

Lemma 2.1.3. *If $\mathscr{T}$ is a directed tree and $X \subseteq V$ is such that $\mathsf{par}(x) \in X$ for all $x \in X$, then*

$$u \in V \setminus X \implies \mathsf{Chi}(u) \cap \big(X \cup \mathsf{Chi}(X)\big) = \varnothing.$$

We now prove an important property of directed trees.

Proposition 2.1.4. *If $\mathscr{T}$ is a directed tree, then for every finite subset W of V there exists $u \in V$ such that $W \subseteq \mathsf{Des}(u)$.*

Proof. If $W = \{w\}$ with some $w \in V$, then setting $u = w$ does the job. If $W = \{a, b\}$ with some distinct $a, b \in V$, then we proceed as follows. Since $\mathscr{T}$ is connected, there exists a finite sequence $v_1, \ldots, v_n$ of vertexes of $\mathcal{G}$ ($n \geqslant 2$) such that $a = v_1$, $\{v_j, v_{j+1}\} \in \widetilde{E}$ for all $j = 1, \ldots, n-1$, and $v_n = b$. Denote by $\mathcal{W}(a, b)$ the set of all such sequences. Without loss of generality we can assume that the length n of our sequence $v_1, \ldots, v_n$ is the smallest among lengths of all sequences from $\mathcal{W}(a, b)$. We first show that the vertexes $v_1, \ldots, v_n$ are distinct. Suppose that, contrary to our claim, there exist $i, j \in \{1, \ldots, n\}$ such that $i < j$ and $v_i = v_j$. Since evidently the sequence $v_1, \ldots, v_i, v_{j+1}, \ldots, v_n$ belongs to $\mathcal{W}(a, b)$, we are led to a contradiction.

We now consider two disjunctive cases which cover all possibilities.

Case 1. Suppose first that $(v_1, v_2) \in E$. Then there exists the largest integer $k \in \{2, \ldots, n\}$ such that $(v_{j-1}, v_j) \in E$ for all $j \in \{2, \ldots, k\}$. We claim that $k = n$. Indeed, otherwise by the maximality of k, $(v_{k+1}, v_k) \in E$, which together with $(v_{k-1}, v_k) \in E$ and $v_{k-1} \neq v_{k+1}$, contradicts the definition of $\mathsf{par}(v_k)$. Hence, $v_{j-1} = \mathsf{par}(v_j)$ for all $j \in \{2, \ldots, n\}$, which implies that $b \in \mathsf{Chi}^{\langle n-1\rangle}(a) \subseteq \mathsf{Des}(a)$. Then $u = a$ meets our requirements.

Case 2. Assume now that Case 1 does not hold. This implies that $(v_2, v_1) \in E$. Then there exists the largest integer $p \in \{2, \ldots, n\}$ such that $(v_j, v_{j-1}) \in E$ for all $j \in \{2, \ldots, p\}$. If $p = n$, then $v_j = \mathsf{par}(v_{j-1})$ for all $j \in \{2, \ldots, n\}$, which yields $a \in \mathsf{Chi}^{\langle n-1\rangle}(b) \subseteq \mathsf{Des}(b)$. Therefore $u = b$ meets our requirements. In turn, if $p < n$, then by the maximality of p, $(v_p, v_{p+1}) \in E$. Arguing as in Case 1, we show that $b \in \mathsf{Chi}^{\langle n-p\rangle}(v_p) \subseteq \mathsf{Des}(v_p)$. Since $v_j = \mathsf{par}(v_{j-1})$ for all $j \in \{2, \ldots, p\}$, we see that $a \in \mathsf{Chi}^{\langle p-1\rangle}(v_p) \subseteq \mathsf{Des}(v_p)$. Hence $a, b \in \mathsf{Des}(v_p)$, which completes the proof of the case when W is a two-point set.

Finally, we have to consider W of cardinality m, which is greater than 2. We use an induction on m. If $\widehat{W} = W \cup \{w\}$ for some $w \notin W$, then by the induction hypothesis, there exists $u \in V$ such that $W \subseteq \mathsf{Des}(u)$. By the first part of the proof, there exists $u' \in V$ such that $u, w \in \mathsf{Des}(u')$. Thus, by (2.1.9), we have

$$\widehat{W} = W \cup \{w\} \subseteq \mathsf{Des}(u) \cup \mathsf{Des}(u') \subseteq \mathsf{Des}(\mathsf{Des}(u')) \cup \mathsf{Des}(u') = \mathsf{Des}(u').$$

This completes the proof. □

COROLLARY 2.1.5. *If $\mathscr{T}$ is a directed tree with root, then*

$$V = \mathsf{Des}(\mathsf{root}) = \bigsqcup_{n=0}^{\infty} \mathsf{Chi}^{\langle n\rangle}(\mathsf{root}).$$

PROOF. If $w \in V$, then by Proposition 2.1.4 there exists $u \in V$ such that $\{\mathsf{root}, w\} \subseteq \mathsf{Des}(u)$. By (2.1.10), this implies that $u = \mathsf{root}$, and consequently $w \in \mathsf{Des}(\mathsf{root})$. An application of (2.1.10) completes the proof. □

It turns out that the set V can be described with the help of the operation $\mathsf{Des}(\cdot)$ even in the case of a rootless directed tree.

PROPOSITION 2.1.6. *Let $\mathscr{T}$ be a rootless directed tree and $u \in V$. Then*

(i) $\mathsf{par}^k(u)$ *make sense for all* $k \in \mathbb{N}$, $\mathsf{par}^k(u) \neq \mathsf{par}^l(u)$ *for all nonnegative integers* $k \neq l$,
(ii) $\mathsf{Des}(\mathsf{par}^l(u)) \subseteq \mathsf{Des}(\mathsf{par}^j(u))$ *for all nonnegative integers* $l < j$,
(iii) $V = \bigcup_{k\in J} \mathsf{Des}(\mathsf{par}^k(u))$ *for every infinite subset* J *of* $\mathbb{N}$,
(iv) *if* $\operatorname{card}(\mathsf{Chi}(\mathsf{par}^k(u))) = 1$ *for all* $k \in \mathbb{N}$, *then* $V = \{\mathsf{par}^k(u)\}_{k=1}^{\infty} \sqcup \mathsf{Des}(u)$.

PROOF. Condition (i) follows from (2.1.10) and the fact that $\mathscr{T}$ is rootless.

(iii) We only have to prove the inclusion "$\subseteq$". Take $v \in V$. Then, by Proposition 2.1.4, there exists $w \in V$ such that $v, u \in \mathsf{Des}(w)$. Owing to (2.1.10), there exists a unique $l \in \mathbb{Z}_+$ such that $u \in \mathsf{Chi}^{\langle l\rangle}(w)$. This implies that $w = \mathsf{par}^l(u)$. Since J is infinite, there exists $j \in J$ such that $l < j$. Hence $\mathsf{par}^l(u) \in \mathsf{Chi}^{\langle j-l\rangle}(\mathsf{par}^j(u)) \subseteq \mathsf{Des}(\mathsf{par}^j(u))$, which implies that

$$v \in \mathsf{Des}(\mathsf{par}^l(u)) \overset{(2.1.5)}{\subseteq} \mathsf{Des}(\mathsf{Des}(\mathsf{par}^j(u))) \overset{(2.1.9)}{=} \mathsf{Des}(\mathsf{par}^j(u)) \subseteq \bigcup_{k\in J} \mathsf{Des}(\mathsf{par}^k(u)).$$

Looking more closely at the last line, we get (ii).

(iv) Put $W = \{\mathsf{par}^n(u)\}_{n=1}^{\infty} \sqcup \mathsf{Des}(u)$. It follows from (2.1.8) that $\mathsf{Chi}(W) \subseteq W$. Hence, by (2.1.7), $\mathsf{Des}(\mathsf{par}^k(u)) \subseteq W$ for all $k \in \mathbb{N}$. This combined with the equality $V = \bigcup_{k=1}^{\infty} \mathsf{Des}(\mathsf{par}^k(u))$ (see (iii)) yields $V = \{\mathsf{par}^n(u)\}_{n=1}^{\infty} \sqcup \mathsf{Des}(u)$, which completes the proof. □

Using Corollary 2.1.5 and arguing as in the proof of Proposition 2.1.6 (iv), we obtain a version of the latter for a directed tree with root.

PROPOSITION 2.1.7. *Let $\mathscr{T}$ be a directed tree with root, and let $u \in V^\circ$. Then there exists a unique $m \in \mathbb{N}$ such that $\mathsf{par}^m(u) = \mathsf{root}$; moreover, $\mathsf{par}^k(u) \neq \mathsf{par}^l(u)$ for all $k, l \in \{0, \ldots, m\}$ such that $k \neq l$. If $\operatorname{card}(\mathsf{Chi}(\mathsf{par}^j(u))) = 1$ for all $j \in \{1, \ldots, m\}$, then $V = \{\mathsf{par}^j(u) \colon j = 1, \ldots, m\} \sqcup \mathsf{Des}(u)$.*

As will be shown below, descenders of a fixed vertex generate a decomposition of a directed tree. The reader is referred to (2.1.1) for the necessary notation.

PROPOSITION 2.1.8. *Let $\mathscr{T}$ be a directed tree and $u \in V$. Then*

(i) $\mathscr{T}_{\mathsf{Des}(u)}$ *is a directed tree with the root* u,

(ii) *$\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ is a directed tree provided $V \setminus \mathsf{Des}(u) \neq \varnothing$; moreover, if the directed tree $\mathscr{T}$ has a root, then so does $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ and*

$$\mathsf{root}(\mathscr{T}) = \mathsf{root}(\mathscr{T}_{V\setminus\mathsf{Des}(u)}).$$

In particular, if $\mathsf{Des}(u) = \mathsf{Des}(v)$ for some $v \in V$, then $u = v$.

Proof. Certainly, the directed graphs $\mathscr{T}_{\mathsf{Des}(u)}$ and $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ satisfy the condition (ii) of Proposition 2.1.1, and they have no circuits. We show that both of them are connected which will imply that they are directed trees. Suppose that u_1 and u_2 are distinct elements of $\mathscr{T}$. By Proposition 2.1.4, there exists $w \in V$ such that $\{u_1, u_2\} \subseteq \mathsf{Des}(w)$. It follows from (2.1.10) that there exist integers $m_1, m_2 \geqslant 0$ such that $u_1 \in \mathsf{Chi}^{\langle m_1\rangle}(w)$ and $u_2 \in \mathsf{Chi}^{\langle m_2\rangle}(w)$. Since $\mathsf{par}(\mathsf{Chi}^{\langle k+1\rangle}(w)) \subseteq \mathsf{Chi}^{\langle k\rangle}(w)$ for all integers $k \geqslant 0$, we see that $\{\mathsf{par}^i(u_1)\colon i = 0, \ldots, m_1\} \subseteq \mathsf{Des}(w)$, $\{\mathsf{par}^j(u_2)\colon j = 0, \ldots, m_2\} \subseteq \mathsf{Des}(w)$ and $\mathsf{par}^{m_1}(u_1) = w = \mathsf{par}^{m_2}(u_2)$. This means that the sequence

$$\mathsf{par}^0(u_1), \mathsf{par}^1(u_1), \ldots, \mathsf{par}^{m_1-1}(u_1), w, \mathsf{par}^{m_2-1}(u_2), \ldots, \mathsf{par}^1(u_2), \mathsf{par}^0(u_2) \tag{2.1.11}$$

is an undirected path joining u_1 and u_2. If $u_1, u_2 \in \mathsf{Des}(u)$, then applying the above to $w = u$, we see that $\mathscr{T}_{\mathsf{Des}(u)}$ is connected. If $u_1, u_2 \in V \setminus \mathsf{Des}(u)$, then no vertex of the undirected path (2.1.11) belongs to $\mathsf{Des}(u)$ which can be deduced from (2.1.9). Hence the graph $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ is connected.

Suppose that, contrary to our claim, u is not a root of $\mathscr{T}_{\mathsf{Des}(u)}$. Then there exists $v \in \mathsf{Des}(u)$ such that $u \in \mathsf{Chi}(v)$. Since, by (2.1.10), there exists an integer $n \geqslant 0$ such that $v \in \mathsf{Chi}^{\langle n\rangle}(u)$, we see that $u \in \mathsf{Chi}^{\langle n+1\rangle}(u) \cap \mathsf{Chi}^{\langle 0\rangle}(u)$, which contradicts (2.1.10). Thus, by Proposition 2.1.1, $u = \mathsf{root}(\mathscr{T}_{\mathsf{Des}(u)})$, which implies the "in particular" part of the conclusion. The "moreover" part of (ii) is easily seen to be true. This completes the proof. □

Remark 2.1.9. Regarding Proposition 2.1.8, note that for every $v \in \mathsf{Des}(u)$, the set of all children of v counted in the graph $\mathscr{T}_{\mathsf{Des}(u)}$ is equal to $\mathsf{Chi}(v)$. In turn, if $v \in V \setminus \mathsf{Des}(u)$, then the set of all children of v counted in the graph $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ is equal to either $\mathsf{Chi}(v)$ if $v \neq \mathsf{par}(u)$, or $\mathsf{Chi}(v) \setminus \{u\}$ otherwise.

A subtree $\mathscr{T}$ of a directed tree $\hat{\mathscr{T}}$ containing all $\hat{\mathscr{T}}$-descendants of each vertex of $\mathscr{T}$ can be characterized as follows.

Proposition 2.1.10. *Let $\mathscr{T} = (V, E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$. Then the following conditions are equivalent*:

(i) *$\mathsf{Chi}_{\mathscr{T}}(u) = \mathsf{Chi}_{\hat{\mathscr{T}}}(u)$ for all $u \in V$,*
(ii) *$\mathsf{Chi}_{\hat{\mathscr{T}}}(u) \subseteq V$ for all $u \in V$,*
(iii) *$\mathsf{Des}_{\mathscr{T}}(u) = \mathsf{Des}_{\hat{\mathscr{T}}}(u)$ for all $u \in V$,*
(iv) *$\mathsf{Des}_{\hat{\mathscr{T}}}(u) \subset V$ for all $u \in V$,*
(v) $V = \begin{cases} \mathsf{Des}_{\hat{\mathscr{T}}}(\mathsf{root}(\mathscr{T})) & \textit{if } \mathscr{T} \textit{ has a root,} \\ \hat{V} & \textit{if } \mathscr{T} \textit{ is rootless.} \end{cases}$

Proof. (i)⇒(iii) An induction argument shows that $\mathsf{Chi}^{\langle n\rangle}_{\mathscr{T}}(u) = \mathsf{Chi}^{\langle n\rangle}_{\hat{\mathscr{T}}}(u)$ for all $n \in \mathbb{Z}_+$ and $u \in V$. This and (2.1.10), applied to $\mathscr{T}$ and $\hat{\mathscr{T}}$, lead to (iii).

(iii)⇒(v) If $\mathscr{T}$ has a root, then we apply (iii) to $u = \mathsf{root}(\mathscr{T})$, and then Corollary 2.1.5 to $\mathscr{T}$. If $\mathscr{T}$ is rootless, then employing Proposition 2.1.6 (iii) to $\mathscr{T}$ and $\hat{\mathscr{T}}$, we get

$$V = \bigcup_{k=1}^{\infty} \mathsf{Des}_{\mathscr{T}}(\mathsf{par}^k(u)) \overset{\text{(iii)}}{=} \bigcup_{k=1}^{\infty} \mathsf{Des}_{\hat{\mathscr{T}}}(\mathsf{par}^k(u)) = \hat{V}, \quad u \in V.$$

(v)⇒(ii) If $\mathscr{T}$ has a root, then by (2.1.8), applied to $\hat{\mathscr{T}}$, we have

$$\mathsf{Chi}_{\hat{\mathscr{T}}}(u) \subseteq \mathsf{Chi}_{\hat{\mathscr{T}}}(V) \overset{\text{(v)}}{=} \mathsf{Chi}_{\hat{\mathscr{T}}}(\mathsf{Des}_{\hat{\mathscr{T}}}(\mathsf{root}(\mathscr{T}))) \subseteq \mathsf{Des}_{\hat{\mathscr{T}}}(\mathsf{root}(\mathscr{T})) \overset{\text{(v)}}{=} V, \quad u \in V.$$

The other case is trivially true.

Since the implications (iii)⇒(iv), (iv)⇒(ii) and (ii)⇒(i) are obvious, the proof is complete. □

We now formulate a useful criterion for a directed tree to have finite number of leaves. Directed trees taken into consideration in Proposition 2.1.11 below are called Fredholm in Section 3.6 (cf. Definition 3.6.3).

Proposition 2.1.11. *If $\mathscr{T}$ is a directed tree such that* $\mathrm{card}(\mathsf{Chi}(u)) < \infty$ *for all $u \in V$ and* $\mathrm{card}(V_{\prec}) < \infty$ (cf. (2.1.2)), *then* $\mathrm{card}(V \setminus V') < \infty$.

Proof. We first show that there is no loss of generality in assuming that $\mathscr{T}$ has a root. Indeed, otherwise $\mathscr{T}$ is rootless, which together with $\mathrm{card}(V_{\prec}) < \infty$ and Proposition 2.1.6 (i) implies that there exists $u \in V$ such that $\mathrm{card}(\mathsf{Chi}(\mathsf{par}^k(u))) = 1$ for all $k \in \mathbb{N}$. By Proposition 2.1.6 (iv), $V = \{\mathsf{par}^k(u)\}_{k=1}^{\infty} \sqcup \mathsf{Des}(u)$. In view of (2.1.8), we have $V' = \{\mathsf{par}^k(u)\}_{k=1}^{\infty} \sqcup V'_{\mathsf{Des}(u)}$ and thus $V \setminus V' = V_{\mathsf{Des}(u)} \setminus V'_{\mathsf{Des}(u)}$. Moreover, $\mathsf{Chi}_{\mathscr{T}}(v) = \mathsf{Chi}_{\mathscr{T}_{\mathsf{Des}(u)}}(v)$ for every $v \in \mathsf{Des}(u)$, and the directed trees $\mathscr{T}$ and $\mathscr{T}_{\mathsf{Des}(u)}$ have the same branching vertexes.

Suppose now that $\mathscr{T}$ has a root. Certainly, we can assume that $\mathscr{T}$ is infinite and $V \setminus V' \neq \varnothing$. Take $w \in V \setminus V'$. Then there exists a positive integer n such that $\mathsf{par}^n(w) \in V_{\prec}$. If not, then by Corollary 2.1.5 there would exist $n \in \mathbb{N}$ such that $\mathsf{par}^n(w) = \mathsf{root}$ and $\mathrm{card}(\mathsf{Chi}(\mathsf{par}^j(w))) = 1$ for $j = 1, \ldots, n$. Since $\mathrm{card}(\mathsf{Chi}(w)) = 0$, we would deduce that $V = \{\mathsf{par}^j(w)\}_{j=0}^{n}$, a contradiction. Let $k(w)$ be the least positive integer such that $\mathsf{par}^{k(w)}(w) \in V_{\prec}$. Set $\Theta(w) = \mathsf{par}^{k(w)}(w)$. Define the equivalence relation $\mathcal{R}$ on $V \setminus V'$ by $w_1 \mathcal{R}\, w_2$ if and only if $\Theta(w_1) = \Theta(w_2)$. Denote by $[w]_{\mathcal{R}}$ the equivalence class of $w \in V \setminus V'$ with respect to $\mathcal{R}$. Using the minimality of $k(v)$ and the fact that v is a leaf of $\mathscr{T}$, one can show that the mapping $[w]_{\mathcal{R}} \ni v \mapsto \mathsf{par}^{k(v)-1}(v) \in \mathsf{Chi}(\Theta(w))$ is injective. This implies that $\mathrm{card}([w]_{\mathcal{R}}) \leqslant \mathrm{card}(\mathsf{Chi}(\Theta(w))) < \infty$. Since the mapping $(V \setminus V')/\mathcal{R} \ni [w]_{\mathcal{R}} \mapsto \Theta(w) \in V_{\prec}$ is a well defined injection, the proof is complete. □

We conclude this section by introducing an equivalence relation partitioning the given directed tree into disjoint classes composed of vertexes of the same generation.

Suppose that $\mathscr{T}$ is a directed tree. We say that vertexes $u, v \in V$ are of the *same generation*, and write $u \sim_{\mathscr{T}} v$, or shortly $u \sim v$, if there exists $n \in \mathbb{Z}_+$ such that $\mathsf{par}^n(u) = \mathsf{par}^n(v)$ (and both sides of the equality make sense). It is easily seen that $\sim$ is an equivalence relation in V. Denote by $[u]_\sim$ the equivalence class of $u \in V$ with respect to $\sim$. Note that if $u \in V^\circ$ and $v \in [u]_\sim$, then $v \in V^\circ$. Evidently $[\mathsf{root}]_\sim = \{\mathsf{root}\}$ if $\mathscr{T}$ has a root. However, if $u \in V'$ and $v \in [u]_\sim$, then it may happen that $v \notin V'$.

Given $u \in V$, we define $N(u) = N_{\mathscr{T}}(u) = \sup\{n \in \mathbb{Z}_+ \colon \mathsf{par}^n(u) \text{ makes sense}\}$. Clearly, if $n \in \mathbb{Z}_+$ and $n \leqslant N(u)$, then $\mathsf{par}^n(u)$ makes sense. Let us collect the basic properties of the relation $\sim$.

Proposition 2.1.12. *If $\mathscr{T}$ is a directed tree, then the equivalence relation $\sim$ has the following properties*[4]:

(i) $\mathsf{par}([u]_\sim) \subseteq [\mathsf{par}(u)]_\sim$ *for* $u \in V^\circ$,

(ii) *for all* $u, v \in V$, $u \sim v$ *if and only if* $\mathsf{par}([u]_\sim) = \mathsf{par}([v]_\sim)$*; moreover, if* $u, v \in V^\circ$, *then* $u \sim v$ *if and only if* $\mathsf{par}([u]_\sim) \cap \mathsf{par}([v]_\sim) \neq \varnothing$,

(iii) *if* $V^\circ \neq \varnothing$, *then* $V^\circ/\sim \ni [u]_\sim \mapsto [\mathsf{par}(u)]_\sim \in V/\sim$ *is an injection,*

(iv) $\mathsf{Chi}([\mathsf{par}(u)]_\sim) = [u]_\sim$ *for* $u \in V^\circ$,

(v) $[u]_\sim = \bigcup_{n=0}^{N(u)} \mathsf{Chi}^{\langle n\rangle}(\mathsf{par}^n(u))$ *for* $u \in V$,

(vi) $V = \bigsqcup_{n=0}^{N(u)} [\mathsf{par}^n(u)]_\sim \sqcup \bigsqcup_{n=1}^{\infty} \mathsf{Chi}^{\langle n\rangle}([u]_\sim)$ *for* $u \in V$,

(vii) *if* $u \in V$, $n \in \mathbb{N}$ *and* $w \in \mathsf{Chi}^{\langle n\rangle}([u]_\sim)$, *then* $\mathsf{Chi}^{\langle n\rangle}([u]_\sim) = [w]_\sim$,

(viii) $\mathsf{Chi}([\mathsf{par}^n(u)]_\sim) = [\mathsf{par}^{n-1}(u)]_\sim$ *for all integers n such that* $1 \leqslant n \leqslant N(u)$, *and* $\mathsf{Chi}(\mathsf{Chi}^{\langle n\rangle}([u]_\sim)) = \mathsf{Chi}^{\langle n+1\rangle}([u]_\sim)$ *for all integers* $n \geqslant 0$.

Proof. The proof of (i)–(v), being standard, is omitted.

(vi) The inclusion "$\subseteq$" (and consequently the equality) can be justified as follows. If $v \in V$, then by Proposition 2.1.4 and (2.1.10) there exist $w \in V$ and $k, l \in \mathbb{Z}_+$ such that $\mathsf{par}^k(u) = w = \mathsf{par}^l(v)$. If $k \geqslant l$, then $v \in [\mathsf{par}^{k-l}(u)]_\sim$. In the opposite case, $x := \mathsf{par}^{l-k}(v) \sim u$ and consequently $v \in \mathsf{Chi}^{\langle l-k\rangle}(x) \subseteq \mathsf{Chi}^{\langle l-k\rangle}([u]_\sim)$.

It remains to prove that the terms in (vi) are pairwise disjoint. Take $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+$ such that $m \leqslant N(u)$. We show that $[\mathsf{par}^m(u)]_\sim \cap \mathsf{Chi}^{\langle n\rangle}([u]_\sim) = \varnothing$. Indeed, otherwise there exists $w \in [\mathsf{par}^m(u)]_\sim \cap \mathsf{Chi}^{\langle n\rangle}([u]_\sim)$, which implies that $\mathsf{par}^s(w) = \mathsf{par}^{m+s}(u)$ and $\mathsf{par}^{n+t}(w) = \mathsf{par}^t(u)$ for some $s, t \in \mathbb{Z}_+$. Hence, if $m + s \leqslant t$, then

$$\mathsf{par}^{t-m}(w) = \mathsf{par}^{s+(t-(m+s))}(w) = \mathsf{par}^t(u) = \mathsf{par}^{n+t}(w).$$

By (2.1.10), this implies that $m + n = 0$, which is a contradiction. By the same kind of reasoning we see that $m + s > t$ leads to a contradiction as well.

Using (2.1.10), we verify that the sets $[\mathsf{par}^m(u)]_\sim$, $0 \leqslant m \leqslant N(u)$, are pairwise disjoint. Likewise, we show that the sets $\mathsf{Chi}^{\langle m\rangle}([u]_\sim)$, $m \in \mathbb{N}$, are pairwise disjoint.

(vii) Apply (iv) and induction on n.

(viii) is a direct consequence of (iv) and the definition of $\mathsf{Chi}^{\langle n\rangle}([u]_\sim)$. This completes the proof. □

It follows from Proposition 2.1.12 (vii) that the partition of V appearing in (vi) coincides with the one generated by the equivalence relation $\sim$.

As shown in Example 2.1.13 below, the inclusion in Proposition 2.1.12 (i) may be proper. The conditions (vi), (vii) and (viii) of Proposition 2.1.12 may suggest that there exists a sequence (finite or infinite) $\{u_n\}_n \subseteq V$ such that $\mathsf{par}(u_n) = u_{n-1}$ for all admissible n's, and $V = \bigsqcup_n [u_n]_\sim$. However, this is not always the case.

Example 2.1.13. Consider the tree $\mathscr{T} = (V, E)$ with root defined by

$$V = \{\mathsf{root}\} \sqcup \{(i, j) \colon i, j \in \mathbb{N},\, i \leqslant j\},$$

[4] $\mathsf{par}(X) := \{v \in V \colon \text{ there exists } x \in X \text{ such that } \mathsf{par}(x) \text{ makes sense and } v = \mathsf{par}(x)\}$ for $X \subseteq V$.

$$E = \big\{\big(\mathsf{root}, (1,j)\big) \colon j \in \mathbb{N}\big\} \sqcup \bigsqcup_{j=2}^{\infty} \big\{\big((i,j),(i+1,j)\big) \colon i = 1, \ldots, j-1\big\}.$$

Then $\mathsf{par}([u]_\sim) = [\mathsf{par}(u)]_\sim \setminus \{(j-1, j-1)\} \subsetneq [\mathsf{par}(u)]_\sim$ for $u = (j,j)$ with $j \geqslant 2$. One can verify that a sequence $\{u_n\}_{n=0}^\infty$ with the properties mentioned above does not exist.

2.2. Operator theory. By an operator *in* a complex Hilbert space $\mathcal{H}$ we understand a linear mapping $A\colon \mathcal{H} \supseteq \mathscr{D}(A) \to \mathcal{H}$ defined on a linear subspace $\mathscr{D}(A)$ of $\mathcal{H}$, called the *domain* of A. The kernel, the range and the adjoint of A are denoted by $\mathscr{N}(A)$, $\mathscr{R}(A)$ and A^*, respectively. A densely defined operator A in $\mathcal{H}$ is called *selfadjoint* (respectively: *normal*) if $A^* = A$ (respectively: $A^*A = AA^*$), cf. [**14, 80**]. We denote by $\|\cdot\|_A$ and $\langle\cdot,\text{-}\rangle_A$ the graph norm and the graph inner product of A, respectively, i.e., $\|f\|_A^2 = \|f\|^2 + \|Af\|^2$ and $\langle f,g\rangle_A = \langle f,g\rangle + \langle Af,Ag\rangle$ for $f,g \in \mathscr{D}(A)$. If A is closable, then the closure of A will be denoted by $\bar{A}$. A linear subspace $\mathcal{E}$ of $\mathscr{D}(A)$ is called a *core* of a closed operator A in $\mathcal{H}$ if $\overline{A|_{\mathcal{E}}} = A$ or equivalently if $\mathcal{E}$ is dense in the graph norm $\|\cdot\|_A$ in $\mathscr{D}(A)$. If A is a closed densely defined operator in $\mathcal{H}$, then $|A|$ stands for the square root of the positive selfadjoint operator A^*A. For real $\alpha > 0$, the α-root $|A|^\alpha$ of $|A|$ is defined by the Stone-von Neumann operator calculus, i.e.,

$$|A|^\alpha = \int_0^\infty x^\alpha E(\mathrm{d}x),$$

where E is the spectral measure of $|A|$ (from now on, we abbreviate $\int_{[0,\infty)}$ to $\int_0^\infty$). The operator $|A|^\alpha$ is certainly positive and selfadjoint. Given operators A and B in $\mathcal{H}$, we write $A \subseteq B$ if $\mathscr{D}(A) \subseteq \mathscr{D}(B)$ and $Ah = Bh$ for all $h \in \mathscr{D}(A)$.

In what follows, $\boldsymbol{B}(\mathcal{H})$ stands for the C^*-algebra of all bounded operators in $\mathcal{H}$ with domain $\mathcal{H}$. We write $I = I_{\mathcal{H}}$ for the identity operator on $\mathcal{H}$. Given $f,g \in \mathcal{H}$, we define the operator $f \otimes g \in \boldsymbol{B}(\mathcal{H})$ by

$$(f \otimes g)(h) = \langle h, g\rangle f, \quad h \in \mathcal{H}.$$

We say that a closed linear subspace $\mathcal{M}$ of $\mathcal{H}$ *reduces* an operator A in $\mathcal{H}$ if $PA \subseteq AP$, where $P \in \boldsymbol{B}(\mathcal{H})$ is the orthogonal projection of $\mathcal{H}$ onto $\mathcal{M}$. If $\mathcal{M}$ reduces A, then $A|_{\mathcal{M}}$ stands for the restriction of A to $\mathcal{M}$. To be more precise, $A|_{\mathcal{M}}$ is an operator in $\mathcal{M}$ such that $\mathscr{D}(A|_{\mathcal{M}}) = \mathscr{D}(A) \cap \mathcal{M}$ and $A|_{\mathcal{M}}h = Ah$ for $h \in \mathscr{D}(A|_{\mathcal{M}})$.

For the reader's convenience, we include the proof of the following result which is surely folklore.

LEMMA 2.2.1. *Let $\{e_\iota\}_{\iota\in\varXi}$ be an orthonormal basis of $\mathcal{H}$, A be a positive selfadjoint operator in $\mathcal{H}$ and $\{t_\iota\}_{\iota\in\varXi}$ be a family of nonnegative real numbers such that $e_\iota \in \mathscr{D}(A)$ and $Ae_\iota = t_\iota e_\iota$ for all $\iota \in \varXi$. Then for every real $\alpha > 0$,*

(i) *the linear span $\mathscr{E}$ of $\{e_\iota\}_{\iota\in\varXi}$ is contained in $\mathscr{D}(A^\alpha)$,*
(ii) *$\mathscr{E}$ is a core of A^α, i.e., $A^\alpha = \overline{A^\alpha|_{\mathscr{E}}}$,*
(iii) *$A^\alpha e_\iota = t_\iota^\alpha e_\iota$ for all $\iota \in \varXi$.*

Moreover, $\mathscr{R}(A)$ is closed if and only if there exists a real number $\delta > 0$ such that $t_\iota \geqslant \delta$ for every $\iota \in \varXi$ for which $t_\iota > 0$.

The operator A appearing in Lemma 2.2.1 will be called a *diagonal* operator (subordinated to the orthonormal basis $\{e_\iota\}_{\iota\in\varXi}$) with diagonal elements $\{t_\iota\}_{\iota\in\varXi}$.

Proof of Lemma 2.2.1. (i) & (iii) Define the spectral measure E on $[0,\infty)$ by

$$E(\sigma)f = \sum_{\iota\in\varXi} \chi_\sigma(t_\iota)\langle f, e_\iota\rangle e_\iota, \quad f\in\mathcal{H},\ \sigma\in\mathfrak{B}([0,\infty)), \tag{2.2.1}$$

where the above series is unconditionally convergent in norm (equivalently: convergent in norm in a generalized sense, cf. [**15**]). Using a standard measure theoretic argument, we deduce from (2.2.1) that for every Borel function $\varphi\colon [0,\infty)\to[0,\infty)$,

$$\int_0^\infty \varphi(x)\langle E(\mathrm{d}x)f, f\rangle = \sum_{\iota\in\varXi}\varphi(t_\iota)|\langle f, e_\iota\rangle|^2, \quad f\in\mathcal{H}. \tag{2.2.2}$$

It follows from (2.2.2), the selfadjointness of A and Parseval's identity that

$$\int_0^\infty x^2\langle E(\mathrm{d}x)f, f\rangle = \sum_{\iota\in\varXi} t_\iota^2|\langle f, e_\iota\rangle|^2 = \sum_{\iota\in\varXi}|\langle Af, e_\iota\rangle|^2 = \|Af\|^2<\infty, \quad f\in\mathscr{D}(A),$$

which means that $\mathscr{D}(A)\subseteq\mathscr{D}(\int_0^\infty x\,E(\mathrm{d}x))$. Arguing as above, we see that

$$\begin{aligned}\langle Af, f\rangle = \Big\langle \sum_{\iota\in\varXi}\langle Af, e_\iota\rangle e_\iota, f\Big\rangle = \sum_{\iota\in\varXi} t_\iota|\langle f, e_\iota\rangle|^2 &\overset{(2.2.2)}{=} \int_0^\infty x\langle E(\mathrm{d}x)f, f\rangle \\ &= \Big\langle \int_0^\infty xE(\mathrm{d}x)f, f\Big\rangle, \quad f\in\mathscr{D}(A).\end{aligned}$$

Both these facts imply that $A\subseteq\int_0^\infty x\,E(\mathrm{d}x)$. Since the considered operators are selfadjoint, we must have $A=\int_0^\infty x\,E(\mathrm{d}x)$ (use [**80**, Theorem 8.14(b)]), which means that E is the spectral measure of A. It follows from the measure transport theorem (cf. [**14**, Theorem 5.4.10]) that the spectral measure E_α of A^α is given by

$$E_\alpha(\sigma) = E\circ\pi_\alpha^{-1}(\sigma), \quad \sigma\in\mathfrak{B}([0,\infty)), \tag{2.2.3}$$

where $\pi_\alpha\colon[0,\infty)\to[0,\infty)$ is defined by $\pi_\alpha(x)=x^\alpha$ for $x\in[0,\infty)$. Hence, by [**14**, Theorem 6.1.3], we have

$$\mathscr{N}(t_\iota^\alpha I_{\mathcal{H}} - A^\alpha) = \mathscr{R}(E_\alpha(\{t_\iota^\alpha\})) \overset{(2.2.3)}{=} \mathscr{R}(E(\{t_\iota\})) = \mathscr{N}(t_\iota I_{\mathcal{H}} - A), \quad \iota\in\varXi.$$

This and our assumptions imposed on the operator A imply (i) and (iii).

(ii) In view of the above, it is enough to show that $\mathscr{E}$ is a core of A. For this, take a vector $f\in\mathscr{D}(A)$ which is orthogonal to $\mathscr{E}$ with respect to the graph inner product $\langle\cdot,\text{-}\rangle_A$. Then

$$0=\langle f, e_\iota\rangle + \langle Af, Ae_\iota\rangle = \langle f, e_\iota\rangle + t_\iota\langle Af, e_\iota\rangle = (1+t_\iota^2)\langle f, e_\iota\rangle, \quad \iota\in\varXi.$$

Since $\{e_\iota\}_{\iota\in\varXi}$ is an orthonormal basis of $\mathcal{H}$, we conclude that $f=0$.

If T is any normal operator in $\mathcal{H}$, then $T=T_0\oplus T_1$, where T_0 is the zero operator on $\mathscr{N}(T)$ and T_1 is an injective normal operator in $\mathcal{H}\ominus\mathscr{N}(T)$ with dense range. Hence, $\mathscr{R}(T)=\mathscr{R}(T_1)=\mathscr{D}(T_1^{-1})$, and consequently, by the inverse mapping theorem, $\mathscr{R}(T)$ is closed if and only if T_1^{-1} is bounded. Applying the above characterization to $T=A$, part (ii) to A_1^{-1} and the fact that $\mathscr{N}(A)$ equals the closed linear span of $\{e_\iota\colon t_\iota=0,\ \iota\in\varXi\}$, we get the "moreover" part of the conclusion. $\square$

Chapter 3. Fundamental properties

3.1. An invitation to weighted shifts. From now on, $\mathscr{T} = (V, E)$ is assumed to be a directed tree. Denote by $\ell^2(V)$ the Hilbert space of all square summable complex functions on V with the standard inner product

$$\langle f, g \rangle = \sum_{u \in V} f(u)\overline{g(u)}, \quad f, g \in \ell^2(V).$$

For $u \in V$, we define $e_u \in \ell^2(V)$ by

$$e_u(v) = \begin{cases} 1 & \text{if } u = v, \\ 0 & \text{otherwise.} \end{cases}$$

The set $\{e_u\}_{u \in V}$ is an orthonormal basis of $\ell^2(V)$. Denote by $\mathscr{E}_V$ the linear span of the set $\{e_u \colon u \in V\}$. Let us point out that $\ell^2(V)$ is a reproducing kernel Hilbert space which is guaranteed by the reproducing property

$$f(u) = \langle f, e_u \rangle, \quad f \in \ell^2(V), \, u \in V. \tag{3.1.1}$$

If W is a nonempty subset of V, then we regard the Hilbert space $\ell^2(W)$ as a closed linear subspace of $\ell^2(V)$ by identifying each $f \in \ell^2(W)$ with the function $\widetilde{f} \in \ell^2(V)$ which extends f and vanishes on the set $V \setminus W$.

Definition 3.1.1. Given $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, a family of complex numbers, we define the operator $S_{\boldsymbol{\lambda}}$ in $\ell^2(V)$ by

$$\begin{aligned} \mathscr{D}(S_{\boldsymbol{\lambda}}) &= \{f \in \ell^2(V) \colon \varLambda_{\mathscr{T}} f \in \ell^2(V)\}, \\ S_{\boldsymbol{\lambda}} f &= \varLambda_{\mathscr{T}} f, \quad f \in \mathscr{D}(S_{\boldsymbol{\lambda}}), \end{aligned} \tag{3.1.2}$$

where $\varLambda_{\mathscr{T}}$ is the mapping defined on functions $f \colon V \to \mathbb{C}$ by

$$(\varLambda_{\mathscr{T}} f)(v) = \begin{cases} \lambda_v \cdot f\big(\mathsf{par}(v)\big) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases} \tag{3.1.3}$$

The operator $S_{\boldsymbol{\lambda}}$ will be called a *weighted shift* on the directed tree $\mathscr{T}$ with weights $\{\lambda_v\}_{v \in V^\circ}$.

It is worth noting that the extremal situation $V^\circ = \varnothing$ is not excluded; then, by (3.1.3), $S_{\boldsymbol{\lambda}}$ is the zero operator on a one-dimensional Hilbert space.

The proof of the following fact is based only on the reproducing property of $\ell^2(V)$. Proposition 3.1.2 can also be deduced from parts (i) and (ii) of Proposition 3.1.3.

Proposition 3.1.2. *Any weighted shift $S_{\boldsymbol{\lambda}}$ on a directed tree $\mathscr{T}$ is a closed operator.*

Proof. Suppose that a sequence $\{f_n\}_{n=1}^\infty \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$ is convergent to a vector $f \in \ell^2(V)$ and the sequence $\{S_{\boldsymbol{\lambda}} f_n\}_{n=1}^\infty$ is convergent to a vector $g \in \ell^2(V)$. Take $u \in V$. By (3.1.1), the sequence $\{(S_{\boldsymbol{\lambda}} f_n)(u)\}_{n=1}^\infty$ is convergent to $g(u)$. If $u \in V^\circ$, then, again by (3.1.1), applied to $(S_{\boldsymbol{\lambda}} f_n)(u) = \lambda_u f_n(\mathsf{par}(u))$, we see that the sequence $\{(S_{\boldsymbol{\lambda}} f_n)(u)\}_{n=1}^\infty$ is convergent to $\lambda_u f(\mathsf{par}(u))$. Thus $(\varLambda_{\mathscr{T}} f)(u) = g(u)$. If $u = \mathsf{root}$, then evidently $(\varLambda_{\mathscr{T}} f)(\mathsf{root}) = 0 = g(\mathsf{root})$. Summarizing, we have shown that $\varLambda_{\mathscr{T}} f = g \in \ell^2(V)$, which means that $f \in \mathscr{D}(S_{\boldsymbol{\lambda}})$ and $g = S_{\boldsymbol{\lambda}} f$. □

Next we describe the domain and the graph norm of the operator $S_{\boldsymbol{\lambda}}$. In what follows, we adopt the conventions that $0 \cdot \infty = 0$ and $\sum_{v \in \varnothing} x_v = 0$.

Proposition 3.1.3. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions hold*:

(i) $\mathscr{D}(S_{\boldsymbol\lambda}) = \big\{f \in \ell^2(V)\colon \sum_{u\in V} \big(\sum_{v\in \mathsf{Chi}(u)} |\lambda_v|^2\big)|f(u)|^2 < \infty\Big\}$,

(ii) $\|f\|^2_{S_{\boldsymbol\lambda}} = \sum_{u\in V} \big(1 + \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2\big)|f(u)|^2$ *for all* $f \in \mathscr{D}(S_{\boldsymbol\lambda})$,

(iii) e_u *is in* $\mathscr{D}(S_{\boldsymbol\lambda})$ *if and only if* $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 < \infty$; *if* $e_u \in \mathscr{D}(S_{\boldsymbol\lambda})$, *then*

$$S_{\boldsymbol\lambda} e_u = \sum_{v\in\mathsf{Chi}(u)} \lambda_v e_v, \quad \|S_{\boldsymbol\lambda} e_u\|^2 = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2, \tag{3.1.4}$$

(iv) *if* $f \in \mathscr{D}(S_{\boldsymbol\lambda})$ *and* W *is a subset of* V, *then* $f\chi_W \in \mathscr{D}(S_{\boldsymbol\lambda})$,

(v) $S_{\boldsymbol\lambda}$ *is densely defined if and only if* $\{e_u\colon u \in V\} \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$,

(vi) $S_{\boldsymbol\lambda} = \overline{S_{\boldsymbol\lambda}|_{\mathscr{E}_V}}$ *provided* $S_{\boldsymbol\lambda}$ *is densely defined.*

Proof. If $f\colon V \to \mathbb{C}$ is any function, then

$$\begin{aligned} \sum_{u\in V} |(\varLambda_{\mathscr{T}} f)(u)|^2 &\overset{(3.1.3)}{=} \sum_{u\in V^\circ} |\lambda_u|^2 |f(\mathsf{par}(u))|^2 \\ &\overset{(2.1.3)}{=} \sum_{u\in V} \Big(\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2\Big)|f(u)|^2, \end{aligned} \tag{3.1.5}$$

which implies (i), (ii) and the first part of (iii). The proof of (3.1.4) is left to the reader (use (3.1.1)).

(iv) is a direct consequence of (i).

(v) The "if" part is clear because $\{e_u\}_{u\in V}$ is an orthonormal basis of $\ell^2(V)$. Now we justify the "only if" part. Suppose that, contrary to our claim, $e_u \notin \mathscr{D}(S_{\boldsymbol\lambda})$ for some $u \in V$. It follows from (i) and (iii) that the vector e_u is orthogonal to $\mathscr{D}(S_{\boldsymbol\lambda})$, which is a contradiction.

(vi) By (i), (ii), (iii) and (v), the Hilbert space $(\mathscr{D}(S_{\boldsymbol\lambda}), \|\cdot\|_{S_{\boldsymbol\lambda}})$ is the weighted ℓ^2 space on V with weights $\big\{1+\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2\big\}_{u\in V}$ in which the set of all complex functions on V vanishing off finite sets is dense. This means that $\mathscr{E}_V$ is a core of $S_{\boldsymbol\lambda}$, which completes the proof. □

It is worth noting that, in general, the linear space $\mathscr{E}_V$ is not invariant for a densely defined weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$. This happens when the set $\mathsf{Chi}(u)$ is infinite for at least one $u \in V$, and all the weights $\{\lambda_v\}_{v\in\mathsf{Chi}(u)}$ are nonzero (use (3.1.4)). However, if the set $\mathsf{Chi}(u)$ is finite for every $u \in V$, then $\mathscr{E}_V$ is invariant for $S_{\boldsymbol\lambda}$.

Remark 3.1.4. The unilateral and bilateral classical weighted shifts fit into our definition. Indeed, it is enough to consider directed trees $(\mathbb{Z}_+, \{(n, n+1)\colon n \in \mathbb{Z}_+\})$ and $(\mathbb{Z}, \{(n, n+1)\colon n \in \mathbb{Z}\})$, respectively (they will be shortly denoted by $\mathbb{Z}_+$ and $\mathbb{Z}$). Then the first equality in (3.1.4) reads as follows:

$$S_{\boldsymbol\lambda} e_n = \lambda_{n+1} e_{n+1}. \tag{3.1.6}$$

The reader should be aware that this is something different from the conventional notation $S_{\boldsymbol\lambda} e_n = \lambda_n e_{n+1}$ which abounds in the literature. In the present paper, we use only the new convention. Let us mention that according to Proposition 3.1.3 any weighted shift $S_{\boldsymbol\lambda}$ on the directed tree $\mathbb{Z}_+$ is densely defined and the linear span of $\{e_n\colon n \in \mathbb{Z}_+\}$ is a core of $S_{\boldsymbol\lambda}$. This fact and (3.1.6) guarantee that $S_{\boldsymbol\lambda}$ is a unilateral classical weighted shift (cf. [**58**, equality (1.7)]). The same reasoning applies to the case of a bilateral classical weighted shift.

NOTATION 3.1.5. Given a weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and $u \in V^\circ$, we denote by $S_{\boldsymbol\lambda_{\to(u)}}$ and $S_{\boldsymbol\lambda_{\leftarrow(u)}}$ the weighted shifts on directed trees $\mathscr{T}_{\mathsf{Des}(u)}$ and $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ with weights $\boldsymbol\lambda_{\to}(u) := \{\lambda_v\}_{v\in\mathsf{Des}(u)\setminus\{u\}}$ and $\boldsymbol\lambda_{\leftarrow}(u) := \{\lambda_v\}_{v\in V\setminus(\mathsf{Des}(u)\cup\mathsf{Root}(\mathscr{T}))}$, respectively (cf. Proposition 2.1.8). If $\mathscr{T}$ has a root and $u = \mathsf{root}$, then we write $S_{\boldsymbol\lambda_{\to(u)}} := S_{\boldsymbol\lambda}$.

We show that if at least one weight of the weighted shift $S_{\boldsymbol\lambda}$ on a directed tree vanishes, then $S_{\boldsymbol\lambda}$ is an orthogonal sum of two weighted shifts on directed trees.

PROPOSITION 3.1.6. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Assume that $\lambda_u = 0$ for some $u \in V^\circ$. Then*

$$S_{\boldsymbol\lambda} = S_{\boldsymbol\lambda_{\to(u)}} \oplus S_{\boldsymbol\lambda_{\leftarrow(u)}}.$$

PROOF. Since $V^\circ \neq \varnothing$, we infer from Proposition 2.1.8 that the graphs $\mathscr{T}_{\mathsf{Des}(u)}$ and $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$ are directed trees. Denote by P_u the orthogonal projection of $\ell^2(V)$ onto $\ell^2(\mathsf{Des}(u))$, i.e., $P_u f = \chi_{\mathsf{Des}(u)} f$ for $f \in \ell^2(V)$. We show that $P_u S_{\boldsymbol\lambda} \subseteq S_{\boldsymbol\lambda} P_u$. For this, let $f \in \mathscr{D}(S_{\boldsymbol\lambda})$. By Proposition 3.1.3 (iv) $P_u f \in \mathscr{D}(S_{\boldsymbol\lambda})$. If $v \in V^\circ$, then either $v \in V^\circ \setminus \{u\}$ and, consequently, by Proposition 2.1.8 $\chi_{\mathsf{Des}(u)}(v) = \chi_{\mathsf{Des}(u)}(\mathsf{par}(v))$, or $v = u$ and hence $\lambda_v = 0$. This implies that for all $v \in V^\circ$,

$$\begin{aligned}(P_u S_{\boldsymbol\lambda} f)(v) &= \chi_{\mathsf{Des}(u)}(v)(S_{\boldsymbol\lambda} f)(v) = \lambda_v \chi_{\mathsf{Des}(u)}(v) f(\mathsf{par}(v)) \\ &= \lambda_v \chi_{\mathsf{Des}(u)}(\mathsf{par}(v)) f(\mathsf{par}(v)) = \lambda_v (P_u f)(\mathsf{par}(v)) = (S_{\boldsymbol\lambda} P_u f)(v).\end{aligned}$$

In turn, if $v = \mathsf{root}(\mathscr{T})$, then by (3.1.3) we have $(P_u S_{\boldsymbol\lambda} f)(v) = 0 = (S_{\boldsymbol\lambda} P_u f)(v)$. This means that $P_u S_{\boldsymbol\lambda} \subseteq S_{\boldsymbol\lambda} P_u$. Hence $S_{\boldsymbol\lambda} = S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))} \oplus S_{\boldsymbol\lambda}|_{\ell^2(V\setminus\mathsf{Des}(u))}$. Using Proposition 2.1.8 as well as Remark 2.1.9, one can show that $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(u))} = S_{\boldsymbol\lambda_{\to(u)}}$ and $S_{\boldsymbol\lambda}|_{\ell^2(V\setminus\mathsf{Des}(u))} = S_{\boldsymbol\lambda_{\leftarrow(u)}}$. Looking at the equality $\mathscr{D}(S_{\boldsymbol\lambda}|_{\ell^2(V\setminus\mathsf{Des}(u))}) = \mathscr{D}(S_{\boldsymbol\lambda_{\leftarrow(u)}})$, the reader should be aware of the fact that

$$\sum_{v\in\mathsf{Chi}(\mathsf{par}(u))} |\lambda_v|^2 = \sum_{v\in\mathsf{Chi}_{\leftarrow}(\mathsf{par}(u))} |\lambda_v|^2 + |\lambda_u|^2,$$

where $\mathsf{Chi}_{\leftarrow}(w)$ is the set of all children of w counted in the graph $\mathscr{T}_{V\setminus\mathsf{Des}(u)}$. This completes the proof. $\square$

The injectivity of a weighted shift on a directed tree is characterized by a condition which essentially refers to the graph structure of the tree. In particular, there may happen that an injective weighted shift on a directed tree has many zero weights (which never happens for classical weighted shifts).

PROPOSITION 3.1.7. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is injective,*
(ii) *$\mathscr{T}$ is leafless and $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 > 0$ for all $u \in V$.*

It follows from Proposition 3.1.7 that a directed tree which admits an injective weighted shift must be leafless.

PROOF OF PROPOSITION 3.1.7. (i)$\Rightarrow$(ii) Suppose that contrary to our claim $\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 = 0$ for some $u \in V$ (of course, this includes the case of $\mathsf{Chi}(u) = \varnothing$). Then, by Proposition 3.1.3 (iii), $e_u \in \mathscr{D}(S_{\boldsymbol\lambda})$ and $S_{\boldsymbol\lambda} e_u = 0$, a contradiction.

(ii)⇒(i) Take $f \in \mathscr{D}(S_{\boldsymbol\lambda})$ such that $S_{\boldsymbol\lambda} f = 0$. Then, by (3.1.2) and (3.1.5), we have

$$0 = \|S_{\boldsymbol\lambda} f\|^2 = \sum_{u \in V} \Big(\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \Big) |f(u)|^2,$$

which, together with (ii), implies that $f(u) = 0$ for all $u \in V$. □

In view of Propositions 3.1.6 and 3.1.7, one can construct a reducible injective and bounded weighted shift on a directed tree with root (see (6.2.10) for examples of directed trees admitting such weighted shifts). This is again something which cannot happen for (bounded or unbounded) injective unilateral classical weighted shifts (see [**58**, Theorem (3.0)]).

The question of when a weighted shift $S_{\boldsymbol\lambda}$ on a directed tree is bounded has a simple answer. Let us point out that implication (i)⇒(ii) of Proposition 3.1.8 below is also an immediate consequence of the closed graph theorem and Proposition 3.1.2.

PROPOSITION 3.1.8. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following conditions are equivalent*:

(i) $\mathscr{D}(S_{\boldsymbol\lambda}) = \ell^2(V)$,
(ii) $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$,
(iii) $\sup_{u \in V} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 < \infty$.

If $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$, then

$$\|S_{\boldsymbol\lambda}\| = \sup_{u \in V} \|S_{\boldsymbol\lambda} e_u\| = \sup_{u \in V} \sqrt{\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2}. \tag{3.1.7}$$

PROOF. (i)⇔(iii) It follows from Proposition 3.1.3 (i) that $\mathscr{D}(S_{\boldsymbol\lambda}) = \ell^2(V)$ if and only if for every complex function f on V,

$$\sum_{u \in V} |f(u)|^2 < \infty \iff \sum_{u \in V} \Big(1 + \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2\Big) |f(u)|^2 < \infty.$$

This in turn is easily seen to be equivalent to (iii).

(ii)⇒(iii) If $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$, then by (3.1.4) we have

$$\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 = \|S_{\boldsymbol\lambda} e_u\|^2 \leqslant \|S_{\boldsymbol\lambda}\|^2, \quad u \in V. \tag{3.1.8}$$

(iii)⇒(ii) Setting $c = \sup_{u \in V} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2$, we get

$$\sum_{u \in V} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 |f(u)|^2 \leqslant c \sum_{u \in V} |f(u)|^2, \quad f \in \ell^2(V),$$

which, by Proposition 3.1.3 (i) and (3.1.5), implies that $\mathscr{D}(S_{\boldsymbol\lambda}) = \ell^2(V)$ and $\|S_{\boldsymbol\lambda}\|^2 \leqslant c$. This and (3.1.8) give (3.1.7). □

According to Propositions 2.1.2 and 3.1.8, $\sup_{v \in V^\circ} |\lambda_v| < \infty$ whenever $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$. However, in general, $\sup_{v \in V^\circ} |\lambda_v| < \infty$ does not imply $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$. What is worse, the above inequality may not imply that the operator $S_{\boldsymbol\lambda}$ is densely defined (cf. Proposition 3.1.3).

COROLLARY 3.1.9. *Let $S_{\boldsymbol\lambda}$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$, and let $\sup_{u \in V} \operatorname{card}(\mathsf{Chi}(u)) < \infty$. Then $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ if and only if $\sup_{v \in V^\circ} |\lambda_v| < \infty$.*

If we want to investigate densely defined weighted shifts on a directed tree with nonzero weights, then we have to assume that the tree under consideration is at most countable, and if the latter holds, we can always find a bounded weighted shift on it with nonzero weights.

PROPOSITION 3.1.10. *If there exists a densely defined weighted shift $S_{\boldsymbol{\lambda}}$ on a directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, then* $\mathrm{card}(V) \leqslant \aleph_0$. *Conversely, if $\mathscr{T}$ is a directed tree such that* $\mathrm{card}(V) \leqslant \aleph_0$, *then there exists a weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ with nonzero weights.*

PROOF. Suppose first that there exists a densely defined weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with nonzero weights. It follows from parts (iii) and (v) of Proposition 3.1.3 and [**15**, Corollary 19.5] that $\mathrm{card}(\mathsf{Chi}(u)) \leqslant \aleph_0$ for all $u \in V$. An induction argument combined with (2.1.6) shows that $\mathrm{card}(\mathsf{Chi}^{\langle n \rangle}(u)) \leqslant \aleph_0$ for all $u \in V$ and $n \in \mathbb{Z}_+$. Hence, by (2.1.4), $\mathrm{card}(\mathsf{Des}(u)) \leqslant \aleph_0$ for all $u \in V$. If $\mathscr{T}$ has a root, then Corollary 2.1.5 implies that $\mathrm{card}(V) \leqslant \aleph_0$. If $\mathscr{T}$ is rootless, then the same inequality holds due to Proposition 2.1.6 (iii).

Assume now that $\mathrm{card}(V) \leqslant \aleph_0$. It is then clear that for every $u \in V'$, there exists a system $\{\lambda_{u,v}\}_{v \in \mathsf{Chi}(u)}$ of positive real numbers such that $\sum_{v \in \mathsf{Chi}(u)} \lambda_{u,v}^2 \leqslant 1$. Hence, by (2.1.3), the system $\boldsymbol{\lambda} = \{\lambda_u\}_{u \in V^\circ}$ given by $\lambda_v = \lambda_{u,v}$ for $v \in \mathsf{Chi}(u)$ and $u \in V'$ is well defined, and by Proposition 3.1.8 $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$. □

We now discuss the question of when the space $\ell^2(V)$ built on a subtree $\mathscr{T}$ of a directed tree $\hat{\mathscr{T}}$ is invariant for a weighted shift on $\hat{\mathscr{T}}$ with nonzero weights. Note that if $\mathscr{T} = (V, E)$ is a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$, then $V^\circ \subseteq \hat{V}^\circ$; moreover, if $\hat{\mathscr{T}}$ has a root, so does $\mathscr{T}$. For equivalent forms of the condition (ii) of Proposition 3.1.11 below, we refer the reader to Proposition 2.1.10.

PROPOSITION 3.1.11. *Let $\mathscr{T} = (V, E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat{V}, \hat{E})$, and let $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(\hat{V}))$ be a weighted shift on $\hat{\mathscr{T}}$ with nonzero weights $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_u\}_{u \in \hat{V}^\circ}$. Then the following two conditions are equivalent*:

(i) *$\ell^2(V)$ is invariant for $S_{\hat{\boldsymbol{\lambda}}}$,*

(ii) $V = \begin{cases} \mathsf{Des}_{\hat{\mathscr{T}}}(\mathsf{root}(\mathscr{T})) & \text{if } \mathscr{T} \text{ has a root,} \\ \hat{V} & \text{if } \mathscr{T} \text{ is rootless.} \end{cases}$

Moreover, if (i) *holds, then $S_{\hat{\boldsymbol{\lambda}}}|_{\ell^2(V)} = S_{\boldsymbol{\lambda}}$, where $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a weighted shift on $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_u\}_{u \in V^\circ}$ given by $\lambda_u = \hat{\lambda}_u$ for $u \in V^\circ$.*

Observe that the implication (ii)⇒(i) remains valid without assuming that $S_{\hat{\boldsymbol{\lambda}}}$ has nonzero weights.

PROOF OF PROPOSITION 3.1.11. (i)⇒(ii) It follows from (3.1.4), applied to $S_{\hat{\boldsymbol{\lambda}}}$, that $\mathsf{Chi}_{\hat{\mathscr{T}}}(u) \subseteq V$ for every $u \in V$. Applying Proposition 2.1.10, we get (ii).

(ii)⇒(i) By Proposition 2.1.10, $\mathsf{Chi}_{\mathscr{T}}(u) = \mathsf{Chi}_{\hat{\mathscr{T}}}(u)$ for all $u \in V$. This together with (3.1.4) yields (i).

If the space $\ell^2(V)$ is invariant for $S_{\hat{\boldsymbol{\lambda}}}$, then the equality $S_{\hat{\boldsymbol{\lambda}}}|_{\ell^2(V)} = S_{\boldsymbol{\lambda}}$ can be inferred from Proposition 2.1.10 (i) and (3.1.4). □

In view of Proposition 3.1.10, the situation discussed in Proposition 3.1.11 may happen only if $\mathrm{card}(\hat{V}) \leqslant \aleph_0$.

3.2. Unitary equivalence. We begin by showing that, from the Hilbert space point of view, the study of weighted shifts on directed trees can be reduced to the case of weighted shifts with nonnegative weights. Comparing with the analogical result for classical weighted shifts, the reader will find that in the present situation the proof is much more complicated (mostly because it essentially depends on the complexity of graphs under consideration). To make the proof as clear and short as possible, we have decided to use a topological argument which seems to be of independent interest. We are aware of the fact that a more elementary proof of Theorem 3.2.1 is available. However, it is essentially longer and more technical (compare with the proof of Theorem 3.3.1).

THEOREM 3.2.1. *A weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ is unitarily equivalent to the weighted shift $S_{|\boldsymbol\lambda|}$ on $\mathscr{T}$ with weights $|\boldsymbol\lambda| = \{|\lambda_v|\}_{v\in V^\circ}$.*

PROOF. Set $\mathbb{T} = \{z \in \mathbb{C}\colon |z| = 1\}$. For $\boldsymbol\beta = \{\beta_u\}_{u\in V} \subseteq \mathbb{T}$, we define the unitary operator $U_{\boldsymbol\beta} \in \boldsymbol{B}(\ell^2(V))$ by $(U_{\boldsymbol\beta} f)(u) = \beta_u f(u)$ for $u \in V$ and $f \in \ell^2(V)$. Since $(U^*_{\boldsymbol\beta} f)(u) = \overline{\beta_u} f(u)$ for $u \in V$ and $f \in \ell^2(V)$, we infer from Proposition 3.1.3 (i) that $\mathscr{D}(S_{|\boldsymbol\lambda|}) = \mathscr{D}(S_{\boldsymbol\lambda}) = \mathscr{D}(U_{\boldsymbol\beta} S_{\boldsymbol\lambda} U^*_{\boldsymbol\beta})$. Hence, for every $f \in \mathscr{D}(S_{|\boldsymbol\lambda|})$,

$$(U_{\boldsymbol\beta} S_{\boldsymbol\lambda} U^*_{\boldsymbol\beta} f)(v) \overset{(3.1.3)}{=} \begin{cases} \lambda_v \beta_v (U^*_{\boldsymbol\beta} f)(\mathsf{par}(v)) = \lambda_v \beta_v \bar\beta_{\mathsf{par}(v)} f(\mathsf{par}(v)) & \text{if } v \in V^\circ, \\ 0 & \text{if } v = \mathsf{root}. \end{cases}$$

To complete the proof it is therefore enough to show that there exists a system $\boldsymbol\beta = \{\beta_v\}_{v\in V} \subseteq \mathbb{T}$ such that

$$\lambda_v \beta_v \bar\beta_{\mathsf{par}(v)} = |\lambda_v|, \quad v \in V^\circ. \tag{3.2.1}$$

We do this in two steps.

STEP 1. For each $(u, \gamma) \in V \times \mathbb{T}$, there exists $\{\beta_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{T}$ such that

$$\beta_u = \gamma, \tag{3.2.2}$$

$$\lambda_v \beta_v = |\lambda_v| \beta_{\mathsf{par}(v)}, \quad v \in \mathsf{Des}(u) \setminus \{u\}. \tag{3.2.3}$$

Indeed, since $\mathsf{Des}(u) \setminus \{u\} = \bigsqcup_{n=1}^\infty \mathsf{Chi}^{\langle n\rangle}(u)$ (use the decomposition (2.1.10)) and $\mathsf{par}(\mathsf{Chi}^{\langle n+1\rangle}(u)) \subseteq \mathsf{Chi}^{\langle n\rangle}(u)$, we can define the wanted system $\{\beta_v\}_{v\in\mathsf{Des}(u)}$ recursively. We begin with (3.2.2), and then having defined β_w for all $w \in \mathsf{Chi}^{\langle n\rangle}(u)$, we define β_v for every $v \in \mathsf{Chi}^{\langle n+1\rangle}(u)$ by $\beta_v = \lambda_v^{-1}|\lambda_v|\beta_{\mathsf{par}(v)}$ whenever $\lambda_v \neq 0$ and by $\beta_v = 1$ otherwise. Hence, an induction argument completes the proof of Step 1.

Step 1 and Corollary 2.1.5 enable us to solve (3.2.1) in the case when $\mathscr{T}$ has a root. We now consider the other case when $\mathscr{T}$ has no root.

STEP 2. There exists $\{\beta_v\}_{v\in V} \subseteq \mathbb{T}$ such that

$$\lambda_v \beta_v = |\lambda_v| \beta_{\mathsf{par}(v)}, \quad v \in V. \tag{3.2.4}$$

To prove this, denote by $\mathbb{T}^V$ the set of all functions from V to $\mathbb{T}$, and equip it with the topology of pointwise convergence on V. By Tihonov's theorem, $\mathbb{T}^V$ is a compact Hausdorff space. Given $u \in V$, we set

$$\varOmega_u = \Big\{\{\beta_v\}_{v\in V} \in \mathbb{T}^V \colon \lambda_v\beta_v = |\lambda_v|\beta_{\mathsf{par}(v)} \text{ for all } v \in \mathsf{Des}(u) \setminus \{u\}\Big\}.$$

Plainly, each set $\varOmega_u$ is closed in $\mathbb{T}^V$. We claim that the family $\{\varOmega_u\}_{u\in V}$ has the finite intersection property. Indeed, if W is a finite nonempty subset of V, then by

Proposition 2.1.4 there exists $u \in V$ such that $W \subseteq \mathsf{Des}(u)$. Hence

$$\mathsf{Des}(w) \subseteq \mathsf{Des}(\mathsf{Des}(u)) \overset{(2.1.9)}{=} \mathsf{Des}(u), \quad w \in W. \tag{3.2.5}$$

This implies that $\mathsf{Des}(w) \setminus \{w\} \subseteq \mathsf{Des}(u) \setminus \{u\}$ for all $w \in W$ (because the only dubious case $u \in \mathsf{Des}(w) \setminus \{w\}$, when combined with (2.1.9) and (3.2.5), yields $\mathsf{Des}(u) = \mathsf{Des}(w)$, which contradicts Proposition 2.1.8). As a consequence, $\varOmega_u \subseteq \bigcap_{w\in W} \varOmega_w$. Since, by Step 1, the set $\varOmega_u$ is nonempty, we conclude that the family $\{\varOmega_u\}_{u\in V}$ has the finite intersection property. Thus, by the compactness of $\mathbb{T}^V$, $\bigcap_{u\in V} \varOmega_u \neq \varnothing$. If $\boldsymbol{\beta} \in \bigcap_{u\in V} \varOmega_u$ and $v \in V$, then $\boldsymbol{\beta} \in \varOmega_{\mathsf{par}(v)}$, which implies (3.2.4). □

Remark 3.2.2. We now discuss the question of uniqueness of solutions in Steps 1 and 2 of the proof of Theorem 3.2.1. Certainly, we lose uniqueness if some of the weights λ_v vanish. The situation is quite different if the weights of $S_{\boldsymbol{\lambda}}$ are nonzero.

In Steps 1′ and 2′ below we assuming that $\lambda_v \neq 0$ for all $v \in V^{\circ}$.

Step 1′. If $u \in V$ is fixed and $\{\beta_v\}_{v\in\mathsf{Des}(u)}, \{\beta'_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{T}$ satisfy (3.2.3), then $\beta'_v = \gamma\beta_v$ for all $v \in \mathsf{Des}(u)$ with $\gamma = \beta'_u\overline{\beta_u}$.

The proof of Step 1′ is similar to that of Step 1.

Step 2′. Suppose that $\mathscr{T}$ has no root. If $u \in V$ is fixed and $\boldsymbol{\beta}, \boldsymbol{\beta}' \in \mathbb{T}^V$ satisfy (3.2.4), then $\beta'_v = \gamma\beta_v$ for all $v \in V$ with $\gamma = \beta'_u\overline{\beta_u}$.

Indeed, if $v \in V$, then by Proposition 2.1.4 there exists $w \in V$ such that $\{u, v\} \subseteq \mathsf{Des}(w)$. According to Step 1′, there exists $\gamma \in \mathbb{T}$ such that $\beta'_x = \gamma\beta_x$ for all $x \in \mathsf{Des}(w)$. Since $u, v \in \mathsf{Des}(w)$, we get $\beta'_v = \gamma\beta_v$ and $\beta'_u = \gamma\beta_u$, which yields $\gamma = \beta'_u\overline{\beta_u}$. This means that γ does not depend on w.

3.3. Circularity. We now prove that a densely defined weighted shift on a directed tree is a circular operator. The definition of a circular operator was introduced in [**6**]. As shown in [**6**, Proposition 1.3], an irreducible bounded operator on a complex Hilbert space is circular if and only if it possesses a circulating C_0-group of unitary operators. The latter property was then undertaken by Mlak and used as the definition of circularity in the more general context of unbounded operators (cf. [**59, 60, 61**]).

Theorem 3.3.1. *Let $S_{\boldsymbol{\lambda}}$ be a weighted shift on a directed tree $\mathscr{T}$. Then for every $c \in \mathbb{R}$ there exists $\boldsymbol{\theta} = \{\theta_u\}_{u\in V} \subseteq \mathbb{R}$ such that*

$$\mathrm{e}^{-\mathrm{i}tN} S_{\boldsymbol{\lambda}} \mathrm{e}^{\mathrm{i}tN} = \mathrm{e}^{\mathrm{i}tc} S_{\boldsymbol{\lambda}}, \quad t \in \mathbb{R}, \tag{3.3.1}$$

where $N = N_{\boldsymbol{\theta}}$ is a unique selfadjoint operator in $\ell^2(V)$ such that $\{e_u\}_{u\in V} \subseteq \mathscr{D}(N)$ and $Ne_u = \theta_u e_u$ for all $u \in V$.

Proof. Fix $c \in \mathbb{R}$ and take $\boldsymbol{\theta} = \{\theta_u\}_{u\in V} \subseteq \mathbb{R}$. Define the operator $N = N_{\boldsymbol{\theta}}$ in $\ell^2(V)$ by $\mathscr{D}(N) = \{f \in \ell^2(V) \colon \boldsymbol{\theta} f \in \ell^2(V)\}$ and $Nf = \boldsymbol{\theta} f$ for $f \in \mathscr{D}(N)$, where $(\boldsymbol{\theta} f)(u) = \theta_u f(u)$ for $u \in V$. Clearly, N is selfadjoint, $\{e_v\}_{v\in V} \subseteq \mathscr{D}(N)$ and $Ne_u = \theta_u e_u$ for all $u \in V$. By Lemma 2.2.1, such N is unique. Moreover, $\{\mathrm{e}^{\mathrm{i}tN}\}_{t\in\mathbb{R}}$ is a C_0-group of unitary operators. Using an explicit description of the spectral measure of N (as in (2.2.1)), we verify that $\mathrm{e}^{\mathrm{i}tN} e_u = \mathrm{e}^{\mathrm{i}t\theta_u} e_u$ for all $u \in V$ and $t \in \mathbb{R}$. Hence, for all $u \in V$ and $f \in \ell^2(V)$, we have

$$(\mathrm{e}^{\mathrm{i}tN} f)(u) \overset{(3.1.1)}{=} \langle \mathrm{e}^{\mathrm{i}tN} f, e_u\rangle = \langle f, \mathrm{e}^{-\mathrm{i}tN} e_u\rangle = \mathrm{e}^{\mathrm{i}t\theta_u}\langle f, e_u\rangle = \mathrm{e}^{\mathrm{i}t\theta_u} f(u).$$

In view of Proposition 3.1.3, this implies that $\mathscr{D}(\mathrm{e}^{-\mathrm{i}tN}S_{\boldsymbol\lambda}\mathrm{e}^{\mathrm{i}tN}) = \mathscr{D}(S_{\boldsymbol\lambda})$ for all $t \in \mathbb{R}$. Moreover, if $f \in \mathscr{D}(S_{\boldsymbol\lambda})$, then

$$(\mathrm{e}^{-\mathrm{i}tN}S_{\boldsymbol\lambda}\mathrm{e}^{\mathrm{i}tN}f)(v) = \mathrm{e}^{-\mathrm{i}t\theta_v}\lambda_v(\mathrm{e}^{\mathrm{i}tN}f)(\mathsf{par}(v)) = \mathrm{e}^{\mathrm{i}t(\theta_{\mathsf{par}(v)}-\theta_v)}(S_{\boldsymbol\lambda}f)(v), \quad v \in V^\circ,$$

and $(\mathrm{e}^{-\mathrm{i}tN}S_{\boldsymbol\lambda}\mathrm{e}^{\mathrm{i}tN}f)(v) = (S_{\boldsymbol\lambda}f)(v) = 0$ for $v = \mathsf{root}$. Consequently, it remains to prove that there exists a solution $\{\theta_u\}_{u\in V} \subseteq \mathbb{R}$ of the equation

$$\mathrm{e}^{\mathrm{i}t(\theta_{\mathsf{par}(v)}-\theta_v)} = \mathrm{e}^{\mathrm{i}tc}, \quad v \in V^\circ,\ t \in \mathbb{R}. \tag{3.3.2}$$

Differentiating both sides of the above equality with respect to t at $t = 0$, we see that (3.3.2) is equivalent to

$$\theta_{\mathsf{par}(v)} - \theta_v = c, \quad v \in V^\circ. \tag{3.3.3}$$

Take $u \in V$. As in the proof of Step 1 of Theorem 3.2.1, we show that for each $\zeta \in \mathbb{R}$, there exists a unique system $\{\theta_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{R}$ such that $\theta_u = \zeta$ and

$$\theta_{\mathsf{par}(v)} - \theta_v = c, \quad v \in \mathsf{Des}(u) \setminus \{u\}. \tag{3.3.4}$$

Therefore, if $\{\theta_v\}_{v\in\mathsf{Des}(u)}, \{\theta'_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{R}$ are solutions of (3.3.4), then so is the system $\{\theta_v + (\theta'_u - \theta_u)\}_{v\in\mathsf{Des}(u)}$ with the same value at u as $\{\theta'_v\}_{v\in\mathsf{Des}(u)}$. Thus, by uniqueness, we have $\theta'_v = \theta_v + (\theta'_u - \theta_u)$ for all $v \in \mathsf{Des}(u)$.

Suppose now that $u_0 \in \mathsf{Des}(u)$ and $\zeta \in \mathbb{R}$. Take any solution $\{\theta_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{R}$ of (3.3.4). Then $\{\theta_v + (\zeta - \theta_{u_0})\}_{v\in\mathsf{Des}(u)}$ is a solution of (3.3.4) with value ζ at u_0. Note that such solution is unique. Indeed, if $\{\theta_v\}_{v\in\mathsf{Des}(u)}, \{\theta'_v\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{R}$ are solutions of (3.3.4) with the same value ζ at u_0, then by the previous paragraph there exists $a \in \mathbb{R}$ such that $\theta'_v = \theta_v + a$ for all $v \in \mathsf{Des}(u)$. Substituting $v = u_0$, we obtain $a = 0$, which gives the required uniqueness.

In view of the above discussion and Corollary 2.1.5, the equation (3.3.3) has a solution in the case when $\mathscr{T}$ has a root.

Let us pass to the other case when $\mathscr{T}$ has no root. Fix any $u_0 \in V$. If $u \in V$ is such that $u_0 \in \mathsf{Des}(u)$, then by the penultimate paragraph there exists a unique system $\{\theta_{u,v}\}_{v\in\mathsf{Des}(u)} \subseteq \mathbb{R}$ solving (3.3.4) and such that $\theta_{u,u_0} = 0$. We now define the required solution $\{\theta_v\}_{v\in V} \subseteq \mathbb{R}$ of (3.3.3) as follows. If $v \in V$, then by Proposition 2.1.4 there exists $u \in V$ such that $v, u_0 \in \mathsf{Des}(u)$. Define $\theta_v = \theta_{u,v}$ (note that $\{\theta_v\}_{v\in V}$ depends of u_0). First we prove that this definition is correct. So, let $u' \in V$ be such that $v, u_0 \in \mathsf{Des}(u')$. We claim that $\theta_{u,v} = \theta_{u',v}$. Indeed, by Proposition 2.1.4, there exists $w \in V$ such that $u, u' \in \mathsf{Des}(w)$. Then, by (2.1.9), we have $\mathsf{Des}(u) \cup \mathsf{Des}(u') \subseteq \mathsf{Des}(w)$. As in the proof of Theorem 3.2.1, we show that the last inclusion implies $\mathsf{Des}(u) \setminus \{u\} \subseteq \mathsf{Des}(w) \setminus \{w\}$ and $\mathsf{Des}(u') \setminus \{u'\} \subseteq \mathsf{Des}(w)\setminus\{w\}$. Hence the system $\{\theta_{w,x}\}_{x\in\mathsf{Des}(u)}$ is a solution of (3.3.4), and $\theta_{w,u_0} = 0$. By uniqueness property, we must have $\theta_{w,x} = \theta_{u,x}$ for all $x \in \mathsf{Des}(u)$. Substituting $x = v$, we get $\theta_{w,v} = \theta_{u,v}$. Applying similar argument to the system $\{\theta_{w,x}\}_{x\in\mathsf{Des}(u')}$, we obtain $\theta_{w,v} = \theta_{u',v}$, which shows that our definition of $\{\theta_v\}_{v\in V}$ is correct. Using Proposition 2.1.4 again, we find $\tilde u \in V$ such that $\{v, \mathsf{par}(v), u_0\} \subseteq \mathsf{Des}(\tilde u)$. Since $v \in \mathsf{Des}(\mathsf{par}(v)) \setminus \{\mathsf{par}(v)\} \subseteq \mathsf{Des}(\tilde u) \setminus \{\tilde u\}$, we have $\theta_v = \theta_{\tilde u,v}$, $\theta_{\mathsf{par}(v)} = \theta_{\tilde u,\mathsf{par}(v)}$ and consequently $\theta_{\mathsf{par}(v)} - \theta_v = c$. As $v \in V$ is arbitrary, the proof is complete. □

A careful inspection of the proof of Theorem 3.3.1 shows that if a tree $\mathscr{T}$ has no root, then for every $c \in \mathbb{R}$ and for every $(u_0, \zeta) \in V \times \mathbb{R}$ there exists a unique system $\{\theta_v\}_{v\in V} \subseteq \mathbb{R}$ such that $\theta_{u_0} = \zeta$ and $\theta_{\mathsf{par}(v)} - \theta_v = c$ for all $v \in V$.

We conclude this section by mentioning some spectral properties of weighted shifts on a directed tree. The fact that the spectral radius of the weighted adjacency operator $A(G)$ of an infinite directed fuzzy graph G belongs to the approximate point spectrum of $A(G)$ was proved in [**31**, Theorem 6.1] (see also [**62**] for the case of infinite undirected graphs). The weighted adjacency operator $A(G)$ is defined in [**31**] for a directed fuzzy graph a vertex of which may have more than one server (read: parent). The reader should also convince himself that in the case of a directed tree our weighted shift operator coincides with the weighted adjacency operator $A(G)$ (note that only the bounded weighted adjacency operators are taken into consideration in [**31**]).

Given a densely defined closed operator A in a complex Hilbert space $\mathcal{H}$, we denote by $\sigma(A)$ and $\sigma_{\mathrm{ap}}(A)$ the spectrum and the approximate point spectrum of A, respectively. If $A \in \boldsymbol{B}(\mathcal{H})$, then $r(A)$ stands for the spectral radius of A. A subset σ of $\mathbb{C}$ is said to be *circular* if

$$\mathrm{e}^{\mathrm{i}t} z \in \sigma \text{ for all } t \in \mathbb{R} \text{ and } z \in \sigma.$$

COROLLARY 3.3.2. *If $S_{\boldsymbol{\lambda}}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$, then the sets $\sigma(S_{\boldsymbol{\lambda}})$, $\sigma(S_{\boldsymbol{\lambda}}^*)$, $\sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}})$ and $\sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}}^*)$ are circular. Moreover, if $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$, then $\{z \in \mathbb{C} \colon |z| = r(S_{\boldsymbol{\lambda}})\} \subseteq \sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}}) \cap \sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}}^*)$.*

PROOF. Let N be as in Theorem 3.3.1 with $c = 1$. Since the operators $\mathrm{e}^{\mathrm{i}tN}$, $t \in \mathbb{R}$, are unitary and $(\mathrm{e}^{\mathrm{i}tN})^* = \mathrm{e}^{-\mathrm{i}tN}$ for all $t \in \mathbb{R}$, we deduce that

$$\sigma(S_{\boldsymbol{\lambda}}) = \sigma\big((\mathrm{e}^{\mathrm{i}tN})^* S_{\boldsymbol{\lambda}} \mathrm{e}^{\mathrm{i}tN}\big) \overset{(3.3.1)}{=} \mathrm{e}^{\mathrm{i}t} \sigma(S_{\boldsymbol{\lambda}}), \quad t \in \mathbb{R},$$

which means that $\sigma(S_{\boldsymbol{\lambda}})$ and consequently $\sigma(S_{\boldsymbol{\lambda}}^*)$ are circular. The same reasoning shows that the approximate point spectra of $S_{\boldsymbol{\lambda}}$ and $S_{\boldsymbol{\lambda}}^*$ are circular.

Suppose now that the operator $S_{\boldsymbol{\lambda}}$ is bounded. Since $\sigma(S_{\boldsymbol{\lambda}})$ is a nonempty compact subset of $\mathbb{C}$, there exists $z_0 \in \sigma(S_{\boldsymbol{\lambda}})$ such that $|z_0| = r(S_{\boldsymbol{\lambda}})$. By the circularity of $\sigma(S_{\boldsymbol{\lambda}})$, we see that the circle $\varGamma := \{z \in \mathbb{C} \colon |z| = r(S_{\boldsymbol{\lambda}})\}$ is contained in $\sigma(S_{\boldsymbol{\lambda}})$. This means that $\varGamma$ is a subset of the boundary of $\sigma(S_{\boldsymbol{\lambda}})$. Hence, by [**18**, Corollary XI.1.2], $\varGamma \subseteq \sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}}) \cap \sigma_{\mathrm{ap}}(S_{\boldsymbol{\lambda}}^*)$. This completes the proof. $\square$

The properties of spectra mentioned in Corollary 3.3.2 are true for general circular operators. For the reader's convenience we have included their proofs. Certainly, other spectra of a circular operator, like the point spectrum, the continuous spectrum and the residual spectrum are circular.

3.4. Adjoints and moduli. We begin by giving an explicit description of the adjoint $S_{\boldsymbol{\lambda}}^*$ of $S_{\boldsymbol{\lambda}}$. Recall that $\mathscr{E}_V$ is the linear span of the set $\{e_u \colon u \in V\}$.

PROPOSITION 3.4.1. *If $S_{\boldsymbol{\lambda}}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, then the following assertions hold:*

(i) $\sum_{v \in \mathsf{Chi}(u)} |\lambda_v f(v)| < \infty$ *for all $u \in V$ and $f \in \ell^2(V)$,*

(ii) $\mathscr{E}_V \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ *and*

$$S_{\boldsymbol{\lambda}}^* e_u = \begin{cases} \overline{\lambda_u} e_{\mathsf{par}(u)} & \text{if } u \in V^\circ, \\ 0 & \text{if } u = \mathsf{root}, \end{cases} \tag{3.4.1}$$

(iii) $(S_{\boldsymbol{\lambda}}^* f)(u) = \sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v)$ *for all $u \in V$ and $f \in \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$,*

(iv) $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) = \big\{f \in \ell^2(V) \colon \sum_{u \in V} \big|\sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v)\big|^2 < \infty\big\}$,

(v) $\|f\|_{S_{\boldsymbol{\lambda}}^*}^2 = \sum_{u \in V} \big(|f(u)|^2 + \big|\sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v)\big|^2\big)$ *for all $f \in \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$,*

(vi) $\ell^2(\mathsf{Chi}(u)) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ *for every $u \in V$,*

(vii) $S^*_{\boldsymbol\lambda} = \overline{S^*_{\boldsymbol\lambda}|_{\mathscr{E}_V}}$.

Proof. (i) By the Cauchy-Schwarz inequality and Proposition 3.1.3 (iii) and (v), we have

$$\Big(\sum_{v\in\mathsf{Chi}(u)} |\lambda_v f(v)|\Big)^2 \leqslant \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \sum_{v\in\mathsf{Chi}(u)} |f(v)|^2 < \infty, \quad u \in V,\ f \in \ell^2(V).$$

(ii) Since

$$\langle S_{\boldsymbol\lambda} f, e_{\mathsf{root}}\rangle \overset{(3.1.1)}{=} (S_{\boldsymbol\lambda} f)(\mathsf{root}) = 0, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}),$$

we get $e_{\mathsf{root}} \in \mathscr{D}(S^*_{\boldsymbol\lambda})$ and $S^*_{\boldsymbol\lambda} e_{\mathsf{root}} = 0$. Assume now that $u \in V^\circ$. Then

$$\langle S_{\boldsymbol\lambda} f, e_u\rangle = (S_{\boldsymbol\lambda} f)(u) = \lambda_u \cdot f\big(\mathsf{par}(u)\big) = \langle f, \overline{\lambda_u} e_{\mathsf{par}(u)}\rangle, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}),$$

which implies that $e_u \in \mathscr{D}(S^*_{\boldsymbol\lambda})$ and $S^*_{\boldsymbol\lambda} e_u = \overline{\lambda_u} e_{\mathsf{par}(u)}$.

(iii) Applying (3.1.1) and Proposition 3.1.3 (v), we deduce that

$$\begin{aligned}(S^*_{\boldsymbol\lambda} f)(u) = \langle S^*_{\boldsymbol\lambda} f, e_u\rangle &= \langle f, S_{\boldsymbol\lambda} e_u\rangle \overset{(3.1.4)}{=} \Big\langle f, \sum_{v\in\mathsf{Chi}(u)} \lambda_v e_v\Big\rangle \\ &= \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v}\langle f, e_v\rangle = \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v), \quad u \in V, f \in \mathscr{D}(S^*_{\boldsymbol\lambda}).\end{aligned}$$

(iv) If $f \in \mathscr{D}(S^*_{\boldsymbol\lambda})$, then

$$\sum_{u\in V}\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 \overset{\text{(iii)}}{=} \sum_{u\in V} |(S^*_{\boldsymbol\lambda} f)(u)|^2 = \|S^*_{\boldsymbol\lambda} f\|^2 < \infty. \tag{3.4.2}$$

Conversely, if f belongs to the right-hand side of (iv), then (i) enables us to define the function $g\colon V \to \mathbb{C}$ by

$$g(u) := \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v), \quad u \in V. \tag{3.4.3}$$

By our assumption, $g \in \ell^2(V)$. Moreover,

$$\begin{aligned}\langle S_{\boldsymbol\lambda} h, f\rangle &= \sum_{u\in V} (S_{\boldsymbol\lambda} h)(u)\cdot\overline{f(u)} \overset{(3.1.3)}{=} \sum_{u\in V^\circ} h(\mathsf{par}(u))\lambda_u \overline{f(u)} \\ &\overset{(2.1.3)}{=} \sum_{u\in V}\sum_{v\in\mathsf{Chi}(u)} h(\mathsf{par}(v))\lambda_v\overline{f(v)} = \sum_{u\in V} h(u) \sum_{v\in\mathsf{Chi}(u)} \lambda_v \overline{f(v)} \\ &\overset{(3.4.3)}{=} \sum_{u\in V} h(u)\overline{g(u)} = \langle h, g\rangle, \quad h \in \mathscr{D}(S_{\boldsymbol\lambda}),\end{aligned}$$

which implies that $f \in \mathscr{D}(S^*_{\boldsymbol\lambda})$ and $g = S^*_{\boldsymbol\lambda} f$.

Assertion (v) is a direct consequence of (3.4.2).

Assertion (vi) follows from (i), (iv) and Proposition 2.1.2.

(vii) Take $f \in \mathscr{D}(S^*_{\boldsymbol\lambda})$ which is orthogonal to $\{e_w\colon w \in V\}$ with respect to the graph inner product $\langle\cdot,\text{-}\rangle_{S^*_{\boldsymbol\lambda}}$. If $w = \mathsf{root}$, then

$$0 = \langle f, e_{\mathsf{root}}\rangle_{S^*_{\boldsymbol\lambda}} \overset{(3.4.1)}{=} f(\mathsf{root}). \tag{3.4.4}$$

We show that f vanishes on $\mathsf{Chi}(u)$ for every $u \in V$, which in view of (3.4.4) and (2.1.3) will complete the proof. Fixing $u \in V$, we get

$$0 = \langle f, e_w\rangle_{S^*_{\boldsymbol\lambda}} \overset{(3.4.1)}{=} f(w) + \langle S^*_{\boldsymbol\lambda} f, \overline{\lambda_w} e_{\mathsf{par}(w)}\rangle = f(w) + \lambda_w (S^*_{\boldsymbol\lambda} f)(u), \quad w \in \mathsf{Chi}(u).$$

Multiplying the left and the right side of the above chain of equalities by $\overline{f(w)}$ and then summing over all $w \in \mathsf{Chi}(u)$, we deduce from (iii) that

$$0 = \Big(\sum_{w\in\mathsf{Chi}(u)} |f(w)|^2\Big) + |(S^*_{\boldsymbol\lambda} f)(u)|^2,$$

which implies that f vanishes on $\mathsf{Chi}(u)$. This completes the proof. □

It follows from Proposition 3.4.1 (ii) that the linear space $\mathscr{E}_V$ is always invariant for the adjoint $S^*_{\boldsymbol\lambda}$ of a densely defined weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$. This is opposed to the fact that $\mathscr{E}_V$ may not be invariant for $S_{\boldsymbol\lambda}$ (see the comments after Proposition 3.1.3).

Remark 3.4.2. We now show that the adjoint of a unilateral classical weighted shift is a weighted shift on a very particular directed tree. Indeed, let us regard $\mathbb{Z}_- := \{\ldots, -2, -1, 0\}$ as a subtree of the directed tree $\mathbb{Z}$ (cf. Remark 3.1.4). Certainly, $\mathbb{Z}_-$ is a rootless directed tree with only one leaf 0. Let $S_{\boldsymbol\lambda}$ be a weighted shift on $\mathbb{Z}_-$ with weights $\boldsymbol\lambda = \{\lambda_{-n}\}_{n=0}^\infty$. Then by Proposition 3.1.3 and the equality (3.1.4) the operator $S_{\boldsymbol\lambda}$ is densely defined, $S_{\boldsymbol\lambda} e_{-n} = \lambda_{-(n-1)} e_{-(n-1)}$ for all $n \in \mathbb{N}$, and $S_{\boldsymbol\lambda} e_0 = 0$. This fact combined with Proposition 3.1.3 (vi) guarantees that $S_{\boldsymbol\lambda}$ can be thought of as the adjoint of the unilateral classical weighted shift with weights $\{\bar\lambda_{-(n-1)}\}_{n=1}^\infty$ (cf. [**58**, equality (1.11)]).

We now describe powers of the modulus of $S_{\boldsymbol\lambda}$.

PROPOSITION 3.4.3. *If $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then for every real $\alpha > 0$,*

(i) *$\mathscr{E}_V \subseteq \mathscr{D}(|S_{\boldsymbol\lambda}|^\alpha)$,*
(ii) *$\mathscr{E}_V$ is a core of $|S_{\boldsymbol\lambda}|^\alpha$, i.e., $|S_{\boldsymbol\lambda}|^\alpha = \overline{|S_{\boldsymbol\lambda}|^\alpha|_{\mathscr{E}_V}}$,*
(iii) *$|S_{\boldsymbol\lambda}|^\alpha e_u = \|S_{\boldsymbol\lambda} e_u\|^\alpha e_u$ for $u \in V$,*
(iv) *$(|S_{\boldsymbol\lambda}|^\alpha f)(u) = \|S_{\boldsymbol\lambda} e_u\|^\alpha f(u)$ for $u \in V$ and $f \in \mathscr{D}(|S_{\boldsymbol\lambda}|^\alpha)$.*

PROOF. (i)–(iii) Applying Proposition 3.4.1 (iii), we get

$$\begin{aligned}(S^*_{\boldsymbol\lambda} S_{\boldsymbol\lambda} f)(u) &= \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} (S_{\boldsymbol\lambda} f)(v) \overset{(3.1.3)}{=} \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 f(\mathsf{par}(v)) \\ &= \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 f(u) \overset{(3.1.4)}{=} \|S_{\boldsymbol\lambda} e_u\|^2 f(u), \quad u \in V,\ f \in \mathscr{D}(S^*_{\boldsymbol\lambda} S_{\boldsymbol\lambda}).\end{aligned} \tag{3.4.5}$$

Now, we show that $\mathscr{E}_V \subseteq \mathscr{D}(S^*_{\boldsymbol\lambda} S_{\boldsymbol\lambda})$. Indeed, if $u \in V$, then

$$\sum_{w\in V}\Big|\sum_{v\in\mathsf{Chi}(w)} \overline{\lambda_v}(S_{\boldsymbol\lambda} e_u)(v)\Big|^2 \overset{(3.1.3)}{=} \sum_{w\in V}\Big|\sum_{v\in\mathsf{Chi}(w)} |\lambda_v|^2 e_u(w)\Big|^2 \overset{(3.1.4)}{=} \|S_{\boldsymbol\lambda} e_u\|^4 < \infty,$$

which, by Proposition 3.4.1 (iv), implies that $S_{\boldsymbol\lambda} e_u \in \mathscr{D}(S^*_{\boldsymbol\lambda})$. Hence, (3.4.5) leads to

$$S^*_{\boldsymbol\lambda} S_{\boldsymbol\lambda} e_u = \|S_{\boldsymbol\lambda} e_u\|^2 e_u, \quad u \in V. \tag{3.4.6}$$

In view of Proposition 3.1.2 and [**80**, Theorem 5.39], the operator $S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}$ is positive and selfadjoint. Using (3.4.6) and applying Lemma 2.2.1 with $\alpha/2$ in place of α to the operator $S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}$, we obtain (i), (ii) and (iii).

(iv) It follows from (3.1.1) that for all $f \in \mathscr{D}(|S_{\boldsymbol\lambda}|^\alpha)$ and $u \in V$,

$$(|S_{\boldsymbol\lambda}|^\alpha f)(u) = \langle |S_{\boldsymbol\lambda}|^\alpha f, e_u\rangle \overset{\text{(i)}}{=} \langle f, |S_{\boldsymbol\lambda}|^\alpha e_u\rangle \overset{\text{(iii)}}{=} \langle f, \|S_{\boldsymbol\lambda} e_u\|^\alpha e_u\rangle = \|S_{\boldsymbol\lambda} e_u\|^\alpha f(u).$$

This completes the proof. □

COROLLARY 3.4.4. *A weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ is an isometry on $\ell^2(V)$ if and only if $\sum_{v\in \mathsf{Chi}(u)} |\lambda_v|^2 = 1$ for all $u \in V$.*

PROOF. Apply Propositions 3.1.3, 3.1.8 and 3.4.3 (with $\alpha = 2$). □

Since the compactness and the membership in the Schatten-von Neumann p-class depend on analogous properties of the modulus of the operator in question (cf. [**69, 67**]), the following corollary is an immediate consequence of Proposition 3.4.3. Below, the limit $\lim_{u\in V}$ and the sum $\sum_{u\in V}$ are understood in a generalized sense.

COROLLARY 3.4.5. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ and let $p \in [1,\infty)$. Then the following two assertions hold.*

(i) *$S_{\boldsymbol\lambda}$ is compact if and only if $\lim_{u\in V} \|S_{\boldsymbol\lambda} e_u\| = 0$,*
(ii) *$S_{\boldsymbol\lambda}$ is in the Schatten p-class if and only if $\sum_{u\in V} \|S_{\boldsymbol\lambda} e_u\|^p < \infty$.*

3.5. The polar decomposition. We now describe the polar decomposition of a densely defined weighted shift on a directed tree. Recall, that if T is a closed densely defined operator in a complex Hilbert space $\mathcal{H}$, then there exists a unique partial isometry $U \in \boldsymbol{B}(\mathcal{H})$ with initial space $\overline{\mathscr{R}(|T|)}$ such that $T = U|T|$. Such decomposition is called the *polar decomposition* of T (see e.g., [**14**, Theorem 8.1.2]). If $T = U|T|$ is the *polar decomposition* of T, then the final space of U equals $\overline{\mathscr{R}(T)}$.

PROPOSITION 3.5.1. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, and let $S_{\boldsymbol\lambda} = U|S_{\boldsymbol\lambda}|$ be the polar decomposition of $S_{\boldsymbol\lambda}$. Then $|S_{\boldsymbol\lambda}|$ is the diagonal operator subordinated to the orthonormal basis $\{e_u\}_{u\in V}$ with diagonal elements $\{\|S_{\boldsymbol\lambda} e_u\|\}_{u\in V}$, and U is the weighted shift $S_{\boldsymbol\pi}$ on $\mathscr{T}$ with weights $\boldsymbol\pi = \{\pi_u\}_{u\in V^\circ}$ given by*[5]

$$\pi_u = \begin{cases} \dfrac{\lambda_u}{\|S_{\boldsymbol\lambda} e_{\mathsf{par}(u)}\|} & \text{if } \mathsf{par}(u) \in V_{\boldsymbol\lambda}^+, \\ 0 & \text{if } \mathsf{par}(u) \notin V_{\boldsymbol\lambda}^+, \end{cases} \qquad u \in V^\circ, \tag{3.5.1}$$

where $V_{\boldsymbol\lambda}^+ := \{u \in V\colon \|S_{\boldsymbol\lambda} e_u\| > 0\}$. Moreover, the following assertions hold:

(i) *$\mathscr{N}(U) = \mathscr{N}(S_{\boldsymbol\lambda}) = \mathscr{N}(|S_{\boldsymbol\lambda}|) = \ell^2(V \setminus V_{\boldsymbol\lambda}^+)$ and $\overline{\mathscr{R}(S_{\boldsymbol\lambda}^*)} = \ell^2(V_{\boldsymbol\lambda}^+)$,*

(ii) $$\mathscr{N}(S_{\boldsymbol\lambda}^*) = \begin{cases} \langle e_{\mathsf{root}}\rangle \oplus \bigoplus_{u\in V'} \big(\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol\lambda^u\rangle\big) & \text{if } \mathscr{T} \text{ has a root,} \\ \bigoplus_{u\in V'} \big(\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol\lambda^u\rangle\big) & \text{otherwise,} \end{cases}$$

where $\boldsymbol\lambda^u \in \ell^2(\mathsf{Chi}(u))$ is given by $\boldsymbol\lambda^u\colon \mathsf{Chi}(u) \ni v \to \lambda_v \in \mathbb{C}$, and $\langle \boldsymbol\lambda^u\rangle$ is the linear span of $\{\boldsymbol\lambda^u\}$,

(iii) *the initial space of U equals $\ell^2(V_{\boldsymbol\lambda}^+)$,*
(iv) *$\overline{\mathscr{R}(U)} = \overline{\mathscr{R}(S_{\boldsymbol\lambda})} = \bigoplus_{u\in V'} \langle \boldsymbol\lambda^u\rangle$.*

---

[5] For simplicity, we suppress the explicit dependence of $\boldsymbol\pi$ on $\boldsymbol\lambda$ in the notation.

Proof. It follows from Proposition 3.1.2 that $S_{\boldsymbol\lambda}$ is closed. By Lemma 2.2.1 and Proposition 3.4.3, $|S_{\boldsymbol\lambda}|$ is the diagonal operator subordinated to the orthonormal basis $\{e_u\}_{u\in V}$ with diagonal elements $\{\|S_{\boldsymbol\lambda} e_u\|\}_{u\in V}$. Hence, the initial space $\overline{\mathscr{R}(|S_{\boldsymbol\lambda}|)}$ of U equals $\ell^2(V_{\boldsymbol\lambda}^+)$. This means that (iii) holds. As a consequence of (iii), we obtain (i). If $u \in V_{\boldsymbol\lambda}^+$, then

$$(3.5.2)\quad Ue_u = \frac{1}{\|S_{\boldsymbol\lambda} e_u\|} U|S_{\boldsymbol\lambda}|e_u = \frac{1}{\|S_{\boldsymbol\lambda} e_u\|} S_{\boldsymbol\lambda} e_u \overset{(3.1.4)}{=} \sum_{v\in\mathsf{Chi}(u)} \frac{\lambda_v}{\|S_{\boldsymbol\lambda} e_u\|} e_v = \sum_{v\in\mathsf{Chi}(u)} \frac{\lambda_v}{\|S_{\boldsymbol\lambda} e_{\mathsf{par}(v)}\|} e_v.$$

In turn, if $u \in V \setminus V_{\boldsymbol\lambda}^+$, then by (i) $e_u \in \ell^2(V \setminus V_{\boldsymbol\lambda}^+) = \mathscr{N}(U)$, and so $Ue_u = 0$. Using (3.5.1) and Proposition 3.1.8, we see that $S_{\boldsymbol\pi} \in \boldsymbol{B}(\ell^2(V))$. Since, by (3.1.4), (3.5.1) and (3.5.2), both operators U and $S_{\boldsymbol\pi}$ coincide on basic vectors e_u, $u \in V$, we deduce that $U = S_{\boldsymbol\pi}$.

We now describe the final space of U. Consider first the case when $\mathscr{T}$ has a root. It follows from Proposition 3.4.1 (iii) and (iv), and Proposition 2.1.2 that

$$(3.5.3)\quad \mathscr{N}(S_{\boldsymbol\lambda}^*) = \{f \in \ell^2(V) \colon \langle f|_{\mathsf{Chi}(u)}, \boldsymbol\lambda^u\rangle = 0 \text{ for all } u \in V'\} = \langle e_{\mathsf{root}}\rangle \oplus \bigoplus_{u\in V'} \Big(\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol\lambda^u\rangle\Big).$$

Since, by Proposition 2.1.2, $\ell^2(V) = \langle e_{\mathsf{root}}\rangle \oplus \bigoplus_{u\in V'} \ell^2(\mathsf{Chi}(u))$, we deduce from (3.5.3) that $\overline{\mathscr{R}(S_{\boldsymbol\lambda})} = \bigoplus_{u\in V'} \langle \boldsymbol\lambda^u\rangle$. If $\mathscr{T}$ is rootless, then (3.5.3) holds with $\langle e_{\mathsf{root}}\rangle$ removed. As a consequence, we get the same formula for $\overline{\mathscr{R}(S_{\boldsymbol\lambda})}$. This proves assertions (ii) and (iv), and hence completes the proof. □

Using the description of the polar decomposition of $S_{\boldsymbol\lambda}$ given in Proposition 3.5.1 and the last paragraph of the proof of Lemma 2.2.1 (with $A = |S_{\boldsymbol\lambda}|$), we see that $\mathscr{R}(S_{\boldsymbol\lambda}) = S_{\boldsymbol\pi}(\mathscr{R}(|S_{\boldsymbol\lambda}|)) = S_{\boldsymbol\pi}(\mathscr{D}(A_1^{-1}))$, where A_1^{-1} is a diagonal operator with diagonal elements $\big\{\frac{1}{\|S_{\boldsymbol\lambda} e_u\|}\big\}_{u\in V_{\boldsymbol\lambda}^+}$. The details are left to the reader.

3.6. Fredholm directed trees. We begin by describing weighted shifts on directed trees with closed ranges. Recall that $V_{\boldsymbol\lambda}^+ = \{u \in V \colon \|S_{\boldsymbol\lambda} e_u\| > 0\} \subseteq V'$.

Proposition 3.6.1. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent*:

(i) *$\mathscr{R}(S_{\boldsymbol\lambda})$ is closed,*
(ii) *$\mathscr{R}(S_{\boldsymbol\lambda}^*)$ is closed,*
(iii) *there exists a real number $\delta > 0$ such that $\|S_{\boldsymbol\lambda} e_u\| \geqslant \delta$ for every $u \in V_{\boldsymbol\lambda}^+$.*

Proof. Equivalence (i)⇔(ii) holds for general closed densely defined Banach space operators (cf. [**38**, Theorem IV.1.2]).

(i)⇔(iii) It follows from the polar decomposition of $S_{\boldsymbol\lambda}$ that $\mathscr{R}(S_{\boldsymbol\lambda})$ is closed if and only if $\mathscr{R}(|S_{\boldsymbol\lambda}|)$ is closed. This fact combined with Proposition 3.5.1 and Lemma 2.2.1 completes the proof. □

Recall that a closed densely defined operator T in a complex Hilbert space $\mathcal{H}$ is said to be *Fredholm* if its range is closed and the spaces $\mathscr{N}(T)$ and $\mathscr{N}(T^*)$ are finite dimensional. It is well known that a closed densely defined operator T in $\mathcal{H}$ is Fredholm if and only if the spaces $\mathscr{N}(T)$ and $\mathcal{H}/\mathscr{R}(T)$ are finite dimensional (cf.

[**38**, Corollary IV.1.13]). The *index* $\mathrm{ind}(T)$ of a Fredholm operator T is given by $\mathrm{ind}(T) = \dim \mathscr{N}(T) - \dim \mathscr{N}(T^*)$ (see [**38**, §4.2] and [**37**] for the case of unbounded operators and [**18**] for bounded ones).

Fredholm weighted shifts on directed trees and their indexes can be characterized as follows (below we adopt the convention that $\inf \varnothing = \infty$).

PROPOSITION 3.6.2. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is a Fredholm operator,*
(ii) *$\mathfrak{c}(S_{\boldsymbol\lambda}) > 0$ and $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$, where*

$$\begin{aligned}\mathfrak{b}(S_{\boldsymbol\lambda}) &:= \sum_{u \in V_{\boldsymbol\lambda}^+} (\mathrm{card}(\mathsf{Chi}(u)) - 1) + \sum_{u \in V' \setminus V_{\boldsymbol\lambda}^+} \mathrm{card}(\mathsf{Chi}(u)),\\ \mathfrak{c}(S_{\boldsymbol\lambda}) &:= \inf\{|\lambda_u|\colon u \in V^\circ, \lambda_u \neq 0, \mathrm{card}(\mathsf{Chi}(\mathsf{par}(u))) = 1\},\end{aligned}$$

(iii) *$\mathfrak{c}(S_{\boldsymbol\lambda}) > 0$, $\mathrm{card}(\mathsf{Chi}(u)) < \infty$ for all $u \in V$, $\mathrm{card}(V_\prec) < \infty$ (see (2.1.2) for the definition of $V_\prec$) and $\mathrm{card}(V' \setminus V_{\boldsymbol\lambda}^+) < \infty$.*

If $S_{\boldsymbol\lambda}$ is Fredholm, then $\mathfrak{a}(S_{\boldsymbol\lambda}) := \mathrm{card}(V \setminus V_{\boldsymbol\lambda}^+) < \infty$ and

$$\mathrm{ind}(S_{\boldsymbol\lambda}) = \begin{cases} \mathfrak{a}(S_{\boldsymbol\lambda}) - \mathfrak{b}(S_{\boldsymbol\lambda}) - 1 & \text{if } \mathscr{T} \text{ has a root,}\\ \mathfrak{a}(S_{\boldsymbol\lambda}) - \mathfrak{b}(S_{\boldsymbol\lambda}) & \text{otherwise.}\end{cases} \tag{3.6.1}$$

PROOF. First we prove that if $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$, then $\mathfrak{a}(S_{\boldsymbol\lambda}) < \infty$, $\mathrm{card}(\mathsf{Chi}(u)) < \infty$ for all $u \in V$ and $\mathrm{card}(V_\prec) < \infty$. We begin by recalling that $V_{\boldsymbol\lambda}^+ \subseteq V' \subseteq V$. Then an easy computation shows that

$$\begin{aligned}&\mathrm{card}(V_\prec) \leqslant \mathfrak{b}(S_{\boldsymbol\lambda}) < \infty,\\ &\mathrm{card}(\mathsf{Chi}(u)) \leqslant \mathfrak{b}(S_{\boldsymbol\lambda}) + 1 < \infty, \quad u \in V.\end{aligned}$$

Hence, by Proposition 2.1.11, $\mathrm{card}(V \setminus V') < \infty$. Since $\mathrm{card}(V' \setminus V_{\boldsymbol\lambda}^+) \leqslant \mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$, we get $\mathfrak{a}(S_{\boldsymbol\lambda}) = \mathrm{card}(V \setminus V') + \mathrm{card}(V' \setminus V_{\boldsymbol\lambda}^+) < \infty$. This proves our claim.

Reversing the above reasoning we deduce that if $\mathrm{card}(\mathsf{Chi}(u)) < \infty$ for all $u \in V$, $\mathrm{card}(V_\prec) < \infty$ and $\mathrm{card}(V' \setminus V_{\boldsymbol\lambda}^+) < \infty$, then $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$ and consequently $\mathfrak{a}(S_{\boldsymbol\lambda}) < \infty$.

(i)⇒(ii) Employing Proposition 3.5.1, we see that $\mathfrak{a}(S_{\boldsymbol\lambda}) < \infty$, $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$ and (3.6.1) holds. By Proposition 3.6.1, we conclude that $\mathfrak{c}(S_{\boldsymbol\lambda}) > 0$.

(ii)⇒(i) In view of the first paragraph of this proof, the implication (ii)⇒(i) can be deduced from Propositions 3.5.1 and 3.6.1.

(ii)⇔(iii) Apply two first paragraphs of this proof. □

Owing to Proposition 3.6.2, a densely defined weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and with closed range is Fredholm if and only if $\mathrm{card}(\mathsf{Chi}(u)) < \infty$ for all $u \in V$ and $\mathrm{card}(V_\prec) < \infty$. Moreover,

$$\mathrm{ind}(S_{\boldsymbol\lambda}) = \begin{cases} \mathrm{card}(V \setminus V') - 1 - \sum_{u \in V'} \big(\mathrm{card}(\mathsf{Chi}(u)) - 1\big) & \text{if } \mathscr{T} \text{ has a root,}\\[2ex] \mathrm{card}(V \setminus V') - \sum_{u \in V'} \big(\mathrm{card}(\mathsf{Chi}(u)) - 1\big) & \text{otherwise.}\end{cases} \tag{3.6.2}$$

A trivial verification shows that

(3.6.3)

$$\operatorname{ind}(S_{\boldsymbol\lambda}) = \begin{cases} \operatorname{card}((V\setminus V')\sqcup V_{\prec}) - 1 - \sum\limits_{u\in V_{\prec}} \operatorname{card}(\mathsf{Chi}(u)) & \text{if } \mathscr{T} \text{ has a root,}\\ \operatorname{card}((V\setminus V')\sqcup V_{\prec}) - \sum\limits_{u\in V_{\prec}} \operatorname{card}(\mathsf{Chi}(u)) & \text{otherwise.}\end{cases}$$

Noting that the right-hand side of (3.6.2) does not depend on $S_{\boldsymbol\lambda}$, we propose the following definition.

Definition 3.6.3. A directed tree $\mathscr{T}$ such that $\operatorname{card}(\mathsf{Chi}(u)) < \infty$ for all $u \in V$ and $\operatorname{card}(V_{\prec}) < \infty$ is called *Fredholm.* The right-hand side of (3.6.2) (which is equal to the right-hand side of (3.6.3)) is denoted by $\operatorname{ind}(\mathscr{T})$ and called the *index* of a Fredholm directed tree $\mathscr{T}$.

The definition of $\operatorname{ind}(S_{\boldsymbol\lambda})$ is correct due to Proposition 2.1.11. We can rephrase Definition 3.6.3 as follows: a directed tree is Fredholm if and only if it has finitely many branching vertexes and each branching vertex has finitely many children. Moreover, each Fredholm directed tree has finitely many leaves. As a consequence, we see that there exists countably many non-isomorphic Fredholm directed trees.

The following is a beneficial excerpt from the proof of Proposition 3.6.2.

Proposition 3.6.4. *If $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$, then $\mathscr{T}$ is Fredholm and $\mathfrak{a}(S_{\boldsymbol\lambda}) < \infty$.*

It is worth mentioning that for every Fredholm directed tree $\mathscr{T}$ we may construct a bounded Fredholm weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ with nonzero weights. Indeed, in view of Propositions 3.1.8 and 3.6.2, the weighted shift $S_{\boldsymbol\lambda}$ with weights $\lambda_u \equiv 1$ meets our requirements.

The assertion (i) of Lemma 3.6.5 below shows that after cutting off a leaf of a Fredholm directed tree $\mathscr{T}$ the index of the trimmed tree remains the same as that of $\mathscr{T}$. The assertion (ii) states that after cutting off a straight infinite branch $\mathsf{Des}(u)$ from a simply branched leafless subtree $\mathsf{Des}(w)$ of $\mathscr{T}$, where u is a child of w, the index of the trimmed tree enlarges by 1. Finally, the assertion (iii) says that after cutting off a trunk of a rootless $\mathscr{T}$ the index of the trimmed tree decreases by 1. For the definition of the subgraph $\mathcal{G}_W$, we refer the reader to (2.1.1).

Lemma 3.6.5. *If $\mathscr{T}$ is a Fredholm directed tree, then the following assertions hold.*

(i) *If $w \in V\setminus V'$ and $V_w := V\setminus\{w\} \neq \varnothing$, then $\operatorname{ind}(\mathscr{T}) = \operatorname{ind}(\mathscr{T}_{V_w})$.*
(ii) *If $w \in V$ is such that $\operatorname{card}(\mathsf{Chi}(w)) \geqslant 2$ and $\operatorname{card}(\mathsf{Chi}(v)) = 1$ for all $v \in \mathsf{Des}(w)\setminus\{w\}$, then $\operatorname{ind}(\mathscr{T}_{V_{[u]}}) = \operatorname{ind}(\mathscr{T}) + 1$ for every $u \in \mathsf{Chi}(w)$, where $V_{[u]} := V\setminus\mathsf{Des}(u)$.*
(iii) *If $\mathscr{T}$ is rootless and $w\in V$ is such that $V = \{\mathsf{par}^n(w)\}_{n=1}^\infty \sqcup \mathsf{Des}(w)$ (such w always exists), then $\operatorname{ind}(\mathscr{T}_{\mathsf{Des}(w)}) = \operatorname{ind}(\mathscr{T}) - 1$.*

Note that by Proposition 2.1.8, $\mathscr{T}_{V_w}$, $\mathscr{T}_{V_{[u]}}$ and $\mathscr{T}_{\mathsf{Des}(w)}$ are directed trees.

Proof of Lemma 3.6.5. (i) It is clear that

$$E_{\mathscr{T}} = E_{\mathscr{T}_{V_w}} \sqcup \{(\mathsf{par}(w), w)\}$$

($\mathsf{par}(w)$ makes sense because $V_w \neq \varnothing$). This implies that $\mathsf{Chi}_{\mathscr{T}}(u) = \mathsf{Chi}_{\mathscr{T}_{V_w}}(u)$ for $u \in V_w\setminus\{\mathsf{par}(w)\}$ and $\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w)) = \mathsf{Chi}_{\mathscr{T}_{V_w}}(\mathsf{par}(w)) \sqcup \{w\}$. Consider first the case when $\mathrm{card}(\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w))) \geqslant 2$. Then $V' = V'_w$ and consequently $\mathrm{card}(V \setminus V') = \mathrm{card}(V_w \setminus V'_w) + 1$. This altogether implies that $\mathrm{ind}(\mathscr{T}) = \mathrm{ind}(\mathscr{T}_{V_w})$. In turn, if $\mathrm{card}(\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w))) = 1$, then arguing as above, we see that $V' = V'_w \sqcup \{\mathsf{par}(w)\}$ and $\mathsf{par}(w) \in V_w \setminus V'_w$, hence that $V \setminus V' = \{w\} \sqcup \big((V_w \setminus V'_w) \setminus \{\mathsf{par}(w)\}\big)$ and $\mathrm{card}(V \setminus V') = \mathrm{card}(V_w \setminus V'_w)$, and finally that $\mathrm{ind}(\mathscr{T}) = \mathrm{ind}(\mathscr{T}_{V_w})$. In particular, we have the following equalities

$$\mathrm{card}(V_w \setminus V'_w) = \begin{cases} \mathrm{card}(V \setminus V') - 1 & \text{if } \mathrm{card}(\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w))) \geqslant 2, \\ \mathrm{card}(V \setminus V') & \text{if } \mathrm{card}(\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w))) = 1. \end{cases} \tag{3.6.4}$$

(ii) It is plain that $V' = V'_{[u]} \sqcup \mathsf{Des}(u)$ and consequently that $V \setminus V' = V_{[u]} \setminus V'_{[u]}$. If $v \in V'_{[u]} \setminus \{w\}$, then $\mathsf{Chi}_{\mathscr{T}}(v) = \mathsf{Chi}_{\mathscr{T}_{V_{[u]}}}(v)$. In turn, if $v = w$, then $\mathsf{Chi}_{\mathscr{T}}(v) = \mathsf{Chi}_{\mathscr{T}_{V_{[u]}}}(v) \sqcup \{u\}$. Finally, if $v \in \mathsf{Des}(u)$, then $\mathrm{card}(\mathsf{Chi}_{\mathscr{T}}(v)) = 1$. All this implies that $\mathrm{ind}(\mathscr{T}_{V_{[u]}}) = \mathrm{ind}(\mathscr{T}) + 1$.

(iii) Arguing as in the first paragraph of the proof of Proposition 2.1.11 and using (3.6.3), we show that $\mathrm{ind}(\mathscr{T}_{\mathsf{Des}(w)}) = \mathrm{ind}(\mathscr{T}) - 1$ (remember that the directed tree $\mathscr{T}_{\mathsf{Des}(w)}$ has a root, while $\mathscr{T}$ not). This completes the proof. □

We now show that the index of a Fredholm directed tree does not exceed 1.

LEMMA 3.6.6. *If $\mathscr{T}$ is a Fredholm directed tree, then*

(i) $\mathrm{ind}(\mathscr{T}) \leqslant 1$ *provided $\mathscr{T}$ is rootless,*
(ii) $\mathrm{ind}(\mathscr{T}) = 0$ *provided $\mathscr{T}$ is finite,*
(iii) $\mathrm{ind}(\mathscr{T}) \leqslant -1$ *provided $\mathscr{T}$ has a root and is infinite.*

PROOF. Without loss of generality, we can assume that $\mathscr{T}$ is infinite (otherwise $\mathrm{ind}(\mathscr{T}) = 0$). If $\mathscr{T}$ is rootless, then by using assertion (iii) of Lemma 3.6.5 we are reduced to showing that $\mathrm{ind}(\mathscr{T}) \leqslant -1$ whenever $\mathscr{T}$ has a root and is infinite (note that the trimmed tree may be finite, however this case has been covered).

If $\mathscr{T}$ is leafless, then by (3.6.2) we have

$$\mathrm{ind}(\mathscr{T}) = -1 - \sum_{u \in V_{\prec}} \big(\mathrm{card}(\mathsf{Chi}(u)) - 1\big) \leqslant -1. \tag{3.6.5}$$

Suppose now that $\mathscr{T}$ is not leafless (however still infinite and with root). Then $\varkappa(\mathscr{T}) := \mathrm{card}(V \setminus V') \geqslant 1$. We claim that there exists an infinite subtree $\check{\mathscr{T}}$ of $\mathscr{T}$ such that $\varkappa(\check{\mathscr{T}}) = \varkappa(\mathscr{T}) - 1$ and $\mathrm{ind}(\check{\mathscr{T}}) = \mathrm{ind}(\mathscr{T})$. To prove our claim, take any $w \in V \setminus V'$. Let k be the least positive integer such that $\mathsf{par}^k(w) \in V_{\prec}$ (the existence of such k is justified in the second paragraph of the proof of Proposition 2.1.11). Applying (3.6.4) and assertion (i) of Lemma 3.6.5 k times, we get the required subtree $\check{\mathscr{T}}$, which proves our claim. Finally, employing the reduction procedure $\mathscr{T} \rightsquigarrow \check{\mathscr{T}}$ $\varkappa(\mathscr{T})$ times, we find an infinite subtree $\check{\mathscr{T}}$ of $\mathscr{T}$ such that $\mathrm{ind}(\mathscr{T}) = \mathrm{ind}(\check{\mathscr{T}})$ and $\varkappa(\check{\mathscr{T}}) = 0$. Since the latter means that $\check{\mathscr{T}}$ is leafless, we deduce from (3.6.5) that $\mathrm{ind}(\mathscr{T}) = \mathrm{ind}(\check{\mathscr{T}}) \leqslant -1$. This completes the proof. □

Applying (3.6.3) and Lemma 3.6.6 we get the following estimates for Fredholm directed trees.

COROLLARY 3.6.7. *If $\mathscr{T}$ is an infinite Fredholm directed tree, then*

$$\operatorname{card}(V \setminus V') + \operatorname{card}(V_{\prec}) \leqslant \sum_{u \in V_{\prec}} \operatorname{card}(\mathsf{Chi}(u)) \quad \textit{if } \mathscr{T} \textit{ has a root,}$$

$$\operatorname{card}(V \setminus V') + \operatorname{card}(V_{\prec}) \leqslant 1 + \sum_{u \in V_{\prec}} \operatorname{card}(\mathsf{Chi}(u)) \quad \textit{otherwise.}$$

Note that for every integer $k \leqslant 1$ there exists a Fredholm directed tree $\mathscr{T}$ such that $\operatorname{ind}(\mathscr{T}) = k$. Indeed, it is clear that $\mathbb{Z}_-$, $\mathbb{Z}$ and $\mathbb{Z}_+$ are Fredholm directed trees (see Remarks 3.1.4 and 3.4.2 for appropriate definitions), and $\operatorname{ind}(\mathbb{Z}_-) = 1$, $\operatorname{ind}(\mathbb{Z}) = 0$ and $\operatorname{ind}(\mathbb{Z}_+) = -1$. The directed trees $\mathbb{Z}$ and $\mathbb{Z}_+$ are leafless and they have no branching vertexes. If we consider any rootless and leafless directed tree $\mathscr{T}$ with $|k|$ branching vertexes ($k \leqslant 0$) each of which having exactly two children, then $\operatorname{ind}(\mathscr{T}) = k$.

It turns out that the index of a Fredholm weighted shift on a directed tree does not exceed 1 even if some weights of $S_{\boldsymbol{\lambda}}$ are zero.

THEOREM 3.6.8. *If $S_{\boldsymbol{\lambda}}$ is a Fredholm weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^{\circ}}$, then $\mathscr{T}$ is Fredholm and* $\operatorname{ind}(S_{\boldsymbol{\lambda}}) = \operatorname{ind}(\mathscr{T}) \leqslant 1$.

PROOF. It follows from Proposition 3.6.2 that $\mathscr{T}$ is Fredholm and

$$\operatorname{card}(V' \setminus V_{\boldsymbol{\lambda}}^{+}) \leqslant \mathfrak{a}(S_{\boldsymbol{\lambda}}) < \infty. \tag{3.6.6}$$

Hence, the following equalities hold

$$\begin{aligned}
\mathfrak{a}(S_{\boldsymbol{\lambda}}) - \mathfrak{b}(S_{\boldsymbol{\lambda}}) \quad &= \operatorname{card}(V \setminus V_{\boldsymbol{\lambda}}^{+}) - \sum_{u \in V_{\boldsymbol{\lambda}}^{+}} (\operatorname{card}(\mathsf{Chi}(u)) - 1) - \sum_{u \in V' \setminus V_{\boldsymbol{\lambda}}^{+}} \operatorname{card}(\mathsf{Chi}(u)) \\
&\overset{(3.6.6)}{=} \operatorname{card}(V \setminus V') + \operatorname{card}(V' \setminus V_{\boldsymbol{\lambda}}^{+}) \\
&\qquad - \sum_{u \in V'} (\operatorname{card}(\mathsf{Chi}(u)) - 1) + \sum_{u \in V' \setminus V_{\boldsymbol{\lambda}}^{+}} (\operatorname{card}(\mathsf{Chi}(u)) - 1) \\
&\qquad\qquad - \sum_{u \in V' \setminus V_{\boldsymbol{\lambda}}^{+}} \operatorname{card}(\mathsf{Chi}(u)) \\
&= \operatorname{card}(V \setminus V') - \sum_{u \in V'} (\operatorname{card}(\mathsf{Chi}(u)) - 1).
\end{aligned}$$

By (3.6.1) and (3.6.2), this implies that $\operatorname{ind}(S_{\boldsymbol{\lambda}}) = \operatorname{ind}(\mathscr{T})$. Employing Lemma 3.6.6 completes the proof. $\square$

It may happen that a directed tree $\mathscr{T}$ is Fredholm but not every densely defined weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with closed range is Fredholm (e.g., the directed tree $\mathscr{T} = \mathscr{T}_{2,0}$ defined in (6.2.10) has the required properties; it is enough to consider a weighted shift $S_{\boldsymbol{\lambda}}$ on $\mathscr{T}$ with infinite number of zero weights, whose nonzero weights are uniformly separated from zero).

The theory of semi-Fredholm operators can be implemented into the context of weighted shift operators on directed trees as well. Recall that a closed densely defined operator T in a complex Hilbert space $\mathcal{H}$ is said to be *left semi-Fredholm* if its range is closed and $\dim \mathscr{N}(T) < \infty$. If T^* is left semi-Fredholm, then T is called *right semi-Fredholm*. It is well known that a closed densely defined operator T in $\mathcal{H}$ is right semi-Fredholm if and only if T has closed range and $\dim \mathscr{N}(T^*) < \infty$. Hence, in view of [**38**, Corollary IV.1.13], a closed densely defined operator T in $\mathcal{H}$ is right semi-Fredholm if and only if the quotient space $\mathcal{H}/\mathscr{R}(T)$ is finite dimensional. If T is either left semi-Fredholm or right semi-Fredholm, then its *index* $\mathrm{ind}(T)$ is defined by $\mathrm{ind}(T) = \dim \mathscr{N}(T) - \dim \mathscr{N}(T^*)$. In general, $\mathrm{ind}(T) \in \mathbb{Z} \sqcup \{\pm\infty\}$ (cf. [**38, 18**]).

Proposition 3.6.9. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions hold.*

(i) *$S_{\boldsymbol\lambda}$ is left semi-Fredholm if and only if $S_{\boldsymbol\lambda}$ has closed range* (*cf.* Proposition 3.6.1) *and* $\mathrm{card}(V \setminus V_{\boldsymbol\lambda}^+) < \infty$. *Moreover, if $S_{\boldsymbol\lambda}$ is left semi-Fredholm, then* $\mathrm{ind}(S_{\boldsymbol\lambda}) \in \{n \in \mathbb{Z}\colon n \leqslant 1\} \sqcup \{-\infty\}$.

(ii) *$S_{\boldsymbol\lambda}$ is right semi-Fredholm if and only if it is Fredholm.*

Proof. Assertion (i) is a direct consequence Proposition 3.5.1 (i) and Theorem 3.6.8.

(ii) If $S_{\boldsymbol\lambda}$ right semi-Fredholm, then by Proposition 3.5.1 (ii) $\mathfrak{b}(S_{\boldsymbol\lambda}) < \infty$. This, in view of Proposition 3.6.4, implies that $\mathfrak{a}(S_{\boldsymbol\lambda}) < \infty$. Thus $S_{\boldsymbol\lambda}$ is Fredholm. $\square$

Suppose that $S_{\boldsymbol\lambda}$ is any densely defined weighted shift on a directed tree *with nonzero weights*. It follows from Proposition 3.6.9 that $S_{\boldsymbol\lambda}$ is left semi-Fredholm if and only if it has closed range and the directed tree $\mathscr{T}$ has finitely many leaves. As a consequence, we see that if $S_{\boldsymbol\lambda}$ has closed range and $\mathscr{T}$ is leafless, then $S_{\boldsymbol\lambda}$ is always left semi-Fredholm. Certainly, it many happen that $S_{\boldsymbol\lambda}$ is left semi-Fredholm but not Fredholm (in fact, this happens more frequently). For example, the isometric weighted shift $S_{\boldsymbol\lambda}$ on the leafless directed tree $\mathscr{T}_{\infty,0}$ defined in Example 6.3.3 (with one branching vertex ω such that $\mathrm{card}(\mathsf{Chi}(\omega)) = \aleph_0$) is left semi-Fredholm and $\mathrm{ind}(S_{\boldsymbol\lambda}) = -\infty$.

Chapter 4. Inclusions of domains

4.1. When $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$? Our next aim is to characterize the circumstances under which the inclusion $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$ holds.

THEOREM 4.1.1. *If $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the following conditions are equivalent*:

(i) $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$,
(ii) *there exists $c > 0$ such that*

$$\sum_{v\in\mathsf{Chi}(u)} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant c, \quad u \in V. \tag{4.1.1}$$

PROOF. (i)$\Rightarrow$(ii) Recall that by Proposition 3.1.3 (v), $\mathscr{E}_V \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. Suppose that $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$. By Proposition 3.1.2, the normed space $(\mathscr{D}(S_{\boldsymbol\lambda}), \|\cdot\|_{S_{\boldsymbol\lambda}})$ is complete. The normed space $(\mathscr{D}(S_{\boldsymbol\lambda}^*), \|\cdot\|_{S_{\boldsymbol\lambda}^*})$ is complete as well (cf. [**14**, Theorems 3.2.1 and 3.3.2]). Applying the closed graph theorem to the identity embedding mapping $(\mathscr{D}(S_{\boldsymbol\lambda}), \|\cdot\|_{S_{\boldsymbol\lambda}}) \to (\mathscr{D}(S_{\boldsymbol\lambda}^*), \|\cdot\|_{S_{\boldsymbol\lambda}^*})$ we see that there exists a positive real number c such that

$$\|f\|_{S_{\boldsymbol\lambda}^*}^2 \leqslant c\,\|f\|_{S_{\boldsymbol\lambda}}^2, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}).$$

By Propositions 3.4.1 (v) and 3.1.3 (ii), the above inequality implies that

$$\begin{aligned} \sum_{u\in V}\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 &\leqslant \|f\|_{S_{\boldsymbol\lambda}^*}^2 \leqslant c\|f\|_{S_{\boldsymbol\lambda}}^2 \\ &\overset{(3.1.4)}{=} c\sum_{u\in V}\big(1+\|S_{\boldsymbol\lambda} e_u\|^2\big)|f(u)|^2, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}). \end{aligned} \tag{4.1.2}$$

First, we consider the case when the tree $\mathscr{T}$ has a root. Employing (2.1.3), we get

$$\begin{aligned} \sum_{u\in V}\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 &\leqslant c(1+\|S_{\boldsymbol\lambda} e_{\mathsf{root}}\|^2)|f(\mathsf{root})|^2 \\ &\quad + c\sum_{u\in V}\sum_{v\in\mathsf{Chi}(u)}\big(1+\|S_{\boldsymbol\lambda} e_v\|^2\big)|f(v)|^2, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}). \end{aligned}$$

Since, by Proposition 3.1.3 (iv), the function $f\cdot\chi_{V^\circ}$ is in $\mathscr{D}(S_{\boldsymbol\lambda})$ for every $f \in \mathscr{D}(S_{\boldsymbol\lambda})$, we see that the above inequality is equivalent to the following one

$$\sum_{u\in V}\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 \leqslant c\sum_{u\in V}\sum_{v\in\mathsf{Chi}(u)}\big(1+\|S_{\boldsymbol\lambda} e_v\|^2\big)|f(v)|^2, \ \ f \in \mathscr{D}(S_{\boldsymbol\lambda}). \tag{4.1.3}$$

If $\mathscr{T}$ is rootless, then similar reasoning leads to the inequality (4.1.3). Since, by Proposition 3.1.3 (iv), $f\cdot\chi_{\mathsf{Chi}(u)} \in \mathscr{D}(S_{\boldsymbol\lambda})$ for all $f \in \mathscr{D}(S_{\boldsymbol\lambda})$ and $u \in V$, we deduce from (4.1.3) and Proposition 2.1.2 that

$$\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 \leqslant c\sum_{v\in\mathsf{Chi}(u)}\big(1+\|S_{\boldsymbol\lambda} e_v\|^2\big)|f(v)|^2, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}), u \in V. \tag{4.1.4}$$

Fix $u \in V'$ and take a finite nonempty subset W of $\mathsf{Chi}(u)$. It follows from Proposition 3.1.3 (v) that $\ell^2(W) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. This allows us to substitute the function

$$f(v) = \begin{cases} \frac{\lambda_v}{1+\|S_{\boldsymbol\lambda} e_v\|^2} & \text{if } v \in W, \\ 0 & \text{otherwise,} \end{cases}$$

into (4.1.4). What we obtain is

$$\Big(\sum_{v\in W} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2}\Big)^2 \leqslant c \sum_{v\in W} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2}.$$

This implies that

$$\sum_{v\in W} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant c.$$

Passing with W to “infinity” if necessary, we get (ii).

(ii)⇒(i) A careful inspection of the proof of (i)⇒(ii), supported by the Cauchy-Schwarz inequality, shows that the reverse implication (ii)⇒(i) is true as well. □

4.2. When $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$? The circumstances under which the inclusion $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ holds are more elaborate and require much more effort to be accomplished. For this reason, we attach to a densely defined weighted shift $S_{\boldsymbol\lambda}$ on a directed tree $\mathscr{T}$ the diagonal operators M_u in $\ell^2(\mathsf{Chi}(u))$, $u \in V'$, given by

$$\begin{aligned} &\mathscr{D}(M_u) = \{g \in \ell^2(\mathsf{Chi}(u))\colon \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |g(v)|^2 < \infty\}, \\ &(M_u g)(v) = \|S_{\boldsymbol\lambda} e_v\| g(v), \quad v \in \mathsf{Chi}(u),\, g \in \mathscr{D}(M_u). \end{aligned} \tag{4.2.1}$$

If $u \in V'$ is such that the function $\boldsymbol\lambda^u\colon \mathsf{Chi}(u) \ni v \to \lambda_v \in \mathbb{C}$ belongs to $\mathscr{D}(M_u)$, then we define the operator T_u in $\ell^2(\mathsf{Chi}(u))$ by

$$T_u = M_u^2 - \frac{1}{1+\|S_{\boldsymbol\lambda} e_u\|^2} M_u(\boldsymbol\lambda^u) \otimes M_u(\boldsymbol\lambda^u), \quad u \in V'. \tag{4.2.2}$$

For simplicity, we suppress the explicit dependence of M_u and T_u on $\boldsymbol\lambda$ in the notation. We gather below indispensable properties of the operators M_u and T_u.

PROPOSITION 4.2.1. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. If $u \in V'$ and $\boldsymbol\lambda^u \in \mathscr{D}(M_u)$, then*

(i) $\{e_v\colon v \in \mathsf{Chi}(u)\} \subseteq \mathscr{D}(T_u) = \mathscr{D}(M_u^2)$,
(ii) *M_u and T_u are positive selfadjoint operators in $\ell^2(\mathsf{Chi}(u))$,*
(iii) *M_u is bounded if and only if T_u is bounded,*
(iv) *$\|S_{\boldsymbol\lambda} e_v\| \leqslant \sqrt{2\|T_u\|}$ for all but at most one vertex $v \in \mathsf{Chi}(u)$ whenever the operator T_u is bounded.*

PROOF. (i) Apply parts (iii) and (v) of Proposition 3.1.3.

(ii) It is easily seen that M_u and T_u are selfadjoint operators, and M_u is positive (see e.g., [**80**, Theorem 6.20]). Moreover, we have

$$\begin{aligned} \big\langle\big(M_u(\boldsymbol\lambda^u) \otimes M_u(\boldsymbol\lambda^u)\big)g, g\big\rangle &= |\langle g, M_u(\boldsymbol\lambda^u)\rangle|^2 = |\langle M_u(g), \boldsymbol\lambda^u\rangle|^2 \\ &\leqslant \|\boldsymbol\lambda^u\|^2 \|M_u(g)\|^2 \overset{(3.1.4)}{\leqslant} (1+\|S_{\boldsymbol\lambda} e_u\|^2)\langle M_u^2(g), g\rangle, \quad g \in \mathscr{D}(T_u), \end{aligned}$$

which implies that T_u is positive.

(iii) Evident.

(iv) Take vertexes $v_1, v_2 \in \mathsf{Chi}(u)$ such that $v_1 \neq v_2$. Without loss of generality, we may assume that $\|S_{\boldsymbol\lambda} e_{v_1}\| \leqslant \|S_{\boldsymbol\lambda} e_{v_2}\|$. It follows that

$$\frac{\|S_{\boldsymbol\lambda} e_v\|^2 \big(1 + \sum_{w\in\mathsf{Chi}(u)\setminus\{v\}} |\lambda_w|^2\big)}{1 + \|S_{\boldsymbol\lambda} e_u\|^2} \overset{(4.2.2)}{=} \langle T_u e_v, e_v\rangle \leqslant \|T_u\|, \quad v \in \mathsf{Chi}(u). \tag{4.2.3}$$

Hence, we have

$$\begin{aligned} \|S_{\boldsymbol\lambda} e_{v_1}\|^2 &\leqslant \frac{\|S_{\boldsymbol\lambda} e_{v_1}\|^2 \big(2 + |\lambda_{v_1}|^2 + |\lambda_{v_2}|^2 + 2\sum_{w\in\mathsf{Chi}(u)\setminus\{v_1,v_2\}} |\lambda_w|^2\big)}{1 + \|S_{\boldsymbol\lambda} e_u\|^2} \\ &\leqslant \frac{\|S_{\boldsymbol\lambda} e_{v_1}\|^2 \big(1 + \sum_{w\in\mathsf{Chi}(u)\setminus\{v_1\}} |\lambda_w|^2\big)}{1 + \|S_{\boldsymbol\lambda} e_u\|^2} \\ &\qquad + \frac{\|S_{\boldsymbol\lambda} e_{v_2}\|^2 \big(1 + \sum_{w\in\mathsf{Chi}(u)\setminus\{v_2\}} |\lambda_w|^2\big)}{1 + \|S_{\boldsymbol\lambda} e_u\|^2} \\ &\overset{(4.2.3)}{\leqslant} 2\|T_u\|. \end{aligned}$$

This enables us to deduce that for every two distinct vertexes $v_1, v_2 \in \mathsf{Chi}(u)$ either $\|S_{\boldsymbol\lambda} e_{v_1}\| \leqslant \sqrt{2\|T_u\|}$ or $\|S_{\boldsymbol\lambda} e_{v_2}\| \leqslant \sqrt{2\|T_u\|}$. As is easily seen, this implies (iv). $\square$

Now we characterize all weighted shifts $S_{\boldsymbol\lambda}$ on directed trees which have the property that $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. We do this with the help of the operators T_u defined in (4.2.2). The proof of this characterization relies heavily on the fact that for every $u \in V'$, the operator T_u factorizes as $T_u = R_u R_u^*$ whenever T_u is bounded (see (4.2.16)).

THEOREM 4.2.2. *If $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the following two conditions are equivalent*:

(i) $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$,

(ii) $T_u \in \boldsymbol{B}(\ell^2(\mathsf{Chi}(u)))$ *for all* $u \in V'$, *and*

$$\sup_{u\in V'} \|T_u\| < \infty. \tag{4.2.4}$$

PROOF. (i)$\Rightarrow$(ii) Suppose that $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. Since both operators are closed, we can argue as in the proof of Theorem 4.1.1. Thus, there exists a constant $c > 0$ such that

$$\|S_{\boldsymbol\lambda} f\|^2 \leqslant c\,(\|f\|^2 + \|S_{\boldsymbol\lambda}^* f\|^2), \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}^*).$$

Similarly to (4.1.2), we see that

$$\sum_{u\in V} \|S_{\boldsymbol\lambda} e_u\|^2 |f(u)|^2 \leqslant c \sum_{u\in V} |f(u)|^2 + c \sum_{u\in V} \Big| \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2, \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}^*). \tag{4.2.5}$$

If the tree $\mathscr{T}$ has a root, then applying (2.1.3) to the left-hand side of (4.2.5) and to the first term of the right-hand side of (4.2.5), we obtain

$$\begin{aligned} &\sum_{u\in V} \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |f(v)|^2 \\ &\quad\leqslant c|f(\mathsf{root})|^2 + c \sum_{u\in V} \Big(\sum_{v\in\mathsf{Chi}(u)} |f(v)|^2 + \Big| \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2\Big), \quad f \in \mathscr{D}(S_{\boldsymbol\lambda}^*). \end{aligned}$$

Since, by Proposition 3.4.1 (iv), there is no restriction on the value of f at root, we see that the above inequality is equivalent to the following one

$$\sum_{u\in V}\sum_{v\in\mathsf{Chi}(u)}\|S_{\boldsymbol\lambda}e_v\|^2|f(v)|^2 \leqslant c\sum_{u\in V}\Big(\sum_{v\in\mathsf{Chi}(u)}|f(v)|^2+\Big|\sum_{v\in\mathsf{Chi}(u)}\overline{\lambda_v}f(v)\Big|^2\Big),\quad f\in\mathscr{D}(S^*_{\boldsymbol\lambda}). \tag{4.2.6}$$

Clearly, inequalities (4.2.5) and (4.2.6) coincide if $\mathscr{T}$ is rootless.

Fix $u\in V'$. Recall that by Proposition 3.1.3, $\boldsymbol\lambda^u\in\ell^2(\mathsf{Chi}(u))$. In view of Proposition 2.1.2 and Proposition 3.4.1 (vi), the inequality (4.2.6) yields

$$\sum_{v\in\mathsf{Chi}(u)}\|S_{\boldsymbol\lambda}e_v\|^2|f(v)|^2\leqslant c(\|f\|^2+|\langle f,\boldsymbol\lambda^u\rangle|^2),\quad f\in\ell^2(\mathsf{Chi}(u)). \tag{4.2.7}$$

If $\boldsymbol\lambda^u=0$, then by (4.2.7), $M_u\in\boldsymbol B(\ell^2(\mathsf{Chi}(u)))$ and $\|M_u\|\le\sqrt{c}$, and consequently by (4.2.2), $\|T_u\|\le c$. Assume now that $\boldsymbol\lambda^u\neq0$. Define the new inner product $\langle\cdot,\text{-}\rangle_u$ on $\ell^2(\mathsf{Chi}(u))$ by

$$\langle f,g\rangle_u=\langle f,g\rangle+\langle f,\boldsymbol\lambda^u\rangle\cdot\langle\boldsymbol\lambda^u,g\rangle,\quad f,g\in\ell^2(\mathsf{Chi}(u)), \tag{4.2.8}$$

and denote by $\mathcal{K}_u$ the Hilbert space $(\ell^2(\mathsf{Chi}(u)),\langle\cdot,\text{-}\rangle_u)$. It follows from (4.2.7) that the operator $R_u\colon\mathcal{K}_u\to\ell^2(\mathsf{Chi}(u))$ defined by

$$(R_uf)(v)=\|S_{\boldsymbol\lambda}e_v\|f(v),\quad v\in\mathsf{Chi}(u),\ f\in\mathcal{K}_u, \tag{4.2.9}$$

is bounded and

$$\|R_u\|\leqslant\sqrt{c}. \tag{4.2.10}$$

Using the Cauchy-Schwarz inequality, we deduce from (4.2.8), (4.2.9) and (4.2.10) that $M_u\in\boldsymbol B(\ell^2(\mathsf{Chi}(u)))$ and $\|M_u\|^2\leqslant c(1+\|\boldsymbol\lambda^u\|^2)$. Hence, $T_u\in\boldsymbol B(\ell^2(\mathsf{Chi}(u)))$.

We now compute the adjoint $R_u^*\colon\ell^2(\mathsf{Chi}(u))\to\mathcal{K}_u$ of R_u. Take $g\in\ell^2(\mathsf{Chi}(u))$. Then, we have

$$\begin{aligned}\sum_{v\in\mathsf{Chi}(u)}\|S_{\boldsymbol\lambda}e_v\|g(v)\overline{h(v)}&=\langle g,R_uh\rangle=\langle R_u^*g,h\rangle_u\\&=\langle R_u^*g,h\rangle+\langle R_u^*g,\boldsymbol\lambda^u\rangle\cdot\langle\boldsymbol\lambda^u,h\rangle\\&=\sum_{v\in\mathsf{Chi}(u)}(R_u^*g)(v)\overline{h(v)}+\langle R_u^*g,\boldsymbol\lambda^u\rangle\cdot\langle\boldsymbol\lambda^u,h\rangle,\quad h\in\ell^2(\mathsf{Chi}(u)).\end{aligned} \tag{4.2.11}$$

Set $\zeta_{u,g}(v)=\|S_{\boldsymbol\lambda}e_v\|g(v)-(R_u^*g)(v)$ for $v\in\mathsf{Chi}(u)$. By (4.2.7), $\zeta_{u,g}\in\ell^2(\mathsf{Chi}(u))$. It follows from (4.2.11) that

$$\langle\zeta_{u,g},h\rangle=\sum_{v\in\mathsf{Chi}(u)}\zeta_{u,g}(v)\overline{h(v)}=\langle R_u^*g,\boldsymbol\lambda^u\rangle\cdot\langle\boldsymbol\lambda^u,h\rangle,\quad h\in\ell^2(\mathsf{Chi}(u)). \tag{4.2.12}$$

This implies that $\{\boldsymbol\lambda^u\}^\perp\subseteq\{\zeta_{u,g}\}^\perp$, where the orthogonality refers to the original inner product of $\ell^2(\mathsf{Chi}(u))$. As a consequence, $\{\zeta_{u,g}\}^{\perp\perp}\subseteq\{\boldsymbol\lambda^u\}^{\perp\perp}$, and so there exists $\alpha_{u,g}\in\mathbb{C}$ such that $\zeta_{u,g}=-\alpha_{u,g}\boldsymbol\lambda^u$. Hence, by the definition of $\zeta_{u,g}$, we have

$$(R_u^*g)(v)=\|S_{\boldsymbol\lambda}e_v\|g(v)+\alpha_{u,g}\lambda_v,\quad v\in\mathsf{Chi}(u). \tag{4.2.13}$$

Since $M_u\in\boldsymbol B(\ell^2(\mathsf{Chi}(u)))$, we see that

$$-\alpha_{u,g}\langle\boldsymbol\lambda^u,h\rangle\overset{(4.2.12)}{=}\langle R_u^*g,\boldsymbol\lambda^u\rangle\langle\boldsymbol\lambda^u,h\rangle$$

$$\begin{aligned}&\overset{(4.2.13)}{=} \langle M_u g, \boldsymbol{\lambda}^u\rangle\langle \boldsymbol{\lambda}^u, h\rangle + \alpha_{u,g}\|\boldsymbol{\lambda}^u\|^2 \langle \boldsymbol{\lambda}^u, h\rangle\\ &= \langle g, M_u(\boldsymbol{\lambda}^u)\rangle\langle \boldsymbol{\lambda}^u, h\rangle + \alpha_{u,g}\|\boldsymbol{\lambda}^u\|^2 \langle \boldsymbol{\lambda}^u, h\rangle, \quad h \in \ell^2(\mathsf{Chi}(u)).\end{aligned}$$

This and (3.1.4) imply that

$$-\alpha_{u,g}(1+\|S_{\boldsymbol{\lambda}} e_u\|^2)\langle \boldsymbol{\lambda}^u, h\rangle = \langle g, M_u(\boldsymbol{\lambda}^u)\rangle\langle \boldsymbol{\lambda}^u, h\rangle, \quad h \in \ell^2(\mathsf{Chi}(u)). \tag{4.2.14}$$

Substituting $h = \boldsymbol{\lambda}^u$ into (4.2.14) and dividing both sides of (4.2.14) by $\|\boldsymbol{\lambda}^u\|^2$ (which, according to our assumption, is nonzero), we get

$$\alpha_{u,g} = -\frac{\langle g, M_u(\boldsymbol{\lambda}^u)\rangle}{1+\|S_{\boldsymbol{\lambda}} e_u\|^2}, \quad h \in \ell^2(\mathsf{Chi}(u)).$$

Hence

$$R_u^* g \overset{(4.2.13)}{=} M_u g - \frac{\langle g, M_u(\boldsymbol{\lambda}^u)\rangle}{1+\|S_{\boldsymbol{\lambda}} e_u\|^2}\, \boldsymbol{\lambda}^u, \quad g \in \ell^2(\mathsf{Chi}(u)). \tag{4.2.15}$$

As a consequence, we have

$$R_u R_u^* g \overset{(4.2.9)}{=} M_u^2 g - \frac{\langle g, M_u(\boldsymbol{\lambda}^u)\rangle}{1+\|S_{\boldsymbol{\lambda}} e_u\|^2}\, M_u(\boldsymbol{\lambda}^u) \overset{(4.2.2)}{=} T_u g, \quad g \in \ell^2(\mathsf{Chi}(u)). \tag{4.2.16}$$

This, together with (4.2.10), implies that $\|T_u\| = \|R_u\|^2 \leqslant c$ for all $u \in V'$.

(ii)$\Rightarrow$(i) Since the operator M_u is bounded, we easily check that the operator $R_u\colon \mathcal{K}_u \to \ell^2(\mathsf{Chi}(u))$ defined by equalities (4.2.8) and (4.2.9) is bounded as well. As in the proof of implication (i)$\Rightarrow$(ii), we verify that the formulas (4.2.15) and (4.2.16) are valid. Hence, by (4.2.16), $\|R_u\| \leqslant \sqrt{c}$ for all $u \in V'$, where $c := \sup_{v\in V'} \|T_v\|$. Now, by reviewing in reverse order this part of the proof of (i)$\Rightarrow$(ii) which begins with (4.2.10) and ends with (4.2.6), we see that (i) holds. $\square$

Remarks 4.2.3. 1) It follows from Proposition 4.2.1 (iv) and Theorem 4.2.2 that if $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$, then for every $u \in V'$ and for all but at most one vertex $v \in \mathsf{Chi}(u)$, $\|S_{\boldsymbol{\lambda}} e_v\|^2 \leqslant C$ with $C = 2\sup_{v\in V'} \|T_v\| < \infty$. This fact, when combined with Proposition 3.1.8, may suggest that in the case of directed binary (or more complicated) trees the inclusion $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$ forces the operator $S_{\boldsymbol{\lambda}}$ to be bounded. However, this is not the case, which is illustrated in Example 4.3.1 below. It is well known that in the case of classical weighted shifts none of the inclusions $\mathscr{D}(S_{\boldsymbol{\lambda}}) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ and $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$ implies the boundedness of the operator $S_{\boldsymbol{\lambda}}$.

2) Looking at the formula (4.2.2) and Theorem 4.2.2, it is tempting to expect that the uniform boundedness of the operators $\{T_u\}_{u\in V'}$, which completely characterizes the inclusion $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$, is equivalent to the uniform boundedness of the operators $\{M_u\}_{u\in V'}$. However, this is not the case. Indeed, assuming that the operator $S_{\boldsymbol{\lambda}}$ is densely defined and all the operators $\{M_u\}_{u\in V'}$ are bounded, one can infer from (4.2.2) and Proposition 4.2.1 (ii) that $0 \leqslant T_u \leqslant M_u^2$ and consequently $\|T_u\| \leqslant \|M_u\|^2$. This means that $\sup_{u\in V'} \|M_u\| < \infty$ implies $\sup_{u\in V'} \|T_u\| < \infty$. In view of (4.2.1), the inequality $\sup_{u\in V'} \|M_u\| < \infty$ is equivalent to $\sup_{u\in V^\circ} \|S_{\boldsymbol{\lambda}} e_u\| < \infty$. Since $e_{\mathsf{root}} \in \mathscr{D}(S_{\boldsymbol{\lambda}})$ whenever $\mathscr{T}$ has a root, the last inequality is equivalent to $\sup_{u\in V} \|S_{\boldsymbol{\lambda}} e_u\| < \infty$, which, by Proposition 3.1.8, is equivalent to $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$. Summarizing, we have shown that if the operator $S_{\boldsymbol{\lambda}}$ is densely defined, then the inequality $\sup_{u\in V'} \|M_u\| < \infty$ is equivalent to the boundedness of $S_{\boldsymbol{\lambda}}$.

3) According to (4.2.2) and Theorem 4.2.2, the question of when $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ reduces to estimating the norms of rank one perturbations of positive diagonal operators. The class of such operators is still not very well understood, despite the structural simplicity of diagonal operators (cf. [**43, 28, 29**] and references therein; see also [**73**] for a more general situation[6]). As will be seen in Example 4.3.1, the norms of unperturbed operators M_u^2 may tend to ∞, while the norms of perturbed operators T_u may be uniformly bounded.

To make the condition (4.2.4) of Theorem 4.2.2 more explicit and calculable, we consider, instead of the usual operator norm, the Hilbert-Schmidt norm and the trace norm, respectively.

PROPOSITION 4.2.4. *Let $S_{\boldsymbol\lambda}$ be a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. If there exists a constant $C > 0$ such that*

$$\begin{aligned} &\sum_{v\in\mathsf{Chi}(u)} \Big[\|S_{\boldsymbol\lambda} e_v\|^2\Big(1-\frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_u\|^2}\Big)\Big]^2 \leqslant C, \quad u\in V',\\ &\sum_{\substack{v,w\in\mathsf{Chi}(u)\\ v\neq w}} \Big[\frac{\|S_{\boldsymbol\lambda} e_v\||\lambda_v|\|S_{\boldsymbol\lambda} e_w\||\lambda_w|}{1+\|S_{\boldsymbol\lambda} e_u\|^2}\Big]^2 \leqslant C, \quad u\in V',\end{aligned} \tag{4.2.17}$$

then $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. In particular, this is the case if for some constant $C>0$,

$$\sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2\Big(1-\frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_u\|^2}\Big) \leqslant C, \quad u\in V'. \tag{4.2.18}$$

PROOF. First we consider the case when (4.2.17) holds. It follows from Proposition 3.1.3 that $\{e_u \colon u\in V\}\subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. We show that $\boldsymbol\lambda^u \in \mathscr{D}(M_u)$ for every $u\in V'$. Indeed, there are two possibilities, either $\|S_{\boldsymbol\lambda} e_v\||\lambda_v| = 0$ for all $v\in\mathsf{Chi}(u)$ and hence $\boldsymbol\lambda^u\in\mathscr{D}(M_u)$, or $\|S_{\boldsymbol\lambda} e_w\||\lambda_w|>0$ for some $w\in\mathsf{Chi}(u)$ which together with the second inequality in (4.2.17) implies that $\boldsymbol\lambda^u\in\mathscr{D}(M_u)$. This means that we can consider the operators T_u, $u\in V'$. Owing to Proposition 4.2.1, for every $u\in V'$, the operator T_u is closed and its domain contains the basis vectors e_v, $v\in\mathsf{Chi}(u)$. It follows from (4.2.2) that

$$\langle T_u e_w, e_v\rangle = \|S_{\boldsymbol\lambda} e_v\|^2\delta_{v,w} - \frac{\|S_{\boldsymbol\lambda} e_v\|\lambda_v\|S_{\boldsymbol\lambda} e_w\|\overline{\lambda_w}}{1+\|S_{\boldsymbol\lambda} e_u\|^2}, \quad v,w\in\mathsf{Chi}(u),\ u\in V', \tag{4.2.19}$$

where $\delta_{v,w}$ is the usual Kronecker delta. Thus, by (4.2.19) and (4.2.17), we have

$$\begin{aligned}\sum_{v,w\in\mathsf{Chi}(u)} |\langle T_u e_v, e_w\rangle|^2 &= \sum_{v\in\mathsf{Chi}(u)} \Big[\|S_{\boldsymbol\lambda} e_v\|^2\Big(1-\frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_u\|^2}\Big)\Big]^2\\ &\quad + \sum_{\substack{v,w\in\mathsf{Chi}(u)\\ v\neq w}} \Big[\frac{\|S_{\boldsymbol\lambda} e_v\||\lambda_v|\|S_{\boldsymbol\lambda} e_w\||\lambda_w|}{1+\|S_{\boldsymbol\lambda} e_u\|^2}\Big]^2 \leqslant 2C, \quad u\in V'.\end{aligned} \tag{4.2.20}$$

We deduce from (4.2.20) that the operator $T_u|_{\mathscr{E}_V}$ is bounded and $\|T_u|_{\mathscr{E}_V}\|\leqslant\sqrt{2C}$. Since T_u is closed, we see that $T_u\in\boldsymbol{B}(\ell^2(\mathsf{Chi}(u)))$ and $\|T_u\|\leqslant\sqrt{2C}$ for all $u\in V'$.

[6] In [**73**] Barry Simon wrote: *Finally to rank one perturbations—maybe something so easy that I can say something useful! Alas, we'll see even this is hard and exceedingly rich.*

By Theorem 4.2.2, we get $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$. Note that T_u is a Hilbert-Schmidt operator with Hilbert-Schmidt norm $\|T_u\|_2$ (cf. [**76**, page 66]) not exceeding $\sqrt{2C}$.

We now claim that (4.2.18) implies (4.2.17) with the new constant C^2. Fix $u \in V'$. For $v, w \in \mathsf{Chi}(u)$, we denote by $\varDelta_{v,w}$ the right-hand side of the equality (4.2.19). First, we show that

$$|\varDelta_{v,w}|^2 \leqslant \varDelta_{v,v}\varDelta_{w,w}, \quad v, w \in \mathsf{Chi}(u). \tag{4.2.21}$$

Since, $\varDelta_{v,v} \geqslant 0$ for all $v \in \mathsf{Chi}(u)$, it is enough to consider the case when $v \neq w$. Under this assumption, we have

$$|\lambda_v|^2 \overset{(3.1.4)}{\leqslant} \|S_{\boldsymbol\lambda} e_u\|^2 - |\lambda_w|^2 < 1 + \|S_{\boldsymbol\lambda} e_u\|^2 - |\lambda_w|^2, \tag{4.2.22}$$

and, by symmetry,

$$|\lambda_w|^2 < 1 + \|S_{\boldsymbol\lambda} e_u\|^2 - |\lambda_v|^2. \tag{4.2.23}$$

It is now easily seen that (4.2.22) and (4.2.23) imply (4.2.21). Combining (4.2.21) with (4.2.18), we get

$$\sum_{v,w\in\mathsf{Chi}(u)} |\varDelta_{v,w}|^2 \leqslant \Big(\sum_{v\in\mathsf{Chi}(u)} \varDelta_{v,v}\Big)^2 \leqslant C^2. \tag{4.2.24}$$

Since the left-hand side of (4.2.24) is equal to the right-hand side of the equality in (4.2.20), our claim is established. In view of (4.2.24) and the discussion in the previous paragraph, we see that $T_u \in \boldsymbol{B}(\ell^2(\mathsf{Chi}(u)))$ and $\|T_u\| \leqslant C$ for all $u \in V'$, which completes the proof. Looking at (4.2.18) and (4.2.19), we deduce that $\sum_{v\in\mathsf{Chi}(u)} \langle T_u e_v, e_v\rangle \leqslant C$, which means that the operator T_u is of trace class and its trace norm $\|T_u\|_1$ is less than or equal to C. $\square$

An inspection of the proof of Proposition 4.2.4 shows that (4.2.17) is equivalent to $\sup_{u\in V'} \|T_u\|_2 < \infty$, while (4.2.18) is equivalent to $\sup_{u\in V'} \|T_u\|_1 < \infty$. Since $\|T\| \leqslant \|T\|_2$ and $\|T\|_2 \leqslant \|T\|_1$ for Hilbert-Schmidt and trace class operators, respectively, we see why implications (4.2.18)$\Rightarrow$(4.2.17) (possibly with different constants C) and (4.2.17)$\Rightarrow$(4.2.4) are true.

COROLLARY 4.2.5. *If $S_{\boldsymbol\lambda}$ is a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, and $\sup_{u\in V} \operatorname{card}(\mathsf{Chi}(u)) < \infty$, then the following two assertions hold*:

(i) *$S_{\boldsymbol\lambda}$ is densely defined.*
(ii) *$\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ if and only if there exists a positive constant C such that*

$$\|S_{\boldsymbol\lambda} e_v\|^2 \Big(1 - \frac{|\lambda_v|^2}{1 + \|S_{\boldsymbol\lambda} e_u\|^2}\Big) \leqslant C, \quad v \in \mathsf{Chi}(u),\ u \in V'. \tag{4.2.25}$$

PROOF. That $S_{\boldsymbol\lambda}$ is densely defined follows immediately from Proposition 3.1.3.

The "if" part of the assertion (ii) is a direct consequence of Proposition 4.2.4. In turn, the "only if" part follows from Theorem 4.2.2 because, by (4.2.19), the left-hand side of inequality (4.2.25) is equal to $\langle T_u e_v, e_v\rangle$. $\square$

The classical weighted shifts fall within the scope of Theorem 4.1.1 and Corollary 4.2.5. We leave it to the reader to write the explicit inequalities characterizing the inclusions $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$ and $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ for classical weighted shifts $S_{\boldsymbol\lambda}$.

4.3. An example. In this section we give an example highlighting the possible relationships between domains of unbounded weighted shifts on directed trees and their adjoints. Since classical weighted shifts are well developed, we concentrate on weighted shifts on directed binary trees, the graphs which are essentially more complicated than "line trees" involved in the definition of classical weighted shifts. The other reason for this is that Propositions 3.1.8 and 4.2.1, and Theorem 4.2.2 may suggest at first glance that weighted shifts $S_{\boldsymbol\lambda}$ on directed binary trees (or more complicated directed trees) satisfying the inclusion $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ are almost always bounded. Fortunately, as shown below, this is not the case.

Example 4.3.1. Let $\mathscr{T}$ be the directed binary tree as in Figure 1 with V° given by

$$V^\circ = \{(i,j)\colon i = 1,2,\ldots,\ j = 1,\ldots,2^i\}.$$

Define the sets $V_{\mathrm{g}}^\circ = \{(i,j) \in V^\circ\colon j = 1\}$ (gray filled ellipses in Figure 1 without root) and $V_{\mathrm{w}}^\circ = V^\circ \setminus V_{\mathrm{g}}^\circ$ (white filled ellipses in Figure 1).

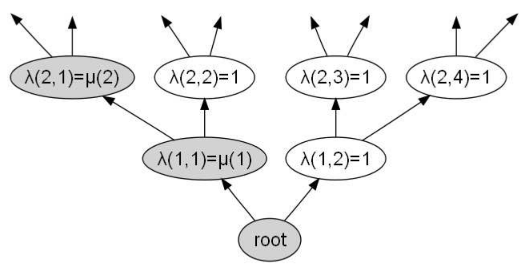

Figure 1

Let $\{\mu(i)\}_{i=1}^\infty$ be a sequence of complex numbers and let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda(i,j)\}_{(i,j)\in V^\circ}$ given by $\lambda(i,j) = 1$ for $(i,j) \in V_{\mathrm{w}}^\circ$, and $\lambda(i,j) = \mu(i)$ for $(i,j) \in V_{\mathrm{g}}^\circ$. In view of Corollary 4.2.5, $S_{\boldsymbol\lambda}$ is densely defined, and $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ if and only if there exists a positive constant C such that for all $u \in V = V'$,

$$\frac{\|S_{\boldsymbol\lambda} e_{v_k}\|^2 (1 + |\lambda_{v_l}|^2)}{1 + |\lambda_{v_1}|^2 + |\lambda_{v_2}|^2} \leqslant C, \quad k,l = 1,2,\ k \neq l,\ \mathsf{Chi}(u) = \{v_1, v_2\}. \tag{4.3.1}$$

If $u \in V_{\mathrm{w}}^\circ$, then (4.3.1) holds with $C = 4/3$. It is also easily seen that (4.3.1) is valid for all $u \in V_{\mathrm{g}}^\circ$ with some positive constant C if and only if

$$\sup_{i\geqslant 1} \frac{1 + |\mu(i+1)|^2}{1 + |\mu(i)|^2} < \infty \quad \Big(\Longleftrightarrow \sup_{i\geqslant 1} \frac{|\mu(i+1)|^2}{1 + |\mu(i)|^2} < \infty\Big). \tag{4.3.2}$$

Hence, $\mathscr{D}(S_{\boldsymbol\lambda}^*) \subseteq \mathscr{D}(S_{\boldsymbol\lambda})$ if and only if (4.3.2) holds. Similar analysis based upon Theorem 4.1.1 shows that $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S_{\boldsymbol\lambda}^*)$ if and only if

$$\sup_{i\geqslant 1} \frac{1 + |\mu(i)|^2}{1 + |\mu(i+1)|^2} < \infty \quad \Big(\Longleftrightarrow \sup_{i\geqslant 1} \frac{|\mu(i)|^2}{1 + |\mu(i+1)|^2} < \infty\Big). \tag{4.3.3}$$

According to Corollary 3.1.9, $S_{\boldsymbol\lambda}$ is bounded if and only if $\sup_{i\geqslant 1} |\mu(i)| < \infty$. Clearly, if $\inf_{i\geqslant 1} |\mu(i)| > 0$, then (4.3.2) is equivalent to $\sup_{i\geqslant 1} |\mu(i+1)/\mu(i)| < \infty$, while (4.3.3) is equivalent to $\sup_{i\geqslant 1} |\mu(i)/\mu(i+1)| < \infty$. This simple observation enables us to find unbounded sequences $\{\mu(i)\}_{i=1}^{\infty} \subseteq \mathbb{C}$ (read: unbounded $S_{\boldsymbol{\lambda}}$'s) for which any of the following four mutually exclusive conditions may hold

1° $\mathscr{D}(S_{\boldsymbol{\lambda}}) = \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ (e.g., $\mu(i) = i^s$ for some $s \in (0,\infty)$, or $\mu(i) = q^i$ for some $q \in (1,\infty)$),

2° $\mathscr{D}(S_{\boldsymbol{\lambda}}) \subsetneq \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ (e.g., $\mu(i) = i!$),

3° $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \subsetneq \mathscr{D}(S_{\boldsymbol{\lambda}})$ (e.g., $\mu(i) = i + 1 - k_n$ if $k_n \leqslant i < k_{n+1}$, where $\{k_n\}_{n=1}^{\infty}$ is a strictly increasing sequence of integers such that $k_1 = 1$ and $\lim_{n\to\infty}(k_{n+1} - k_n) = \infty$),

4° $\mathscr{D}(S_{\boldsymbol{\lambda}}) \nsubseteq \mathscr{D}(S_{\boldsymbol{\lambda}}^*)$ and $\mathscr{D}(S_{\boldsymbol{\lambda}}^*) \nsubseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$ (e.g., $\mu(i) = (i + 1 - k_n)!$ if $k_n \leqslant i < k_{n+1}$, where $\{k_n\}_{n=1}^{\infty}$ is as in 3°).

The examples mentioned in 3° and 4° fit into a more general scheme. Given a strictly increasing function $\phi\colon \{1, 2, \ldots\} \to (0,\infty)$ such that $\lim_{i\to\infty} \phi(i) = \infty$, we set $\mu(i) = \phi(i + 1 - k_n) + l_n$ if $k_n \leqslant i < k_{n+1}$, where $\{k_n\}_{n=1}^{\infty}$ is as in 3° and $\{l_n\}_{n=1}^{\infty} \subseteq [0,\infty)$. Choosing appropriate ϕ, $\{k_n\}_{n=1}^{\infty}$ and $\{l_n\}_{n=1}^{\infty}$, we can find examples of unbounded $S_{\boldsymbol{\lambda}}$'s satisfying 3° (e.g., $\phi(i) = i$, $k_n = n^3$ and $l_n = n$) and 4° ($\phi(i) = i!$, $k_n = n^2$ and $l_n = n$) with the additional property that $\lim_{i\to\infty} |\mu(i)| = \infty$.

One can construct unbounded weighted shifts on directed binary trees with the required properties mentioned in conditions 1°-4° whose weights are more complicated than those explicated in Figure 1. The simplest way of doing this is to draw a directed binary tree with gray and white vertexes (read: filled ellipses) following the rule that only one child of the gray vertex is gray, the other being white, and that both children of the white vertex are white.

Chapter 5. Hyponormality and cohyponormality

5.1. Hyponormality. Starting from this section, we shall concentrate mostly on investigating bounded weighted shifts on directed trees. We begin with recalling definitions of some important classes of operators (see also Corollary 3.4.4, Proposition 8.1.4, Lemma 8.1.5 and Proposition 8.1.7 for characterizations of isometric, coisometric, normal and quasinormal weighted shifts, respectively). Let $\mathcal{H}$ be a complex Hilbert space. An operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *subnormal* if there exists a complex Hilbert space $\mathcal{K}$ and a normal operator $N \in \boldsymbol{B}(\mathcal{K})$ such that $\mathcal{H} \subseteq \mathcal{K}$ (isometric embedding) and $Ah = Nh$ for all $h \in \mathcal{H}$ (cf. [**39, 19**]). An operator $A \in \boldsymbol{B}(\mathcal{H})$ is called *hyponormal* if $\|A^*f\| \leqslant \|Af\|$ for all $f \in \mathcal{H}$. We say that $A \in \boldsymbol{B}(\mathcal{H})$ is *paranormal* if $\|Af\|^2 \leqslant \|A^2f\|\|f\|$ for all $f \in \mathcal{H}$. It is a well known fact that subnormal operators are always hyponormal, but not conversely. The latter can be easily seen by considering classical weighted shifts. Also, it is well known that hyponormal operators are paranormal (cf. [**44, 34**]), but not conversely (cf. [**34**, Theorem 2]). As noticed by Furuta, the latter reduces to finding an example of a hyponormal operator whose square is not hyponormal; the square is just the wanted paranormal operator. Since paranormal unilateral classical weighted shifts are automatically hyponormal, they are not proper candidates for operators with the aforesaid property. Example 5.3.2 shows that weighted shifts on directed trees are prospective candidates for this purpose.

As pointed out below, hyponormal weighted shifts on a directed tree with nonzero weights must be injective.

Proposition 5.1.1. *Let $\mathscr{T}$ be a directed tree with $V^\circ \neq \varnothing$. If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a hyponormal weighted shift on $\mathscr{T}$ whose all weights are nonzero, then $\mathscr{T}$ is leafless. In particular, $S_{\boldsymbol{\lambda}}$ is injective and* $\mathrm{card}(V) = \aleph_0$.

Proof. Suppose that, contrary to our claim, $\mathsf{Chi}(u) = \varnothing$ for some $u \in V$. It follows from $V^\circ \neq \varnothing$ and Corollary 2.1.5 that $u \in V^\circ$. Then we have (see also (5.1.1))

$$|\lambda_u|^2 \overset{(3.4.1)}{=} \|S_{\boldsymbol{\lambda}}^* e_u\|^2 \leqslant \|S_{\boldsymbol{\lambda}} e_u\|^2 \overset{(3.1.4)}{=} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 = 0,$$

which is a contradiction. Since each leafless directed tree is infinite, Propositions 3.1.7 and 3.1.10 complete the proof. $\square$

We now characterize the hyponormality of weighted shifts on directed trees in terms of their weights (see Theorem 8.2.1 for the case of p-hyponormality). Given a directed tree $\mathscr{T}$ and $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$, we define $\mathsf{Chi}_{\boldsymbol{\lambda}}^+(u)$

$$\mathsf{Chi}_{\boldsymbol{\lambda}}^+(u) = \{v \in \mathsf{Chi}(u)\colon \sum_{w \in \mathsf{Chi}(v)} |\lambda_w|^2 > 0\}, \quad u \in V.$$

If $\{e_v\}_{v \in V} \subseteq \mathscr{D}(S_{\boldsymbol{\lambda}})$ and $u \in V$, then by (3.1.4),

$$\mathsf{Chi}_{\boldsymbol{\lambda}}^+(u) = \{v \in \mathsf{Chi}(u)\colon \|S_{\boldsymbol{\lambda}} e_v\| > 0\} = \mathsf{Chi}(u) \cap V_{\boldsymbol{\lambda}}^+,$$

and for every $v \in \mathsf{Chi}(u)$, $\|S_{\boldsymbol{\lambda}} e_v\| = 0$ if and only if $\lambda_w = 0$ for all $w \in \mathsf{Chi}(v)$ (of course, the case of $\mathsf{Chi}(v) = \varnothing$ is not excluded).

Theorem 5.1.2. *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is hyponormal,*

(ii) *for every $u \in V$, the following two conditions hold*:

$$\text{if } v \in \mathsf{Chi}(u) \text{ and } \|S_{\boldsymbol\lambda} e_v\| = 0, \text{ then } \lambda_v = 0, \tag{5.1.1}$$

$$\sum_{v \in \mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant 1. \tag{5.1.2}$$

Proof. Suppose that $S_{\boldsymbol\lambda}$ is hyponormal. Arguing as in the proof of Theorem 4.1.1 up to the inequality (4.1.4) (with $c = 1$ and $\|S_{\boldsymbol\lambda} e_v\|^2$ in place of $1 + \|S_{\boldsymbol\lambda} e_v\|^2$), we deduce that

$$\Big|\sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 \leqslant \sum_{v \in \mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |f(v)|^2, \quad f \in \ell^2(\mathsf{Chi}(u)),\, u \in V. \tag{5.1.3}$$

If $v \in \mathsf{Chi}(u)$ is such that $\|S_{\boldsymbol\lambda} e_v\| = 0$, then by substituting $f = e_v$ into (5.1.3), we obtain $\lambda_v = 0$, which proves (5.1.1). In view of (5.1.1) and (5.1.3), we have

$$\Big|\sum_{v \in \mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \overline{\lambda_v} f(v)\Big|^2 \leqslant \sum_{v \in \mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |f(v)|^2, \quad f \in \ell^2(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)),\, u \in V. \tag{5.1.4}$$

This in turn implies (5.1.2) (consult the part of the proof of Theorem 4.1.1 which comes after the inequality (4.1.4)). It is a matter of routine to verify that the above reasoning can be reversed. □

Remark 5.1.3. In view of Proposition 5.1.1, it is tempting to expect that weights of an injective hyponormal weighted shift on a directed tree are all nonzero. However, this is not the case. Indeed, consider two injective hyponormal weighted shifts $S_{\boldsymbol\lambda_1}$ and $S_{\boldsymbol\lambda_2}$ on directed trees $\mathscr{T}_1 = (V_1, E_1)$ and $\mathscr{T}_2 = (V_2, E_2)$ with nonzero weights $\boldsymbol\lambda_1 = \{\lambda_{1,v}\}_{v \in V_1^\circ}$ and $\boldsymbol\lambda_2 = \{\lambda_{2,v}\}_{v \in V_2^\circ}$, respectively, and assume that the directed tree $\mathscr{T}_2$ has a root denoted by root_2 (cf. Examples 5.3.2 and 5.3.3). Set $V = V_1 \sqcup V_2$ and $E = E_1 \sqcup E_2 \sqcup \{(\omega_1, \mathsf{root}_2)\}$, where ω_1 is an arbitrarily fixed vertex of $\mathscr{T}_1$. Then clearly $\mathscr{T} = (V, E)$ is a directed tree and $V^\circ = V_1^\circ \sqcup V_2$. Define a family $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ} \subseteq \mathbb{C}$ by setting $\lambda_v = \lambda_{1,v}$ for $v \in V_1^\circ$, $\lambda_v = \lambda_{2,v}$ for $v \in V_2^\circ$ and $\lambda_{\mathsf{root}_2} = 0$. Applying Proposition 3.1.6 to $u = \mathsf{root}_2$, we deduce that $S_{\boldsymbol\lambda} = S_{\boldsymbol\lambda_1} \oplus S_{\boldsymbol\lambda_2}$, which implies that $S_{\boldsymbol\lambda}$ is an injective hyponormal weighted shifts with exactly one vanishing weight.

Remark 5.1.4. It is worth mentioning that the conclusion of Proposition 5.1.1 is no longer true if some weights of $S_{\boldsymbol\lambda}$ vanish. Indeed, it is enough to modify the example in Remark 5.1.3 simply by considering an injective hyponormal weighted shift $S_{\boldsymbol\lambda_1}$ with nonzero weights and $V_2 = \{\mathsf{root}_2\}$ (see the comment following Definition 3.1.1). Then the weighted shift $S_{\boldsymbol\lambda} = S_{\boldsymbol\lambda_1} \oplus S_{\boldsymbol\lambda_2} = S_{\boldsymbol\lambda_1} \oplus 0$ is hyponormal, it has exactly one vanishing weight and $\mathsf{Chi}(\mathsf{root}_2) = \varnothing$.

Remark 5.1.5. The notion of hyponormality can be extended to the case of unbounded operators. A densely defined operator A in a complex Hilbert space $\mathcal{H}$ is said to be *hyponormal* if $\mathscr{D}(A) \subseteq \mathscr{D}(A^*)$ and $\|A^* f\| \leqslant \|Af\|$ for all $f \in \mathscr{D}(A)$. It is known that hyponormal operators are closable and their closures are hyponormal as well (see [**66, 50, 51, 52, 74**] for elements of the theory of unbounded hyponormal operators). A close inspection of the proof reveals that *Theorem* 5.1.2 *remains true for densely defined weighted shifts on directed trees* (in the present more general context the inequalities (5.1.3) and (5.1.4), which play a pivotal role in the proof, have to be considered for $f \in \mathscr{D}(S_{\boldsymbol\lambda})$). Note also that if $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the conditions (5.1.1) and (5.1.2) imply (4.1.1) with $c = 1$. Indeed, for all $u \in V$,

$$\sum_{v\in\mathsf{Chi}(u)} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2} \overset{(5.1.1)}{=} \sum_{v\in\mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \frac{|\lambda_v|^2}{1+\|S_{\boldsymbol\lambda} e_v\|^2} \leqslant \sum_{v\in\mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol\lambda} e_v\|^2} \overset{(5.1.2)}{\leqslant} 1.$$

As a consequence of Theorem 4.1.1, we get $\mathscr{D}(S_{\boldsymbol\lambda}) \subseteq \mathscr{D}(S^*_{\boldsymbol\lambda})$.

5.2. Cohyponormality. Recall that an operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *cohyponormal* if its adjoint A^* is hyponormal. The question of cohyponormality of weighted shifts on directed trees is more delicate than hyponormality. It requires considering two distinct cases.

Lemma 5.2.1. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions hold*:

(i) *if $\mathscr{T}$ has a root, then $S_{\boldsymbol\lambda}$ is cohyponormal if and only if $S_{\boldsymbol\lambda} = 0$,*

(ii) *if $\mathscr{T}$ is rootless, then $S_{\boldsymbol\lambda}$ is cohyponormal if and only if for every $u \in V$ the following two conditions are satisfied*:

 (a) $\operatorname{card}(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) \leqslant 1$,

 (b) *if* $\operatorname{card}(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) = 1$, *then* $0 < \|S_{\boldsymbol\lambda} e_v\| \leqslant |\lambda_v|$ *for* $v \in \mathsf{Chi}^+_{\boldsymbol\lambda}(u)$ *and* $\lambda_w = 0$ *for* $w \in \mathsf{Chi}(u) \setminus \mathsf{Chi}^+_{\boldsymbol\lambda}(u)$.

Proof. (i) Assume that $\mathscr{T}$ has a root and $S_{\boldsymbol\lambda}$ is cohyponormal. Note that if $S^*_{\boldsymbol\lambda} e_u = 0$ for some $u \in V$, then by the cohyponormality of $S_{\boldsymbol\lambda}$ we have $S_{\boldsymbol\lambda} e_u = 0$, which together with (3.1.4) implies that $\lambda_v = 0$ for all $v \in \mathsf{Chi}(u)$, or equivalently that $S^*_{\boldsymbol\lambda} e_v = 0$ for all $v \in \mathsf{Chi}(u)$ (use (3.4.1)). Since $S^*_{\boldsymbol\lambda} e_{\mathsf{root}} = 0$, an induction argument shows that $\lambda_v = 0$ for all $v \in \mathsf{Chi}^{\langle n\rangle}(u)$ and for all $n = 1, 2, \ldots$ Applying Corollary 2.1.5, wee see that $\lambda_v = 0$ for all $v \in V^\circ$, which means that $S_{\boldsymbol\lambda} = 0$.

(ii) Assume that $\mathscr{T}$ is rootless. Suppose first that $S_{\boldsymbol\lambda}$ is cohyponormal. Arguing as in the proof of Theorem 4.2.2 up to the inequality (4.2.5), we deduce that

$$\sum_{u\in V} \|S_{\boldsymbol\lambda} e_u\|^2 |f(u)|^2 \leqslant \sum_{u\in V} \Big| \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2, \quad f \in \ell^2(V), \tag{5.2.1}$$

which by Proposition 2.1.2 takes the form

$$\sum_{u\in V} \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |f(v)|^2 \leqslant \sum_{u\in V} \Big| \sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2, \quad f \in \ell^2(V). \tag{5.2.2}$$

This in turn is equivalent to

$$\sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 |f(v)|^2 \leqslant |\langle f, \boldsymbol\lambda^u\rangle|^2, \quad f \in \ell^2(\mathsf{Chi}(u)),\ u \in V', \tag{5.2.3}$$

where $\boldsymbol\lambda^u = \{\lambda_v\}_{v\in\mathsf{Chi}(u)}$. It follows from (5.2.3) that

$$\ell^2(\mathsf{Chi}(u)) \ominus \langle \boldsymbol\lambda^u\rangle \subseteq \ell^2(\mathsf{Chi}(u)) \ominus \ell^2(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)), \quad u \in V',$$

where $\langle \boldsymbol\lambda^u\rangle$ stands for the linear span of $\{\boldsymbol\lambda^u\}$. Taking orthogonal complements in $\ell^2(\mathsf{Chi}(u))$, we obtain $\ell^2(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) \subseteq \langle \boldsymbol\lambda^u\rangle$. This implies that $\operatorname{card}(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) \leqslant 1$. If $\operatorname{card}(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) = 1$, say $\mathsf{Chi}^+_{\boldsymbol\lambda}(u) = \{v\}$, then $\ell^2(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)) = \langle \boldsymbol\lambda^u\rangle$, which guarantees that $\lambda_v \neq 0$ and $\lambda_w = 0$ for all $w \in \mathsf{Chi}(u) \setminus \{v\}$. In turn, by the cohyponormality of $S_{\boldsymbol\lambda}$ we have

$$\|S_{\boldsymbol\lambda} e_v\| \leqslant \|S_{\boldsymbol\lambda}^* e_v\| \overset{(3.4.1)}{=} |\lambda_v|.$$

Reversing the above reasoning completes the proof. □

Lemma 5.2.1 enables us to give a complete description of cohyponormal weighted shifts on a directed tree. In particular, the structure of a subtree corresponding to nonzero weights of a cohyponormal weighted shift is established.

THEOREM 5.2.2. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a nonzero weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then $S_{\boldsymbol\lambda}$ is cohyponormal if and only if the tree $\mathscr{T}$ is rootless and one of the following two disjunctive conditions holds*:

(i) *there exists a sequence $\{u_n\}_{n=-\infty}^{\infty} \subseteq V$ such that*

$$0 < |\lambda_{u_n}| \leqslant |\lambda_{u_{n-1}}| \text{ and } u_{n-1} = \mathsf{par}(u_n) \tag{5.2.4}$$

for all $n \in \mathbb{Z}$, and $\lambda_v = 0$ for all $v \in V \setminus \{u_n \colon n \in \mathbb{Z}\}$,

(ii) *there exist a sequence $\{u_n\}_{n=-\infty}^{0} \subseteq V$ such that*

$$0 < \sum_{v \in \mathsf{Chi}(u_0)} |\lambda_v|^2 \leqslant |\lambda_{u_0}|^2,\ 0 < |\lambda_{u_n}| \leqslant |\lambda_{u_{n-1}}| \text{ and } u_{n-1} = \mathsf{par}(u_n) \tag{5.2.5}$$

for all integers $n \leqslant 0$, and $\lambda_v = 0$ for all $v \in V \setminus (\{u_n \colon n \leqslant 0\} \cup \mathsf{Chi}(u_0))$.

PROOF. Suppose that $S_{\boldsymbol\lambda}$ is cohyponormal. Since $S_{\boldsymbol\lambda} \neq 0$, it follows from Lemma 5.2.1 (i) that $\mathscr{T}$ is rootless. Hence, we have

$$|\lambda_u| \overset{(3.1.4)}{\leqslant} \|S_{\boldsymbol\lambda} e_{\mathsf{par}(u)}\| \leqslant \|S_{\boldsymbol\lambda}^* e_{\mathsf{par}(u)}\| \overset{(3.4.1)}{=} |\lambda_{\mathsf{par}(u)}|, \quad u \in V. \tag{5.2.6}$$

Observe that

(5.2.7) if $u \in V$ is such that $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(u)) = 1$, then $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(\mathsf{par}(u))) = 1$ and $u \in \mathsf{Chi}_{\boldsymbol\lambda}^+(\mathsf{par}(u))$.

Indeed, it follows from Lemma 5.2.1 (b) that $0 < \|S_{\boldsymbol\lambda} e_v\| \leqslant |\lambda_v|$, where $v \in \mathsf{Chi}_{\boldsymbol\lambda}^+(u)$. As a consequence, we have $\lambda_v \neq 0$ and $\|S_{\boldsymbol\lambda} e_u\| > 0$. Hence, $u \in \mathsf{Chi}_{\boldsymbol\lambda}^+(\mathsf{par}(u))$ and so, by Lemma 5.2.1 (a), $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(\mathsf{par}(u))) = 1$.

Since $\mathscr{T}$ is rootless and $S_{\boldsymbol\lambda} \neq 0$, there exists $u \in V$ such that $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(u)) = 1$. Applying an induction procedure, we infer from (5.2.7) that there exists a backward sequence $\{u_n\}_{n=-\infty}^{-1}$ such that $u_{n-1} = \mathsf{par}(u_n)$ and $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(u_n)) = 1$ for all integers $n \leqslant -1$ with $u_{-1} = u$ (by Proposition 2.1.6, $u_m \neq u_n$ whenever $m \neq n$). Take $v \in \mathsf{Chi}_{\boldsymbol\lambda}^+(u)$. If $\mathrm{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(v)) = 1$, then the new sequence $\{\ldots, u_{-2}, u_{-1}, u_0\}$ with $u_0 = v$ shares the same properties as $\{u_n\}_{n=-\infty}^{-1}$. There are now two possibilities. The first is that this procedure never terminates. Let us denote by $\{u_n\}_{n=-\infty}^{\infty}$ the resulting sequence. We show that $\{u_n\}_{n=-\infty}^{\infty}$ fulfills (i). For this, take $v \in V \setminus \{u_n \colon n \in \mathbb{Z}\}$. Suppose that, contrary to our claim, $\lambda_v \neq 0$. By Proposition 2.1.4, there exists an integer $k \geqslant 1$ such that $\mathsf{par}^k(v) \in \{u_n \colon n \in \mathbb{Z}\}$, say $\mathsf{par}^k(v) = u_l$ with $l \in \mathbb{Z}$, and $\mathsf{par}^j(v) \notin \{u_n \colon n \in \mathbb{Z}\}$ for $j = 0, \ldots, k-1$. It follows from (5.2.6) that $\lambda_{\mathsf{par}^{k-1}(v)} \neq 0$. Since $\mathsf{Chi}_{\boldsymbol\lambda}^+(u_l) = \{u_{l+1}\}$, $\mathsf{par}^{k-1}(v) \in \mathsf{Chi}(u_l)$ and $\mathsf{par}^{k-1}(v) \neq u_{l+1}$, we get a contradiction with Lemma 5.2.1 (b). The inequalities in (5.2.4) follow from Lemma 5.2.1 (b).

The other possibility is that our procedure terminates, which means that we obtain a sequence $\{u_n\}_{n=-\infty}^{-1}$ such that $u_{n-1} = \mathsf{par}(u_n)$ and $\operatorname{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(u_n)) = 1$ for all integers $n \leqslant -1$, and

$$\operatorname{card}(\mathsf{Chi}_{\boldsymbol\lambda}^+(u_0)) = 0 \text{ for a unique } u_0 \in \mathsf{Chi}_{\boldsymbol\lambda}^+(u_{-1}). \tag{5.2.8}$$

We show that $\{u_n\}_{n=-\infty}^{0}$ fulfills (ii). Take $v \in V \setminus (\{u_n\colon n \leqslant 0\} \cup \mathsf{Chi}(u_0))$. Suppose that, contrary to our claim, $\lambda_v \neq 0$. As in the previous case, we find $k \geqslant 1$ such that $\mathsf{par}^k(v) \in \{u_n\colon n \leqslant 0\} \cup \mathsf{Chi}(u_0)$ and $\mathsf{par}^j(v) \notin \{u_n\colon n \leqslant 0\} \cup \mathsf{Chi}(u_0)$ for $j = 0, \ldots, k-1$. By (5.2.6), we get $\lambda_{\mathsf{par}^{k-1}(v)} \neq 0$. Now we have two possibilities, either $\mathsf{par}^k(v) \in \mathsf{Chi}(u_0)$ which implies that $\mathsf{par}^k(v) \in \mathsf{Chi}_{\boldsymbol\lambda}^+(u_0)$, in contradiction with (5.2.8), or $\mathsf{par}^k(v) \in \{u_n\colon n \leqslant -1\}$, say $\mathsf{par}^k(v) = u_l$ with $l \leqslant -1$ (note that the case of $l = 0$ is impossible), which implies that $\mathsf{Chi}_{\boldsymbol\lambda}^+(u_l) = \{u_{l+1}\}$ (use (5.2.8) if $l = -1$, and (5.2.7) if $l \leqslant -2$), $\mathsf{par}^{k-1}(v) \in \mathsf{Chi}(u_l)$ and $\mathsf{par}^{k-1}(v) \neq u_{l+1}$, in contradiction with Lemma 5.2.1 (b). Since the inequalities in (5.2.5) hold due to Lemma 5.2.1 (b), the "only if" part of the conclusion is justified.

Finally, applying Lemma 2.1.3 (with $X = \{u_n\colon n \in \mathbb{Z}\}$ or $X = \mathsf{Chi}(u_0) \cup \{u_n\colon n \leqslant 0\}$) and Lemma 5.2.1 (ii) completes the proof of the "if" part. $\square$

The following corollary is a direct consequence of Proposition 3.1.7 and Theorem 5.2.2. It asserts that injective cohyponormal weighted shifts on directed trees are bilateral classical weighted shifts.

Corollary 5.2.3. *A nonzero weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ is injective and cohyponormal if and only if there exists a sequence $\{u_k\}_{k=-\infty}^{\infty}$ such that $V = \{u_k\colon k \in \mathbb{Z}\}$, $u_{n-1} = \mathsf{par}(u_n)$ and $0 < |\lambda_{u_n}| \leqslant |\lambda_{u_{n-1}}|$ for all $n \in \mathbb{Z}$.*

A typical directed tree admitting a nonzero cohyponormal weighted shift $S_{\boldsymbol\lambda}$ that satisfies the condition (ii) of Theorem 5.2.2 is illustrated in Figure 2 below (such directed tree is infinite). Only edges joining vertexes corresponding to nonzero weights of $S_{\boldsymbol\lambda}$ are drawn. Of course, the tree may have more edges however their arrowheads must correspond to zero weights of $S_{\boldsymbol\lambda}$; they are not drawn.

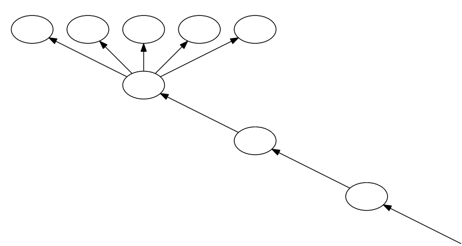

Figure 2

Remark 5.2.4. A closed densely defined operator A in a complex Hilbert space $\mathcal{H}$ is said to be *cohyponormal* if A^* is hyponormal (cf. Remark 5.1.5), i.e., $\mathscr{D}(A^*) \subseteq \mathscr{D}(A)$ and $\|Af\| \leqslant \|A^*f\|$ for all $f \in \mathscr{D}(A^*)$. A thorough inspection of proofs shows that *Lemma* 5.2.1, *Theorem* 5.2.2 *and Corollary* 5.2.3 *remain true for densely defined weighted shifts on directed trees* (now we have to employ Propositions 3.1.3 (v) and 3.4.1 (ii); moreover, the inequalities (5.2.1) and (5.2.2) have to be considered for $f \in \mathscr{D}(S_{\boldsymbol\lambda}^*)$).

5.3. Examples. We begin by giving an example of a bounded injective weighted shift on a directed tree which is paranormal but not hyponormal (see Section 5.1 for definitions). In view of Remark 3.1.4, the directed tree considered in Example 5.3.1 below is one step more complicated than that appearing in the case of classical weighted shifts.

Example 5.3.1. Let $\mathscr{T}$ be the directed tree as in Figure 3 with V° given by $V^\circ = \{(i,j)\colon i=1,2,\ j=1,2,\ldots\}$ (note that $\mathscr{T} = \mathscr{T}_{2,0}$, cf. (6.2.10)).

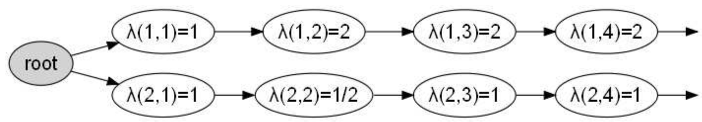

Figure 3

Let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda(i,j)\}_{(i,j)\in V^\circ}$ given by $\lambda(1,1)=1$, $\lambda(1,j)=2$ for all $j \geqslant 2$, $\lambda(2,2)=1/2$ and $\lambda(2,j)=1$ for all $j \neq 2$. By Corollary 3.1.9 and Proposition 3.1.7, $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ and $\mathscr{N}(S_{\boldsymbol\lambda}) = \{0\}$. To make the notation more readable, we write $e_{i,j}$ instead of e_u for $u=(i,j)\in V^\circ$. It follows from (3.1.4) that

$$\begin{cases} S_{\boldsymbol\lambda}(e_{\mathsf{root}}) = e_{1,1} + e_{2,1}, \quad S_{\boldsymbol\lambda}(e_{1,j}) = 2e_{1,j+1} \text{ for } j \geqslant 1, \\ S_{\boldsymbol\lambda}(e_{2,1}) = \frac{1}{2}e_{2,2}, \quad S_{\boldsymbol\lambda}(e_{2,j}) = e_{2,j+1} \text{ for } j \geqslant 2. \end{cases} \tag{5.3.1}$$

Take $f = \alpha e_{\mathsf{root}} + \sum_{j=1}^\infty \alpha_{1,j} e_{1,j} + \sum_{j=1}^\infty \alpha_{2,j} e_{2,j} \in \ell^2(V)$. Then, by (5.3.1), we have

$$S_{\boldsymbol\lambda} f = \alpha e_{1,1} + \alpha e_{2,1} + 2\sum_{j=1}^\infty \alpha_{1,j} e_{1,j+1} + \frac{1}{2}\alpha_{2,1} e_{2,2} + \sum_{j=2}^\infty \alpha_{2,j} e_{2,j+1},$$

$$S_{\boldsymbol\lambda}^2 f = 2\alpha e_{1,2} + \frac{1}{2}\alpha e_{2,2} + 4\sum_{j=1}^\infty \alpha_{1,j} e_{1,j+2} + \frac{1}{2}\alpha_{2,1} e_{2,3} + \sum_{j=2}^\infty \alpha_{2,j} e_{2,j+2}.$$

Putting all this together implies that

$$\|f\|^2 = |\alpha|^2 + \sum_{j=1}^\infty |\alpha_{1,j}|^2 + |\alpha_{2,1}|^2 + \sum_{j=2}^\infty |\alpha_{2,j}|^2,$$

$$\|S_{\boldsymbol\lambda} f\|^2 = 2|\alpha|^2 + 4\sum_{j=1}^\infty |\alpha_{1,j}|^2 + \frac{1}{4}|\alpha_{2,1}|^2 + \sum_{j=2}^\infty |\alpha_{2,j}|^2,$$

$$\|S_{\boldsymbol\lambda}^2 f\|^2 = \frac{17}{4}|\alpha|^2 + 16\sum_{j=1}^\infty |\alpha_{1,j}|^2 + \frac{1}{4}|\alpha_{2,1}|^2 + \sum_{j=2}^\infty |\alpha_{2,j}|^2.$$

Substituting $x_1 = |\alpha|^2$, $x_2 = \sum_{j=1}^\infty |\alpha_{1,j}|^2$, $x_3 = |\alpha_{2,1}|^2$ and $x_4 = \sum_{j=2}^\infty |\alpha_{2,j}|^2$ into the following inequality[7]

$$\Big(2x_1 + 4x_2 + \frac{1}{4}x_3 + x_4\Big)^2 \leqslant \Big(x_1 + x_2 + x_3 + x_4\Big) \cdot \Big(\frac{17}{4}x_1 + 16x_2 + \frac{1}{4}x_3 + x_4\Big),$$

which is true for all nonnegative reals x_1, x_2, x_3, x_4, we get $\|S_{\boldsymbol\lambda} f\|^4 \leqslant \|f\|^2 \|S_{\boldsymbol\lambda}^2 f\|^2$. This means that $S_{\boldsymbol\lambda}$ is paranormal. Since, by (5.3.1) and (3.4.1), $\|S_{\boldsymbol\lambda} e_{2,1}\| = \frac{1}{2}$ and $\|S_{\boldsymbol\lambda}^* e_{2,1}\| = 1$, we conclude that $S_{\boldsymbol\lambda}$ is not hyponormal.

[7] which can be obtained from the Cauchy-Schwarz inequality, or by direct computation

Recall that there are bounded hyponormal operators whose squares are not hyponormal (see [**39, 45, 41, 26**]). This cannot happen for classical weighted shifts. However, this really happens for weighted shifts on directed trees. Our next example is build on a very simple directed tree (though a little bit more complicated than that in Example 5.3.1). It is parameterized by four real parameters q, r, s and t (in fact, by three independent real parameters, cf. (5.3.2)). In Example 5.3.3 below, we demonstrate yet another sample of a hyponormal weighted shift with non-hyponormal square which is indexed only by two real parameters, however it is build on a more complicated directed tree. It seems to be impossible to reduce the number of parameters.

Example 5.3.2. Let $\mathscr{T}$ be the directed tree as in Figure 4 with V° given by $V^\circ = \{(0,0)\} \cup \{(i,j) \colon i = 1,2;\ j = 1,2,\ldots\}$ (note that $\mathscr{T} = \mathscr{T}_{2,1}$, cf. (6.2.10)).

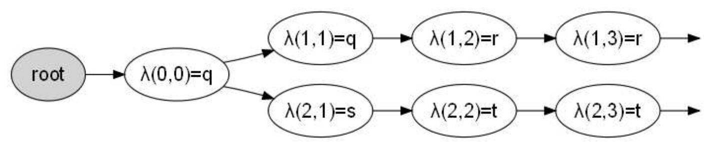

Figure 4

Fix positive real numbers q, r, s, t such that

$$\left(\frac{s}{t}\right)^2\left(\frac{r}{t}\right)^2 + \left(\frac{q}{t}\right)^2 \leqslant \left(\frac{r}{t}\right)^2 \quad \text{and} \quad \left(\frac{s}{t}\right)\left(\frac{q}{t}\right) > 1. \tag{5.3.2}$$

(e.g., $q = 4$, $r = 8$, $s = 1$ and $t \in [2/\sqrt{3}, 2)$ are sample numbers satisfying (5.3.2)). Let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda(i,j)\}_{(i,j)\in V^\circ}$ given by $\lambda(0,0) = \lambda(1,1) = q$, $\lambda(2,1) = s$, $\lambda(1,j) = r$ and $\lambda(2,j) = t$ for $j = 2,3,\ldots$ By Corollary 3.1.9 and Proposition 3.1.7, $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ and $\mathscr{N}(S_{\boldsymbol\lambda}) = \{0\}$. It is a routine matter to verify that $S_{\boldsymbol\lambda}$ satisfies the inequality (5.1.2). Hence, by Theorem 5.1.2, $S_{\boldsymbol\lambda}$ is hyponormal. It follows from (3.1.4) and (3.4.1) that $S_{\boldsymbol\lambda}^2 e_{2,1} = t^2 e_{2,3}$ and $S_{\boldsymbol\lambda}^{*2} e_{2,1} = sqe_{\mathsf{root}}$, which, by the right-hand inequality in (5.3.2), implies that $\|S_{\boldsymbol\lambda}^2 e_{2,1}\| < \|S_{\boldsymbol\lambda}^{*2} e_{2,1}\|$. This means that $S_{\boldsymbol\lambda}^2$ is not hyponormal.

Example 5.3.3. Let $\mathscr{T}$ be the directed binary tree as in Figure 5 with V° given by

$$V^\circ = \{(i,j) \colon i = 1,2,\ldots,\ j = 1,\ldots,2^{i-1}\}.$$

Fix positive real numbers α, β. Let $S_{\boldsymbol\lambda}$ be the weighted shift on $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda(i,j)\}_{(i,j)\in V^\circ}$ given by

$$\lambda(i,j) = \begin{cases} \alpha & \text{if } i = j = 1, \\ \alpha & \text{if } i \geqslant 2 \text{ and } j = 1,\ldots,2^{i-2}, \\ \beta & \text{if } i \geqslant 2 \text{ and } j = 2^{i-2}+1,\ldots,2^{i-1}. \end{cases}$$

By Corollary 3.1.9 and Proposition 3.1.7, $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ and $\mathscr{N}(S_{\boldsymbol\lambda}) = \{0\}$. In virtue of Theorem 5.1.2, $S_{\boldsymbol\lambda}$ is hyponormal. Assume now that $\alpha > 2\beta$. Then $S_{\boldsymbol\lambda}^2$ is not hyponormal because, by (3.1.4) and (3.4.1), we have (see the notational convention used in Example 5.3.1)

$$\|S_{\boldsymbol\lambda}^{*2} e_{2,2}\| = \alpha\beta > 2\beta^2 = \|S_{\boldsymbol\lambda}^2 e_{2,2}\|.$$

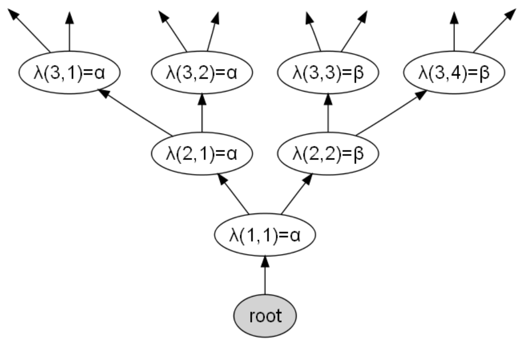

Figure 5

Chapter 6. Subnormality

6.1. A general approach. Our goal in this section is to find a characterization of subnormality of weighted shifts on directed trees (see Section 5.1 for the definition of a subnormal operator). We begin by attaching to a family $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$ of weights of a weighted shift $S_{\boldsymbol{\lambda}}$ on a directed tree $\mathscr{T}$ a new family $\{\lambda_{u|v}\}_{u\in V, v\in \mathsf{Des}(u)}$ defined by

$$\lambda_{u|v} = \begin{cases} 1 & \text{if } v = u, \\ \prod_{j=0}^{n-1} \lambda_{\mathsf{par}^j(v)} & \text{if } v \in \mathsf{Chi}^{\langle n\rangle}(u),\ n \geqslant 1. \end{cases}$$

Owing to (2.1.10), the above definition is correct. It is easily seen that the following recurrence formula holds:

$$\lambda_{u|v} = \lambda_{u|\mathsf{par}(v)}\lambda_v, \quad u \in V,\ v \in \mathsf{Des}(u),\ v \neq u, \tag{6.1.1}$$

$$\lambda_{\mathsf{par}(v)|w} = \lambda_v \lambda_{v|w}, \quad v \in V^\circ,\ w \in \mathsf{Des}(v). \tag{6.1.2}$$

Lemma 6.1.1. *If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$, then the following two conditions hold*:

(i) $S_{\boldsymbol{\lambda}}^n e_u = \sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} \lambda_{u|v}\, e_v$ *for all* $u \in V$ *and* $n \in \mathbb{Z}_+$,
(ii) $\|S_{\boldsymbol{\lambda}}^n e_u\|^2 = \sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} |\lambda_{u|v}|^2$ *for all* $u \in V$ *and* $n \in \mathbb{Z}_+$.

Proof. All we need to prove is the equality in the condition (i). We proceed by induction on n. The case $n = 0$ is obvious. Suppose that (i) holds for a given integer $n \geqslant 0$. It follows from the definition of $\mathsf{Chi}^{\langle n+1\rangle}(u)$ and Proposition 2.1.2 that

$$\mathsf{Chi}^{\langle n+1\rangle}(u) = \bigsqcup_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} \mathsf{Chi}(v). \tag{6.1.3}$$

Then by the induction hypothesis and the continuity of $S_{\boldsymbol{\lambda}}$ we have

$$\begin{aligned} S_{\boldsymbol{\lambda}}^{n+1} e_u = S_{\boldsymbol{\lambda}}(S_{\boldsymbol{\lambda}}^n e_u) &= \sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} \lambda_{u|v}\, S_{\boldsymbol{\lambda}} e_v \\ &\overset{(3.1.4)}{=} \sideset{}{^\oplus}\sum_{v\in\mathsf{Chi}^{\langle n\rangle}(u)} \sum_{w\in\mathsf{Chi}(v)} \lambda_{u|v}\lambda_w e_w \\ &\overset{(6.1.3)}{=} \sum_{w\in\mathsf{Chi}^{\langle n+1\rangle}(u)} \lambda_{u|\mathsf{par}(w)}\lambda_w e_w \overset{(6.1.1)}{=} \sum_{w\in\mathsf{Chi}^{\langle n+1\rangle}(u)} \lambda_{u|w} e_w, \end{aligned}$$

where the symbol $\sum^\oplus$ is reserved for denoting the orthogonal series. This completes the proof. $\square$

We say that a sequence $\{t_n\}_{n=0}^\infty$ of real numbers is a *Hamburger moment sequence* if there exists a positive Borel measure μ on $\mathbb{R}$ such that

$$t_n = \int_{\mathbb{R}} s^n \mathrm{d}\mu(s), \quad n \in \mathbb{Z}_+;$$

μ is called a *representing measure* of $\{t_n\}_{n=0}^\infty$. In view of Hamburger's theorem (cf. [**71**, Theorem 1.2]), a sequence $\{t_n\}_{n=0}^\infty$ of real numbers is a Hamburger moment sequence if and only if it is *positive definite*, i.e.,

$$\sum_{k,l=0}^{n} t_{k+l}\alpha_k\overline{\alpha_l} \geqslant 0, \quad \alpha_0,\ldots,\alpha_n \in \mathbb{C}, \quad n \in \mathbb{Z}_+.$$

A Hamburger moment sequence is said to be *determinate* if it has only one representing measure. Let us recall a useful criterion for determinacy.

(6.1.4) If $\{t_n\}_{n=0}^\infty$ is a Hamburger moment sequence such that $a := \limsup_{n\to\infty} t_{2n}^{1/2n} < \infty$, then it is determinate and its unique representing measure is concentrated on $[-a,a]$.

Indeed, if μ is a representing measure of $\{t_n\}_{n=0}^\infty$, then the closed support of μ is contained in $[-a,a]$ (cf. [**68**, page 71]), and so μ is a unique representing measure of $\{t_n\}_{n=0}^\infty$ (see [**30**]). Yet another approach to (6.1.4) is in [**77**, Theorem 2].

A Hamburger moment sequence having a representing measure concentrated on $[0,\infty)$ is called a *Stieltjes moment sequence*. Clearly

(6.1.5) if $\{t_n\}_{n=0}^\infty$ is a Stieltjes moment sequence, then so is $\{t_{n+1}\}_{n=0}^\infty$.

The above property is no longer valid for Hamburger moment sequences. By Stieltjes theorem (cf. [**71**, Theorem 1.3]), a sequence $\{t_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is a Stieltjes moment sequence if and only if the sequences $\{t_n\}_{n=0}^\infty$ and $\{t_{n+1}\}_{n=0}^\infty$ are positive definite.

The question of backward extendibility of Hamburger or Stieltjes moment sequences has known solutions (see e.g., [**20**, Proposition 8] and [**78**]). What we further need is a variant of this question.

Lemma 6.1.2. *Let $\{t_n\}_{n=0}^\infty$ be a Stieltjes moment sequence. Set $t_{-1} = 1$. Then the following three conditions are equivalent*:

(i) *$\{t_{n-1}\}_{n=0}^\infty$ is a Stieltjes moment sequence,*
(ii) *$\{t_{n-1}\}_{n=0}^\infty$ is positive definite,*
(iii) *there exists a representing measure μ of $\{t_n\}_{n=0}^\infty$ concentrated on $[0,\infty)$ such that*[8] $\int_0^\infty \frac{1}{s}\mathrm{d}\mu(s) \leqslant 1$.

If μ is as in (iii)*, then the positive Borel measure ν on $\mathbb{R}$ defined by*

$$\nu(\sigma) = \int_\sigma \frac{1}{s}\mathrm{d}\mu(s) + \Big(1 - \int_0^\infty \frac{1}{s}\mathrm{d}\mu(s)\Big)\delta_0(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}), \tag{6.1.6}$$

is a representing measure of $\{t_{n-1}\}_{n=0}^\infty$ concentrated on $[0,\infty)$; moreover, $\nu(\{0\}) = 0$ if and only if $\int_0^\infty \frac{1}{s}\mathrm{d}\mu(s) = 1$.

Proof. (i)$\Leftrightarrow$(ii) Apply the Stieltjes theorem.

(i)$\Rightarrow$(iii) Let ρ be a representing measure of $\{t_{n-1}\}_{n=0}^\infty$ concentrated on $[0,\infty)$. Define the positive Borel measure μ on $[0,\infty)$ by $\mathrm{d}\mu(s) = s\mathrm{d}\rho(s)$. Then

$$t_n = t_{(n+1)-1} = \int_0^\infty s^n s\mathrm{d}\rho(s) = \int_0^\infty s^n\mathrm{d}\mu(s), \quad n \in \mathbb{Z}_+.$$

Clearly, $\mu(\{0\}) = 0$ and consequently

$$\int_0^\infty \frac{1}{s}\mathrm{d}\mu(s) = \int_{(0,\infty)} \mathrm{d}\rho(s) = \rho((0,\infty)) = \int_{[0,\infty)} s^0\mathrm{d}\rho(s) - \rho(\{0\}) = 1 - \rho(\{0\}).$$

This implies that $\int_0^\infty \frac{1}{s}\mathrm{d}\mu(s) \leqslant 1$.

[8] We adhere to the convention that $\frac{1}{0} := \infty$. Hence, $\int_0^\infty \frac{1}{s}\mathrm{d}\mu(s) < \infty$ implies $\mu(\{0\}) = 0$.

(iii)$\Rightarrow$(i) It is easily verifiable that the measure ν defined by (6.1.6) is concentrated on $[0,\infty)$ and $t_{n-1} = \int_0^\infty s^n \mathrm{d}\nu(s)$ for all $n \in \mathbb{Z}_+$. Hence $\{t_{n-1}\}_{n=0}^\infty$ is a Stieltjes moment sequence. The "moreover" part of the conclusion is obvious. $\square$

Let us recall Lambert's characterization of subnormality (cf. [**57**]; see also [**75**, Theorem 7] for the general, not necessarily injective, case): an operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if $\{\|T^n f\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for all $f \in \mathcal{H}$. Since $T^{n+1}f = T^n(Tf)$, we infer from the Stieltjes theorem that

(6.1.7) an operator $T \in \boldsymbol{B}(\mathcal{H})$ is subnormal if and only if $\{\|T^n f\|^2\}_{n=0}^\infty$ is a Hamburger moment sequence for all $f \in \mathcal{H}$.

Lambert's theorem enables us to write a complete characterization of subnormality of weighted shifts on directed trees.

Theorem 6.1.3. *If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, then the following conditions are equivalent*:

(i) *$S_{\boldsymbol{\lambda}}$ is subnormal,*
(ii) *$\Big\{ \sum_{v \in \mathsf{Chi}^{\langle n \rangle}(u)} |\lambda_{u|v}|^2 \Big\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $u \in V$,*
(iii) *$\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $u \in V$,*
(iv) *$\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^\infty$ is a Hamburger moment sequence for every $u \in V$,*
(v) *$\sum_{k,l=0}^n \|S_{\boldsymbol{\lambda}}^{k+l} e_u\|^2 \alpha_k \overline{\alpha_l} \geqslant 0$ for all $\alpha_0, \ldots, \alpha_n \in \mathbb{C}$, $n \in \mathbb{Z}_+$ and $u \in V$.*

Proof. (i)$\Rightarrow$(ii) Employ the Lambert theorem (or rather its easier part) and Proposition 6.1.1 (ii).

(ii)$\Rightarrow$(iii) Apply Lemma 6.1.1 (ii).

(iii)$\Rightarrow$(iv) Evident.

(iv)$\Rightarrow$(i) Take $f \in \ell^2(V)$. An induction argument shows that for a fixed integer $n \geqslant 0$, the sets $\mathsf{Chi}^{\langle n \rangle}(u)$, $u \in V$, are pairwise disjoint. By Lemma 6.1.1 (i), this implies that

$$\|S_{\boldsymbol{\lambda}}^n f\|^2 = \Big\| {\sum_{u \in V}}^{\oplus} f(u) S_{\boldsymbol{\lambda}}^n e_u \Big\|^2 = \sum_{u \in V} |f(u)|^2 \|S_{\boldsymbol{\lambda}}^n e_u\|^2, \quad n \in \mathbb{Z}_+. \tag{6.1.8}$$

Since the sequence $\{\|S_{\boldsymbol{\lambda}}^n e_u\|^2\}_{n=0}^\infty$ is positive definite, we can easily infer from (6.1.8) that the sequence $\{\|S_{\boldsymbol{\lambda}}^n f\|^2\}_{n=0}^\infty$ is positive definite as well. Applying the Hamburger theorem and (6.1.7), we get the subnormality of $S_{\boldsymbol{\lambda}}$.

The equivalence (iv)$\Leftrightarrow$(v) is a direct consequence of the Hamburger theorem. This completes the proof. $\square$

One of the consequences of Theorem 6.1.3 is that the study of subnormality of weighted shifts on directed trees reduces to the case of trees with root.

Corollary 6.1.4. *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$. Suppose that X is a subset of V such that $V = \bigcup_{x \in X} \mathsf{Des}(x)$. Then $S_{\boldsymbol{\lambda}}$ is subnormal if and only if the operator $S_{\boldsymbol{\lambda}_{\to(x)}}$ is subnormal for every $x \in X$ (cf. Notation 3.1.5).*

Proof. Note that by (2.1.8) the space $\ell^2(\mathsf{Des}(u))$ is invariant for $S_{\boldsymbol{\lambda}}$ and

$$S_{\boldsymbol{\lambda}_{\to(u)}} = S_{\boldsymbol{\lambda}}|_{\ell^2(\mathsf{Des}(u))}.$$

Hence, by Theorem 6.1.3, $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $S_{\boldsymbol{\lambda}_{\to(u)}}$ is subnormal for every $u \in V$. Since $S_{\boldsymbol{\lambda}_{\to(u)}} \subseteq S_{\boldsymbol{\lambda}_{\to(x)}}$ whenever $u \in \mathsf{Des}(x)$, the conclusion follows from the above characterization of subnormality and the equality $V = \bigcup_{x\in X} \mathsf{Des}(x)$. $\square$

It turns out that in some instances the condition (iii) of Theorem 6.1.3 can be essentially weakened without spoiling the equivalence (i)$\Leftrightarrow$(iii). This effect is similar to that appearing in the case of classical weighted shifts. The result which follows will be referred to as the *small* lemma (see also Lemma 6.1.10).

LEMMA 6.1.5. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and let $u_0, u_1 \in V$ be such that $\mathsf{Chi}(u_0) = \{u_1\}$. If $\{\|S_{\boldsymbol\lambda}^n e_{u_0}\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence and $\lambda_{u_1} \neq 0$, then $\{\|S_{\boldsymbol\lambda}^n e_{u_1}\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence.*

PROOF. Observing that

$$\|S_{\boldsymbol\lambda}^n e_{u_1}\|^2 \overset{(3.1.4)}{=} \frac{1}{|\lambda_{u_1}|^2}\|S_{\boldsymbol\lambda}^{n+1} e_{u_0}\|^2, \quad n \in \mathbb{Z}_+,$$

and applying (6.1.5), we complete the proof. $\square$

Note that Lemma 6.1.5 is no longer true if $\operatorname{card}(\mathsf{Chi}(u_0)) \geqslant 2$.

Example 6.1.6. Let $S_{\boldsymbol\lambda}$ be a weighted shift on the directed tree $\mathscr{T}_{2,0}$ (cf. (6.2.10)) with weights $\{\lambda_v\}_{v\in V_{2,0}^\circ}$ given by $\{\lambda_{1,j}\}_{j=1}^\infty = \{a, \frac{b}{a}, \frac{a}{b}, 1, 1, \ldots\}$ and $\{\lambda_{2,j}\}_{j=1}^\infty = \{b, \frac{a}{b}, \frac{b}{a}, 1, 1, \ldots\}$, where $a, b \in (0,1)$ are such that $a < b$ and $a^2 + b^2 = 1$. Then $S_{\boldsymbol\lambda}$ is bounded, $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty = \{1, 1, \ldots\}$ is a Stieltjes moment sequence and neither of the sequences $\{\|S_{\boldsymbol\lambda}^n e_{1,1}\|^2\}_{n=0}^\infty = \{1, (\frac{b}{a})^2, 1, 1, \ldots\}$ and $\{\|S_{\boldsymbol\lambda}^n e_{2,1}\|^2\}_{n=0}^\infty = \{1, (\frac{a}{b})^2, 1, 1, \ldots\}$ is a Stieltjes moment sequence (consult also Proposition 6.2.4).

As an immediate consequence of Theorem 6.1.3 and Lemma 6.1.5 (see also Remark 3.1.4), we obtain the Berger-Gellar-Wallen criterion for subnormality of injective unilateral classical weighted shifts (cf. [**36, 40**]).

COROLLARY 6.1.7. *A bounded injective unilateral classical weighted shift with weights $\{\lambda_n\}_{n=1}^\infty$ (with notation as in (1.3)) is subnormal if and only if the sequence $\{1, |\lambda_1|^2, |\lambda_1\lambda_2|^2, |\lambda_1\lambda_2\lambda_3|^2, \ldots\}$ is a Stieltjes moment sequence.*

Before formulating the next corollary, we recall that a two-sided sequence $\{t_n\}_{n=-\infty}^\infty$ of real numbers is said to be a *two-sided Stieltjes moment sequence* if there exists a positive Borel measure μ on $(0,\infty)$ such that

$$t_n = \int_{(0,\infty)} s^n \mathrm{d}\mu(s), \quad n \in \mathbb{Z};$$

μ is called a representing measure of $\{t_n\}_{n=-\infty}^\infty$. It is easily seen that

(6.1.9) $\{t_n\}_{n=-\infty}^\infty \subseteq \mathbb{R}$ is a two-sided Stieltjes moment sequence if and only if $\{t_{n+k}\}_{n=-\infty}^\infty$ is a two-sided Stieltjes moment sequence for some (equivalently: for all) $k \in \mathbb{Z}$.

It is known that (cf. [**53**, Theorem 6.3] and [**12**, page 202])

(6.1.10) $\{t_n\}_{n=-\infty}^\infty \subseteq \mathbb{R}$ is a two-sided Stieltjes moment sequence if and only if the sequences $\{t_{n-k}\}_{n=0}^\infty$, $k = 0, 1, 2, \ldots$, are positive definite.

We are now in a position to deduce an analogue of the Berger-Gellar-Wallen criterion for subnormality of injective bilateral classical weighted shifts (cf. [**19**, Theorem II.6.12] and [**75**, Theorem 5]). Another proof of this fact will be given just after Remark 6.1.11.

Corollary 6.1.8. *A bounded injective bilateral classical weighted shift S with weights $\{\lambda_n\}_{n\in\mathbb{Z}}$ (with notation as in (1.3)) is subnormal if and only if the two-sided sequence $\{t_n\}_{n=-\infty}^{\infty}$ defined by*

$$t_n = \begin{cases} |\lambda_1 \cdots \lambda_n|^2 & \text{for } n \geqslant 1, \\ 1 & \text{for } n = 0, \\ |\lambda_{n+1} \cdots \lambda_0|^{-2} & \text{for } n \leqslant -1, \end{cases} \tag{6.1.11}$$

is a two-sided Stieltjes moment sequence.

Proof. By Lemma 6.1.5 and Theorem 6.1.3 (iii), S is subnormal if and only if $\{\|S^n(S^{-k}e_0)\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $k \in \mathbb{Z}_+$. This, when combined with (6.1.10), completes the proof. $\square$

Taking into account (6.1.10), we can rephrase Corollary 6.1.8 as follows: S is subnormal if and only if $\{\ldots, |\lambda_{-1}\lambda_0|^{-2}, |\lambda_0|^{-2}, 1, |\lambda_1|^2, |\lambda_1\lambda_2|^2, \ldots\}$ is a two-sided Stieltjes moment sequence no matter which position is chosen as the zero one.

Another question worth exploring is to find relationships between representing measures of Stieltjes moment sequences $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^{\infty}$, $u \in V$. We begin by fixing notation.

Notation 6.1.9. Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$. If for some $u \in V$, $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence, then, in view of (6.1.4), it is determinate and its unique representing measure is concentrated on $[0, \|S_{\boldsymbol\lambda}\|^2]$. Denote this measure by μ_u (or by $\mu_u^{\mathscr{T}}$ if we wish to make clear the dependence of μ_u on $\mathscr{T}$).

The result which follows will be referred to as the *big* lemma (as opposed to Lemma 6.1.5 which is called the small lemma).

Lemma 6.1.10. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, and let $u \in V'$ be such that $\{\|S_{\boldsymbol\lambda}^n e_v\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence for every $v \in \mathsf{Chi}(u)$. Then the following conditions are equivalent*[9]:

(i) *$\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^{\infty}$ is a Stieltjes moment sequence,*
(ii) *$S_{\boldsymbol\lambda}$ satisfies the consistency condition at u, i.e.,*

$$\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_v(s) \leqslant 1. \tag{6.1.12}$$

If (i) *holds, then $\mu_v(\{0\}) = 0$ for every $v \in \mathsf{Chi}(u)$ such that $\lambda_v \neq 0$, and the representing measure μ_u of $\big\{\|S_{\boldsymbol\lambda}^n e_u\|^2\big\}_{n=0}^{\infty}$ is given by*

(6.1.13)

$$\mu_u(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{s}\mathrm{d}\mu_v(s) + \Big(1 - \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\mathrm{d}\mu_v(s)\Big)\delta_0(\sigma)$$

for $\sigma \in \mathfrak{B}(\mathbb{R})$; moreover, $\mu_u(\{0\}) = 0$ if and only if $S_{\boldsymbol\lambda}$ satisfies the strong consistency condition at u, i.e.,

$$\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_v(s) = 1. \tag{6.1.14}$$

[9] In (6.1.12), we adhere to the standard convention that $0 \cdot \infty = 0$; see also footnote 8.

Proof. Define the set function μ on Borel subsets of $\mathbb{R}$ by

$$\mu(\sigma) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \mu_v(\sigma), \quad \sigma \in \mathfrak{B}(\mathbb{R}).$$

Then μ is a positive Borel measure concentrated on $[0, \infty)$, and[10]

$$\int_0^\infty f \mathrm{d}\mu = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty f \mathrm{d}\mu_v \tag{6.1.15}$$

for every Borel function $f\colon [0,\infty) \to [0,\infty]$. In particular, we have

$$\int_0^\infty \frac{1}{s}\, \mathrm{d}\mu(s) = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_v(s). \tag{6.1.16}$$

Combining (2.1.6) with the fact that the sets $\mathsf{Chi}^{\langle n \rangle}(u)$, $u \in V$, are pairwise disjoint for every fixed integer $n \geqslant 0$, we deduce that

$$\mathsf{Chi}^{\langle n+1 \rangle}(u) = \bigsqcup_{v \in \mathsf{Chi}(u)} \mathsf{Chi}^{\langle n \rangle}(v). \tag{6.1.17}$$

Employing twice Lemma 6.1.1 (ii), we get

$$\begin{aligned} \|S_{\boldsymbol\lambda}^{n+1} e_u\|^2 &= \sum_{w \in \mathsf{Chi}^{\langle n+1 \rangle}(u)} |\lambda_{u|w}|^2 \\ &\overset{(6.1.17)}{=} \sum_{v \in \mathsf{Chi}(u)} \sum_{w \in \mathsf{Chi}^{\langle n \rangle}(v)} |\lambda_{u|w}|^2 \\ &\overset{(6.1.2)}{=} \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \sum_{w \in \mathsf{Chi}^{\langle n \rangle}(v)} |\lambda_{v|w}|^2 \\ &= \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \|S_{\boldsymbol\lambda}^{n} e_v\|^2, \quad n \in \mathbb{Z}_+. \end{aligned} \tag{6.1.18}$$

This implies that

$$\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2 = \sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 \int_0^\infty s^n \, \mathrm{d}\mu_v(s) \overset{(6.1.15)}{=} \int_0^\infty s^n \mathrm{d}\mu(s), \quad n \in \mathbb{Z}_+,$$

which means that $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure μ. Since $\limsup_{n\to\infty} (\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2)^{1/n} \leqslant \|S_{\boldsymbol\lambda}\|^2$, we deduce from (6.1.4) that $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$ is a determinate Hamburger moment sequence, and consequently, μ is its unique representing measure. Employing now the equality (6.1.16) and Lemma 6.1.2 with $t_n = \|S_{\boldsymbol\lambda}^{n+1} e_u\|^2$, we see that the conditions (i) and (ii) are equivalent. The formula (6.1.13) can easily be inferred from (6.1.6) by applying (6.1.15) (consult Notation 6.1.9). The remaining part of conclusion is now obvious. □

Remark 6.1.11. A thorough inspection of the proof of Lemma 6.1.10 reveals that the implication (ii)⇒(i) can be justified without recourse to the determinacy of Stieltjes moment sequences. In particular, the formula (6.1.13) gives a representing measure of $\big\{\|S_{\boldsymbol\lambda}^{n} e_u\|^2\big\}_{n=0}^\infty$ provided μ_v is a representing measure of $\big\{\|S_{\boldsymbol\lambda}^{n} e_v\|^2\big\}_{n=0}^\infty$

---

[10] Apply the Lebesgue monotone convergence theorem to measures μ, μ_v and to the counting measure on $\mathsf{Chi}(u)$; note also that the cardinality of $\mathsf{Chi}(u)$ may be larger than $\aleph_0$.

concentrated on $[0,\infty)$ for every $v \in \mathsf{Chi}(u)$. This observation seems to be of potential relevance because it can be used to produce examples of unbounded weighted shifts on directed trees. However, the proof of the implication (i)⇒(ii) requires using the determinacy of the sequence $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$.

Lemma 6.1.10 turns out to be a useful tool for verifying subnormality of weighted shifts on directed trees. First, we apply it to give another proof (without recourse to (6.1.10)) of the Berger-Gellar-Wallen criterion for subnormality of injective bilateral classical weighted shifts.

Second proof of Corollary 6.1.8. Suppose first that S is subnormal. Applying Lemma 6.1.10 to $u = e_{k-1}$, $k \in \mathbb{Z}$, we deduce that $\mu_k(\{0\}) = 0$ for all $k \in \mathbb{Z}$. As a consequence, we see that the inequality (6.1.12) turns into the equality (6.1.14), and the second term of the right-hand side of the equality (6.1.13) vanishes. This, when applied to $u = e_{-1}$, leads to $\frac{1}{|\lambda_0|^2} = \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_0(s)$ and $\mathrm{d}\mu_{-1}(s) = \frac{|\lambda_0|^2}{s}\mathrm{d}\mu_0(s)$ (be aware of (1.3)). Employing an induction argument, we show that for every $k \in \mathbb{Z}_+$,

$$\frac{1}{|\lambda_{-k}\cdots\lambda_0|^2} = \int_0^\infty \frac{1}{s^{k+1}}\,\mathrm{d}\mu_0(s) \text{ and } \mathrm{d}\mu_{-k-1}(s) = \frac{|\lambda_{-k}\cdots\lambda_0|^2}{s^{k+1}}\mathrm{d}\mu_0(s), \tag{6.1.19}$$

which completes the proof of the "only if" part of the conclusion.

Reversely, if $\{t_n\}_{n=-\infty}^\infty$ in (6.1.11) is a two-sided Stieltjes moment sequence with a representing measure μ_0, then the equality $\|S^n e_{-k}\|^2 = |\lambda_{-k+1}\cdots\lambda_0|^2 t_{n-k}$ which holds for all $k \in \mathbb{N}$ and $n \in \mathbb{Z}_+$ implies that for every $k \in \mathbb{N}$, the sequence $\{\|S^n e_{-k}\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence with a representing measure μ_{-k} defined in (6.1.19). By Lemma 6.1.5, $\{\|S^n e_k\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $k \in \mathbb{Z}_+$. This, together with Theorem 6.1.3, completes the proof of the "if" part. □

The ensuing proposition which concerns subnormal extendibility of weights will be illustrated in Example 6.3.5 in the context of directed trees $\mathscr{T}_{\eta,\kappa}$.

Proposition 6.1.12. *Let $\mathscr{T} = (V,E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat V, \hat E)$ such that $\mathsf{Chi}_{\mathscr{T}}(w) \neq \mathsf{Chi}_{\hat{\mathscr{T}}}(w)$ for some $w \in V \setminus \mathsf{Root}(\mathscr{T})$, and $\mathsf{Des}_{\mathscr{T}}(v) = \mathsf{Des}_{\hat{\mathscr{T}}}(v)$ for all $v \in \mathsf{Chi}_{\mathscr{T}}(w)$. Suppose that $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ is a subnormal weighted shift on the directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_u\}_{u\in V^\circ}$. Then the directed tree $\mathscr{T}$ is leafless and there exists no subnormal weighted shift $S_{\hat{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(\hat V))$ on $\hat{\mathscr{T}}$ with nonzero weights $\hat{\boldsymbol\lambda} = \{\hat\lambda_u\}_{u\in\hat V^\circ}$ such that $\boldsymbol\lambda \subseteq \hat{\boldsymbol\lambda}$, i.e., $\lambda_u = \hat\lambda_u$ for all $u \in V^\circ$.*

Note that the directed tree $\hat{\mathscr{T}}$ in Proposition 6.1.12 may not be leafless.

Proof of Proposition 6.1.12. Suppose that, contrary to our claim, such an $S_{\hat{\boldsymbol\lambda}}$ exists. It follows from Proposition 5.1.1 that $\mathscr{T}$ and $\hat{\mathscr{T}}$ are leafless. Hence $\mathsf{Chi}_{\mathscr{T}}(w) \neq \varnothing$. Applying Lemma 6.1.10 to $S_{\boldsymbol\lambda}$ and $u = \mathsf{par}(w)$, we see that $\mu_w^{\mathscr{T}}(\{0\}) = 0$. Next, applying Lemma 6.1.10 to $S_{\boldsymbol\lambda}$ and $u = w$, we get

$$1 = \sum_{v\in\mathsf{Chi}_{\mathscr{T}}(w)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_v^{\mathscr{T}}(s) \quad \text{(see Notation 6.1.9).} \tag{6.1.20}$$

The same is true for $S_{\hat{\boldsymbol\lambda}}$. Hence, we have

$$
\begin{aligned}
1 = \sum_{v\in \mathsf{Chi}_{\hat{\mathscr{T}}}(w)} |\hat{\lambda}_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_v^{\hat{\mathscr{T}}}(s) &\overset{(\star)}{=} \sum_{v\in \mathsf{Chi}_{\mathscr{T}}(w)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_v^{\mathscr{T}}(s) + \sum_{v\in \mathsf{Chi}_{\hat{\mathscr{T}}}(w)\setminus \mathsf{Chi}_{\mathscr{T}}(w)} |\hat{\lambda}_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_v^{\hat{\mathscr{T}}}(s) \\
&\overset{(6.1.20)}{=} 1 + \sum_{v\in \mathsf{Chi}_{\hat{\mathscr{T}}}(w)\setminus \mathsf{Chi}_{\mathscr{T}}(w)} |\hat{\lambda}_v|^2 \int_0^\infty \frac{1}{s}\,\mathrm{d}\mu_v^{\hat{\mathscr{T}}}(s),
\end{aligned}
$$

which implies that $\hat{\lambda}_v = 0$ for all $v \in \mathsf{Chi}_{\hat{\mathscr{T}}}(w) \setminus \mathsf{Chi}_{\mathscr{T}}(w) \neq \varnothing$, a contradiction. The equality $(\star)$ follows from $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$ and the fact that $\mu_v^{\mathscr{T}} = \mu_v^{\hat{\mathscr{T}}}$ for all $v \in \mathsf{Chi}_{\mathscr{T}}(w)$, the latter being a direct consequence of the equality $S_{\boldsymbol{\lambda}}|_{\ell^2(\mathsf{Des}_{\mathscr{T}}(v))} = S_{\hat{\boldsymbol{\lambda}}}|_{\ell^2(\mathsf{Des}_{\hat{\mathscr{T}}}(v))}$ which holds for all $v \in \mathsf{Chi}_{\mathscr{T}}(w)$. This completes the proof. $\square$

Note that Proposition 6.1.12 is no longer true when $w = \mathsf{root}(\mathscr{T})$ (cf. Example 6.3.5).

6.2. Subnormality on assorted directed trees. Classical weighted shifts are built on very special directed trees which are characterized by the property that each vertex has exactly one child (cf. Remark 3.1.4). In this section, we consider one step more complicated directed trees, namely those with the property that each vertex except one has exactly one child; the exceptional vertex is assumed to be a branching vertex (cf. (2.1.2)).

Below we adhere to Notation 3.1.5. Set $J_\iota = \{k \in \mathbb{N}\colon k \leqslant \iota\}$ for $\iota \in \mathbb{Z}_+ \sqcup \{\infty\}$. Note that $J_0 = \varnothing$ and $J_\infty = \mathbb{N}$.

THEOREM 6.2.1. *Suppose that $\mathscr{T}$ is a directed tree for which there exists $\omega \in V$ such that $\mathrm{card}(\mathsf{Chi}(\omega)) \geqslant 2$ and $\mathrm{card}(\mathsf{Chi}(v)) = 1$ for every $v \in V \setminus \{\omega\}$. Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on the directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V^\circ}$. Then the following assertions are valid.*

- (i) *If $\omega \in \mathsf{Root}(\mathscr{T})$, then $S_{\boldsymbol{\lambda}}$ is subnormal if and only if (6.1.12) holds for $u = \omega$ and $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^\infty$ is a Stieltjes moment sequence for every $v \in \mathsf{Chi}(\omega)$.*
- (ii) *If $\mathscr{T}$ has a root and $\omega \neq \mathsf{root}$, then $S_{\boldsymbol{\lambda}}$ is subnormal if and only if any one of the following two equivalent conditions holds:*
  - (ii-a) *$S_{\boldsymbol{\lambda}_{\to(\omega)}}$ is subnormal, (6.1.14) is valid for $u = \omega$,*

$$
\frac{1}{|\prod_{j=0}^{k-1} \lambda_{\mathsf{par}^j(\omega)}|^2} = \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 \int_0^\infty \frac{1}{s^{k+1}}\mathrm{d}\mu_v(s) \tag{6.2.1}
$$

    for all $k \in J_{\kappa-1}$, and

$$
\frac{1}{|\prod_{j=0}^{\kappa-1} \lambda_{\mathsf{par}^j(\omega)}|^2} \geqslant \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 \int_0^\infty \frac{1}{s^{\kappa+1}}\mathrm{d}\mu_v(s), \tag{6.2.2}
$$

    where κ is a unique positive integer such that $\mathsf{par}^\kappa(\omega) = \mathsf{root}$;
  - (ii-b) *$\{\|S_{\boldsymbol{\lambda}}^n e_{\mathsf{root}}\|^2\}_{n=0}^\infty$ and $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^\infty$ are Stieltjes moment sequences for all $v \in \mathsf{Chi}(\omega)$.*
- (iii) *If $\mathscr{T}$ is rootless, then $S_{\boldsymbol{\lambda}}$ is subnormal if and only if any one of the following two equivalent conditions holds:*
  - (iii-a) *$S_{\boldsymbol{\lambda}_{\to(\omega)}}$ is subnormal, (6.1.14) is valid for $u = \omega$, and (6.2.1) is valid for all $k \in \mathbb{N}$,*

(iii-b) $\{\|S^n_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|^2\}_{n=0}^\infty$ *and* $\{\|S^n_{\boldsymbol\lambda} e_v\|^2\}_{n=0}^\infty$ *are Stieltjes moment sequences for infinitely many integers* $k \geqslant 1$ *and for all* $v \in \mathsf{Chi}(\omega)$.

Proof. (i) To prove the "if" part of the conclusion of (i), denote by $\mathscr{S}$ the set of all $v \in V$ such that $\big\{\|S^n_{\boldsymbol\lambda} e_v\|^2\big\}_{n=0}^\infty$ is a Stieltjes moment sequence. Then, by Lemma 6.1.10, $\{\omega\} \cup \mathsf{Chi}(\omega) \subseteq \mathscr{S}$. In turn, by Corollary 2.1.5 and Lemma 6.1.5, we see that $V \setminus (\{\omega\} \cup \mathsf{Chi}(\omega)) \subseteq \mathscr{S}$, which together with Theorem 6.1.3 completes the proof of the "if" part. The "only if" part is a direct consequence of Theorem 6.1.3 and Lemma 6.1.10.

(ii) It follows from our assumptions on $\mathscr{T}$ and Proposition 2.1.7 that there exists a unique $\kappa \in \mathbb{N}$ satisfying the equality $\mathsf{par}^\kappa(u) = \mathsf{root}$, and that

$$V = \{\mathsf{par}^j(\omega) \colon j = 1, \ldots, \kappa\} \sqcup \mathsf{Des}(\omega). \tag{6.2.3}$$

We first prove the "only if" part of the conclusion of (ii). For this, suppose that $S_{\boldsymbol\lambda}$ is subnormal. Then $S_{\boldsymbol\lambda_{\to(\omega)}}$ is subnormal as a restriction of $S_{\boldsymbol\lambda}$ to its invariant subspace $\ell^2(\mathsf{Des}(\omega))$. Applying Lemma 6.1.10 to $u = \mathsf{par}^k(\omega)$, $k \in J_\kappa$, we deduce that $\mu_{\mathsf{par}^k(\omega)}(\{0\}) = 0$ for all $k = 0, \ldots, \kappa - 1$. This, when combined with Lemma 6.1.10, applied to $u = \mathsf{par}^k(\omega)$ with $k = 0, \ldots, \kappa$, leads to

$$\sum_{v \in \mathsf{Chi}(\omega)} |\lambda_v|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_v(s) = 1, \tag{6.2.4}$$

$$|\lambda_{\mathsf{par}^k(\omega)}|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_{\mathsf{par}^k(\omega)}(s) = 1, \quad k \in \mathbb{Z},\ 0 \leqslant k \leqslant \kappa - 2, \tag{6.2.5}$$

$$|\lambda_{\mathsf{par}^{\kappa-1}(\omega)}|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_{\mathsf{par}^{\kappa-1}(\omega)}(s) \leqslant 1, \tag{6.2.6}$$

$$\mu_\omega(\sigma) = \sum_{v \in \mathsf{Chi}(\omega)} |\lambda_v|^2 \int_\sigma \frac{1}{s} \mathrm{d}\mu_v(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}), \tag{6.2.7}$$

$$\mu_{\mathsf{par}^k(\omega)}(\sigma) = |\lambda_{\mathsf{par}^{k-1}(\omega)}|^2 \int_\sigma \frac{1}{s} \mathrm{d}\mu_{\mathsf{par}^{k-1}(\omega)}(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}),\ k \in J_{\kappa-1}. \tag{6.2.8}$$

Using an induction argument, we deduce from (6.2.5), (6.2.7) and (6.2.8) that (6.2.1) holds for every $k \in J_{\kappa-1}$, and that the measures $\mu_{\mathsf{par}^k(\omega)}$, $k \in J_{\kappa-1}$, are given by

$$\frac{\mu_{\mathsf{par}^k(\omega)}(\sigma)}{|\prod_{j=0}^{k-1} \lambda_{\mathsf{par}^j(\omega)}|^2} = \sum_{v \in \mathsf{Chi}(\omega)} |\lambda_v|^2 \int_\sigma \frac{1}{s^{k+1}}\, \mathrm{d}\mu_v(s), \quad \sigma \in \mathfrak{B}(\mathbb{R}). \tag{6.2.9}$$

(To show that (6.2.1) holds for $k+1$ in place of k, we have to employ the formula (6.2.9).) Next, we infer (6.2.2) from (6.2.6), (6.2.7) (if $\kappa = 1$) and (6.2.9) (if $\kappa \geqslant 2$, with $k = \kappa - 1$). This means that (ii-a) holds. Clearly, the condition (ii-b) is a direct consequence of Theorem 6.1.3.

Let us turn to the proof of the "if" part of the conclusion of (ii). Assume first that (ii-a) holds. By subnormality of $S_{\boldsymbol\lambda_{\to(\omega)}} = S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(\omega))}$, we have $\mathsf{Des}(\omega) \subseteq \mathscr{S}$ (cf. Theorem 6.1.3). This and (6.2.4), when combined with Lemma 6.1.10, yields (6.2.7). If $\kappa = 1$, then the formula (6.2.7) for μ_ω enables us to rewrite the inequality (6.2.2) as $|\lambda_\omega|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_\omega(s) \leqslant 1$. By Lemma 6.1.10 (ii), $\mathsf{root} = \mathsf{par}(\omega) \in \mathscr{S}$, which together with (6.2.3) and Theorem 6.1.3 implies subnormality of $S_{\boldsymbol\lambda}$. If $\kappa \geqslant 2$, then the equality (6.2.1) with $k = 1$ takes the form $|\lambda_\omega|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_\omega(s) = 1$. Applying Lemma 6.1.10 again, we see that $\mathsf{par}(\omega) \in \mathscr{S}$ and the measure $\mu_{\mathsf{par}(\omega)}$ is given by (6.2.9) with $k = 1$. An induction argument shows that for every $k \in J_{\kappa-1}$, $\mathsf{par}^k(\omega) \in \mathscr{S}$ and the measure $\mu_{\mathsf{par}^k(\omega)}$ is given by (6.2.9). Finally, the formula (6.2.9) with $k = \kappa - 1$ for $\mu_{\mathsf{par}^{\kappa-1}(\omega)}$ enables us to rewrite the inequality (6.2.2) as $|\lambda_{\mathsf{par}^{\kappa-1}(\omega)}|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_{\mathsf{par}^{\kappa-1}(\omega)}(s) \leqslant 1$. By Lemma 6.1.10 (ii), $\mathsf{root} = \mathsf{par}^\kappa(\omega) \in \mathscr{S}$. This combined with (6.2.3) and Theorem 6.1.3 shows that $S_{\boldsymbol\lambda}$ is subnormal.

Suppose now that (ii-b) holds. Employing Lemma 6.1.5 repeatedly first to $u_0 = \mathsf{root} = \mathsf{par}^\kappa(\omega)$, then to $u_0 = \mathsf{par}^{\kappa-1}(\omega)$ and so on up to $u_0 = \mathsf{par}^1(\omega)$, we see that $\{\mathsf{par}^j(\omega)\colon j = 0, \ldots, \kappa\} \subseteq \mathscr{S}$. The same procedure applied to members of $\mathsf{Chi}(\omega)$ shows that $\mathsf{Des}(\omega)\setminus\{\omega\} \subseteq \mathscr{S}$, which by (6.2.3) and Theorem 6.1.3 completes the proof of (ii).

(iii) Let J be an infinite subset of $\mathbb{N}$. In view of Proposition 2.1.6 (iii), we can apply Corollary 6.1.4 to $X = \{\mathsf{par}^k(\omega)\colon k \in J\}$. What we get is that $S_{\boldsymbol\lambda}$ is subnormal if and only if $S_{\boldsymbol\lambda_{\to(x)}}$ is subnormal for every $x \in X$. Since for every $x \in X$, $S_{\boldsymbol\lambda_{\to(x)}} = S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(x))}$ and the directed tree $\mathscr{T}_{\mathsf{Des}(x)}$ has the property required in (ii), we see that (iii) can be deduced from (ii) by applying the aforesaid characterization of subnormality of $S_{\boldsymbol\lambda}$. This completes the proof. $\square$

Our next aim is to rewrite Theorem 6.2.1 in terms of weights of weighted shifts being studied. In view of Proposition 5.1.1, there is no loss of generality in assuming that $\operatorname{card}(\mathsf{Chi}(\omega)) \leqslant \aleph_0$. A careful look at the proof of Theorem 6.2.1 reveals that the directed trees considered therein can be modelled as follows (see Figure 6). Given $\eta, \kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$ with $\eta \geqslant 2$, we define the directed tree $\mathscr{T}_{\eta,\kappa} = (V_{\eta,\kappa}, E_{\eta,\kappa})$ by (recall that $J_\iota = \{k \in \mathbb{N}\colon k \leqslant \iota\}$ for $\iota \in \mathbb{Z}_+ \sqcup \{\infty\}$)

$$\begin{aligned} V_{\eta,\kappa} &= \big\{-k\colon k \in J_\kappa\big\} \sqcup \{0\} \sqcup \big\{(i,j)\colon i \in J_\eta,\ j \in \mathbb{N}\big\}, \\ E_{\eta,\kappa} &= E_\kappa \sqcup \big\{(0,(i,1))\colon i \in J_\eta\big\} \sqcup \big\{((i,j),(i,j+1))\colon i \in J_\eta,\ j \in \mathbb{N}\big\}, \\ E_\kappa &= \big\{(-k,-k+1)\colon k \in J_\kappa\big\}. \end{aligned} \tag{6.2.10}$$

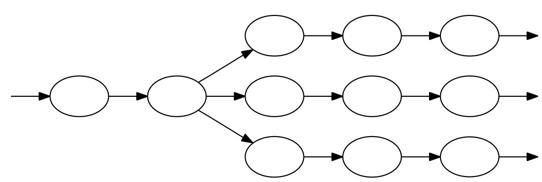

Figure 6

If $\kappa < \infty$, then the directed tree $\mathscr{T}_{\eta,\kappa}$ has a root and $\mathsf{root}(\mathscr{T}_{\eta,\kappa}) = -\kappa$. In turn, if $\kappa = \infty$, then the directed tree $\mathscr{T}_{\eta,\infty}$ is rootless. In all cases, the branching vertex ω is equal to 0. Note that the simplest[11] leafless directed tree which is not isomorphic to $\mathbb{Z}_+$ and $\mathbb{Z}$ (cf. Remark 3.1.4) coincides with $\mathscr{T}_{2,0}$.

We are now ready to reformulate Theorem 6.2.1 in terms of weights. Writing the counterpart of (iii-b), being a little bit too long, is left to the reader. Below we adhere to notation $\lambda_{i,j}$ instead of a more formal expression $\lambda_{(i,j)}$.

Corollary 6.2.2. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ be a weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Then the following assertions hold.*

[11] This means that there is no proper leafless subtree of the underlying directed tree which is not isomorphic to $\mathbb{Z}_+$.

(i) *If $\kappa = 0$, then $S_{\boldsymbol\lambda}$ is subnormal if and only if there exist Borel probability measures $\{\mu_i\}_{i=1}^{\eta}$ on $[0,\infty)$ such that*

$$\int_0^\infty s^n \mathrm{d}\mu_i(s) = \Big|\prod_{j=2}^{n+1} \lambda_{i,j}\Big|^2, \quad n \in \mathbb{N},\ i \in J_\eta, \tag{6.2.11}$$

$$\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_i(s) \leqslant 1. \tag{6.2.12}$$

(ii) *If $0 < \kappa < \infty$, then $S_{\boldsymbol\lambda}$ is subnormal if and only if one of the following two equivalent conditions holds*:

(ii-a) *there exist Borel probability measures $\{\mu_i\}_{i=1}^{\eta}$ on $[0,\infty)$ which satisfy* (6.2.11) *and the following requirements*:

$$\sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s}\, \mathrm{d}\mu_i(s) = 1, \tag{6.2.13}$$

$$\frac{1}{|\prod_{j=0}^{k-1} \lambda_{-j}|^2} = \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{k+1}} \mathrm{d}\mu_i(s), \quad k \in J_{\kappa-1}, \tag{6.2.14}$$

$$\frac{1}{|\prod_{j=0}^{\kappa-1} \lambda_{-j}|^2} \geqslant \sum_{i=1}^{\eta} |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}} \mathrm{d}\mu_i(s); \tag{6.2.15}$$

(ii-b) *there exist Borel probability measures $\{\mu_i\}_{i=1}^{\eta}$ and ν on $[0,\infty)$ which satisfy* (6.2.11) *and the equations below*

$$\int_0^\infty s^n \mathrm{d}\nu(s) = \begin{cases} |\prod_{j=\kappa-n}^{\kappa-1} \lambda_{-j}|^2 & \text{if } n \in J_\kappa, \\ |\prod_{j=0}^{\kappa-1} \lambda_{-j}|^2 \big(\sum_{i=1}^{\eta} |\prod_{j=1}^{n-\kappa} \lambda_{i,j}|^2\big) & \text{if } n \in \mathbb{N} \setminus J_\kappa. \end{cases}$$

(iii) *If $\kappa = \infty$, then $S_{\boldsymbol\lambda}$ is subnormal if and only if there exist Borel probability measures $\{\mu_i\}_{i=1}^{\eta}$ on $[0,\infty)$ satisfying* (6.2.11), (6.2.13) *and* (6.2.14).

Moreover, if $S_{\boldsymbol\lambda}$ is subnormal and $\{\mu_i\}_{i=1}^{\eta}$ are Borel probability measures on $[0,\infty)$ satisfying (6.2.11), *then $\mu_i = \mu_{i,1}$ for all $i \in J_\eta$.*

PROOF. Apply Theorem 6.2.1 (consult also (6.1.4)). □

Corollary 6.2.2 suggests the possibility of singling out a class of subnormal weighted shifts on $\mathscr{T}_{\eta,\kappa}$ (with $\kappa \in \mathbb{N}$) whose behaviour on e_{root} is, in a sense, extreme.

Remark 6.2.3. Suppose that $\kappa \in \mathbb{N}$. We say that a subnormal weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ is *extremal* if

$$\|S_{\boldsymbol\lambda} e_{\mathsf{root}}\| = \max \|S_{\tilde{\boldsymbol\lambda}} e_{\mathsf{root}}\|,$$

where the maximum is taken over all subnormal weighted shifts $S_{\tilde{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\tilde{\boldsymbol\lambda} = \{\tilde\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that $S_{\boldsymbol\lambda_{\to(-\kappa+1)}} = S_{\tilde{\boldsymbol\lambda}_{\to(-\kappa+1)}}$, or equivalently that $\lambda_v = \tilde\lambda_v$ for all $v \neq -\kappa+1$. It follows from Corollary 6.2.2 that a subnormal weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ is extremal if and only if $S_{\boldsymbol\lambda}$ satisfies the condition (ii-a) with the inequality in (6.2.15) replaced by equality; in other words, $S_{\boldsymbol\lambda}$ is extremal if and only if $S_{\boldsymbol\lambda}$ satisfies the strong consistency condition at each vertex $u \in V_{\eta,\kappa}$ (cf. (6.1.14)). Hence, if $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a subnormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$, then any weighted shift $S_{\tilde{\boldsymbol{\lambda}}}$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\tilde{\boldsymbol{\lambda}} = \{\tilde{\lambda}_v\}_{v \in V_{\eta,\kappa}^\circ}$ satisfying the following equalities

$$|\tilde{\lambda}_v| = \begin{cases} |\lambda_v| & \text{for } v \neq -\kappa+1, \\ \Big(\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}} \mathrm{d}\mu_i(s)\Big)^{-1/2} & \text{for } v = -\kappa+1 \text{ if } \kappa = 1, \\ \prod_{j=0}^{\kappa-2} |\lambda_{-j}| \Big(\sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^\infty \frac{1}{s^{\kappa+1}} \mathrm{d}\mu_i(s)\Big)^{-1/2} & \text{for } v = -\kappa+1 \text{ if } \kappa > 1, \end{cases}$$

is bounded (cf. Proposition 3.1.8), subnormal and extremal.

If a weighted shift $S_{\boldsymbol{\lambda}}$ on the directed tree $\mathscr{T}_{\eta,0}$ is an isometry, then $S_{\boldsymbol{\lambda}}$ is subnormal and $\|S_{\boldsymbol{\lambda}}^n e_0\|^2 = 1$ for all $n \in \mathbb{Z}_+$. It turns out that the reverse implication holds as well even if subnormality is relaxed into hyponormality. According to Example 6.1.6 and Proposition 6.2.4 below, the assumption $\|S_{\boldsymbol{\lambda}}^n e_0\|^2 = 1$, $n \in \mathbb{Z}_+$, by itself does not imply the isometricity of $S_{\boldsymbol{\lambda}}$. This phenomenon is quite different comparing with the case of unilateral classical weighted shifts in which the aforesaid assumption always implies isometricity.

Proposition 6.2.4. *If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,0}))$ is a weighted shift on $\mathscr{T}_{\eta,0}$ with positive weights $\{\lambda_v\}_{v \in V_{\eta,0}^\circ}$, then the following conditions are equivalent*:

(i) *$S_{\boldsymbol{\lambda}}$ is an isometry,*
(ii) *$S_{\boldsymbol{\lambda}}$ is hyponormal and $\|S_{\boldsymbol{\lambda}}^n e_0\|^2 = 1$ for all $n \in \mathbb{Z}_+$,*
(iii) *$\sum_{i=1}^\eta \lambda_{i,1}^2 = 1$ and $\lambda_{i,j} = 1$ for all $i \in J_\eta$ and $j = 2, 3, \ldots$*

Proof. (i)⇒(ii) Obvious.

(ii)⇒(iii) Fix an integer $j \geqslant 2$. Consider first the case when $\lambda_{i,j} > 1$ for some $i \in J_\eta$. Since $S_{\boldsymbol{\lambda}}$ is hyponormal, we see that $\lambda_{i,2} \leqslant \lambda_{i,3} \leqslant \lambda_{i,4} \leqslant \ldots$ (cf. Theorem 5.1.2). Hence, we have

$$(\lambda_{i,j}^2)^{n+1} \Big(\prod_{k=1}^{j-1} \lambda_{i,k}^2\Big) \leqslant \prod_{k=1}^{j+n} \lambda_{i,k}^2 \leqslant \sum_{l=1}^{\eta} \prod_{k=1}^{j+n} \lambda_{l,k}^2 = \|S_{\boldsymbol{\lambda}}^{j+n} e_0\|^2 = 1, \quad n \geqslant 0,$$

which contradicts $\lambda_{i,j} > 1$. Thus, we must have $\lambda_{i,j} \leqslant 1$ for all $i \in J_\eta$. This in turn implies that $\lambda_{i,j} = 1$ for $i \in J_\eta$, because if $\lambda_{i,j} < 1$ for some $i \in J_\eta$, then

$$\|S_{\boldsymbol{\lambda}}^{j-1} e_0\|^2 = \sum_{l=1}^{\eta} \prod_{k=1}^{j-1} \lambda_{l,k}^2 < \sum_{l=1}^{\eta} \prod_{k=1}^{j} \lambda_{l,k}^2 = \|S_{\boldsymbol{\lambda}}^{j} e_0\|^2,$$

which is a contradiction.

(iii)⇒(i) Apply Corollary 3.4.4. □

Proposition 6.2.5. *If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ and $\kappa \geqslant 1$, then the following conditions are equivalent*:

(i) *$S_{\boldsymbol{\lambda}}$ is an isometry,*
(ii) *$\|S_{\boldsymbol{\lambda}}^n e_0\|^2 = 1$ for all $n \in \mathbb{Z}_+$, and either $S_{\boldsymbol{\lambda}}$ is subnormal and extremal if $\kappa < \infty$, or $S_{\boldsymbol{\lambda}}$ is subnormal if $\kappa = \infty$,*
(iii) *$\sum_{i=1}^\eta \lambda_{i,1}^2 = 1$, $\lambda_{i,j} = 1$ for all $i \in J_\eta$ and $j = 2, 3, \ldots$, and $\lambda_{-k} = 1$ for all integers k such that $0 \leqslant k < \kappa$.*

Proof. As in the previous proof, we verify that the only implication which requires explanation is (ii)⇒(iii). Applying Proposition 6.2.4 to the subnormal operator $S_{\boldsymbol{\lambda}_{\rightarrow(0)}}$, we see that condition (iii) of this proposition holds. Since evidently

$\mu_0 = \delta_1$, we deduce from the strong consistency condition at $u = -1$ (which is guaranteed either by Lemma 6.1.10, or by the extremality of $S_{\boldsymbol{\lambda}}$, cf. Remark 6.2.3) that $\lambda_0 = 1$. This in turn implies that $\mu_{-1} = \delta_1$. Using this procedure repeatedly completes the proof. $\square$

Example 6.2.6. In general, if $\kappa \in \mathbb{N}$, it is impossible to relax extremal subnormality into subnormality in the condition (ii) of Proposition 6.2.5 without affecting the implication (ii)$\Rightarrow$(i). Indeed, using Proposition 6.2.5, we construct an isometric weighted shift $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$. In particular, $\lambda_{-k} = 1$ for all integers k such that $0 \leqslant k < \kappa$. For a fixed $t \in (0,1)$, we define the weighted shift $S_{\tilde{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\tilde{\boldsymbol{\lambda}} = \{\tilde{\lambda}_v\}_{v\in V_{\eta,\kappa}^\circ}$ by $\tilde{\lambda}_v = \lambda_v$ for every $v \neq -\kappa+1$, and $\tilde{\lambda}_{-\kappa+1} = t$. Since δ_1 is the representing measure of the Stieltjes moment sequence $\{\|S_{\boldsymbol{\lambda}}^n e_v\|^2\}_{n=0}^\infty$ for all $v \in V_{\eta,\kappa}$, we infer from Corollary 6.2.2 (ii-a) that $S_{\tilde{\boldsymbol{\lambda}}}$ is subnormal but not isometric.

It turns out that if $\kappa = \infty$, then the implication (ii)$\Rightarrow$(i) of Proposition 6.2.5 is no longer true if subnormality is replaced by hyponormality (compare with Proposition 6.2.4). Indeed, using Theorem 5.1.2, one can construct a non-isometric hyponormal $S_{\boldsymbol{\lambda}}$ with isometric $S_{\boldsymbol{\lambda}_{\to(0)}}$.

6.3. Modelling subnormality on $\mathscr{T}_{\eta,\kappa}$. Corollary 6.2.2 enables us to give a method of constructing all possible bounded subnormal weighted shifts on the directed tree $\mathscr{T}_{\eta,\kappa}$ with nonzero weights (see (6.2.10) for the definition of $\mathscr{T}_{\eta,\kappa}$). This is done in Procedure 6.3.1 below. By virtue of Theorem 3.2.1, there is no loss of generality in assuming that weighted shifts under consideration have positive weights. Note also that, in view of Notation 6.1.9, the Borel probability measures $\{\mu_i\}_{i=1}^\eta$ appearing in Corollary 6.2.2 are unique, concentrated on a common finite subinterval of $[0,\infty)$ and

$$\int_0^\infty \frac{1}{s^k}\mathrm{d}\mu_i(s) < \infty, \quad k \in J_{\kappa+1},\, i \in J_\eta. \tag{6.3.1}$$

Procedure 6.3.1. Let $\{\mu_i\}_{i=1}^\eta$ be a sequence of Borel probability measures on $[0,\infty)$ satisfying (6.3.1) and concentrated on a common finite subinterval of $[0,\infty)$. It is easily seen that then

$$0 < \int_0^\infty s^n \mathrm{d}\mu_i(s) < \infty, \quad n \in \mathbb{Z},\, n \geqslant -(\kappa+1),\, i \in J_\eta.$$

Set $M = \sup_{i\in J_\eta} \sup \operatorname{supp}(\mu_i)$, where $\operatorname{supp}(\mu_i)$ stands for the closed support of the measure μ_i. Define

$$\lambda_{i,j} = \sqrt{\frac{\int_0^\infty s^{j-1}\mathrm{d}\mu_i(s)}{\int_0^\infty s^{j-2}\mathrm{d}\mu_i(s)}}, \quad j = 2,3,4,\ldots,\, i \in J_\eta. \tag{6.3.2}$$

If $\kappa = 0$, we take a sequence $\{\lambda_{i,1}\}_{i\in J_\eta}$ of positive real numbers satisfying (6.2.12). Then clearly the weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,0}^\circ}$ just defined satisfy (6.2.11) and (6.2.12). If $\kappa \geqslant 1$, we take a sequence $\{\lambda_{i,1}\}_{i\in J_\eta}$ of positive real numbers solving (6.2.13) and satisfying the following condition

$$\sum_{i=1}^\eta \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^k}\mathrm{d}\mu_i(s) < \infty, \quad k \in J_{\kappa+1}. \tag{6.3.3}$$

(The question of the existence of such a sequence is discussed in Lemma 6.3.2 below.) The remaining weights $\{\lambda_{-k}\}_{k=0}^{\kappa-1}$ are defined by

$$\lambda_{-k} = \sqrt{\frac{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+1}} \mathrm{d}\mu_i(s)}{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+2}} \mathrm{d}\mu_i(s)}}, \quad k \in \mathbb{Z}_+,\ 0 \leqslant k < \kappa - 1, \tag{6.3.4}$$

$$\lambda_{-\kappa+1} = \vartheta, \text{ where } 0 < \vartheta \leqslant \sqrt{\frac{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{\kappa}} \mathrm{d}\mu_i(s)}{\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{\kappa+1}} \mathrm{d}\mu_i(s)}}. \tag{6.3.5}$$

One can easily verify that the full system of weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ satisfies (6.2.11), (6.2.13), (6.2.14) and (6.2.15) (of course, if $\kappa = 1$, then (6.2.14) and (6.3.4) do not appear; similarly, if $\kappa = \infty$, then (6.2.15) and (6.3.5) do not appear). Thus we are left with proving the boundedness of the weighted shift $S_{\boldsymbol{\lambda}}$. We will also give an explicit formula for the norm of $S_{\boldsymbol{\lambda}}$.

Fix $i \in J_\eta$ and set $\alpha_{i,n} := \int_0^\infty s^n \mathrm{d}\mu_i(s)$ for $n \in \mathbb{Z}_+$. Applying the Cauchy-Schwarz inequality in $L^2(\mu_i)$, we get

$$\alpha_{i,n+1}^2 \leqslant \alpha_{i,n} \cdot \alpha_{i,n+2}, \quad n \in \mathbb{Z}_+.$$

Hence, the sequence $\{\frac{\alpha_{i,n+1}}{\alpha_{i,n}}\}_{n=1}^\infty$ is monotonically increasing. This combined with [**68**, Exercise 23, page 74] gives

$$\sup_{j \geqslant 1} \|S_{\boldsymbol{\lambda}} e_{i,j}\|^2 = \sup_{j \geqslant 2} \lambda_{i,j}^2 \overset{(6.3.2)}{=} \sup_{n \geqslant 0} \frac{\alpha_{i,n+1}}{\alpha_{i,n}} = \lim_{n \to \infty} \frac{\alpha_{i,n+1}}{\alpha_{i,n}} = \sup \operatorname{supp}(\mu_i). \tag{6.3.6}$$

Since μ_i, $i \in J_\eta$, are probability measures, we deduce from (6.2.12) that

$$\|S_{\boldsymbol{\lambda}} e_0\|^2 = \sum_{i \in J_\eta} \lambda_{i,1}^2 \leqslant M \sum_{i \in J_\eta} \lambda_{i,1}^2 \int_0^M \frac{1}{s} \mathrm{d}\mu_i(s) \leqslant M. \tag{6.3.7}$$

Together with Proposition 3.1.8, this implies that if $\kappa = 0$, then $S_{\boldsymbol{\lambda}}$ is bounded and $\|S_{\boldsymbol{\lambda}}\|^2 = M$. The case of $\kappa \geqslant 1$ needs a little more effort. By the Cauchy-Schwarz inequality for integrals and series, we get

$$\begin{aligned} &\left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+2}} \mathrm{d}\mu_i(s) \right)^2 \\ &\quad \leqslant \left(\sum_{i=1}^{\eta} \left(\lambda_{i,1} \sqrt{\int_0^\infty \frac{1}{s^{k+1}} \mathrm{d}\mu_i(s)} \right) \left(\lambda_{i,1} \sqrt{\int_0^\infty \frac{1}{s^{k+3}} \mathrm{d}\mu_i(s)} \right) \right)^2 \\ &\quad \leqslant \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+1}} \mathrm{d}\mu_i(s) \right) \left(\sum_{i=1}^{\eta} \lambda_{i,1}^2 \int_0^\infty \frac{1}{s^{k+3}} \mathrm{d}\mu_i(s) \right), \quad k \in \mathbb{Z}_+, \end{aligned}$$

which, in view of (6.3.4) and (6.3.5), means that

$$\lambda_{-(k+1)} \leqslant \lambda_{-k} = \|S_{\boldsymbol{\lambda}} e_{-(k+1)}\|, \quad k \in \mathbb{Z}_+,\ 0 \leqslant k < \kappa - 1. \tag{6.3.8}$$

Combining (6.3.6), (6.3.7) and (6.3.8) with Proposition 3.1.8 and Corollary 6.2.2, we see that $S_{\boldsymbol{\lambda}}$ is a bounded subnormal operator and $\|S_{\boldsymbol{\lambda}}\|^2 = \max\{M, \lambda_0^2\}$. Applying Theorem 5.1.2 to $u = -1$ and then (6.3.7), we deduce that $\lambda_0^2 \leqslant \|S_{\boldsymbol{\lambda}} e_0\|^2 \leqslant M$. Summarizing, we have proved that (κ is arbitrary)

$$\|S_{\boldsymbol{\lambda}}\|^2 = \sup_{i \in J_\eta} \sup \operatorname{supp}(\mu_i). \tag{6.3.9}$$

We now discuss the question of the existence of a sequence $\{\lambda_{i,1}\}_{i\in J_\eta}$ of positive real numbers satisfying (6.2.13) and (6.3.3).

LEMMA 6.3.2. *Let $\{\mu_i\}_{i=1}^{\eta}$ be a sequence of Borel probability measures concentrated on a common finite subinterval $[0,M]$ of $\mathbb{R}$. Then a sequence $\{\lambda_{i,1}\}_{i\in J_\eta}\subseteq(0,\infty)$ satisfying* (6.2.13) *and* (6.3.3) *exists if and only if one of the following two disjunctive conditions holds*:

(i) $\int_0^\infty \frac{1}{s^{\kappa+1}}\mathrm{d}\mu_i(s)<\infty$ *for all $i\in J_\eta$, provided $\kappa<\infty$,*
(ii) $\int_0^\infty \frac{1}{s^{k}}\mathrm{d}\mu_i(s)<\infty$ *for all $k\in\mathbb{N}$ and $i\in J_\eta$, provided $\kappa=\infty$.*

PROOF. Since the necessity is obvious, we only need to consider the sufficiency. Assume that $\kappa<\infty$. Since $\operatorname{supp}(\mu_i)\subseteq[0,M]$, we see that $0<\int_0^\infty\frac{1}{s^k}\mathrm{d}\mu_i(s)<\infty$ for all $k\in J_{\kappa+1}$ ($i\in J_\eta$). This enables us to find a sequence $\{\lambda_{i,1}\}_{i=1}^{\eta}$ of positive real numbers such that $t_k:=\sum_{i=1}^{\eta}\lambda_{i,1}^2\int_0^\infty\frac{1}{s^k}\mathrm{d}\mu_i(s)<\infty$ for all $k\in J_{\kappa+1}$. Replacing $\lambda_{i,1}$ by $\lambda_{i,1}/\sqrt{t_1}$, we get the required sequence.

The same reasoning applies in the case of $\kappa=\infty$; now the sequence $\{\lambda_{i,1}\}_{i=1}^{\eta}$ may be chosen as follows: $\lambda_{i,1}:=2^{-i}(\max_{k\in J_i}\int_0^\infty\frac{1}{s^k}\mathrm{d}\mu_i(s))^{-1/2}$ for all $i\in J_\eta$. $\square$

If $S_{\boldsymbol\lambda}$ is a bounded subnormal weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights, then by Proposition 3.1.3 weights corresponding to $\mathsf{Chi}(0)$ are always square summable. We now show that for any square summable sequence $\{x_i\}_{i\in J_\eta}$ of positive real numbers, there exists a bounded subnormal weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V_{\eta,\kappa}^{\circ}}$ such that $\lambda_{i,1}=x_i$ for all $i\in J_\eta$. In fact, $S_{\boldsymbol\lambda}$ can always be chosen to be a scalar multiple of an isometry.

Example 6.3.3. Let $\{x_i\}_{i\in J_\eta}$ be a square summable sequence of positive real numbers. Take any sequence $\{t_i\}_{i\in J_\eta}$ of positive real numbers such that $0<\inf_{i\in J_\eta}t_i$, $\sup_{i\in J_\eta}t_i<\infty$ and $\sum_{i\in J_\eta}\frac{x_i^2}{t_i}=1$ (the simplest possible solution is the constant one $t_i=\sum_{j\in J_\eta}x_j^2$). Consider any family $\{\mu_i\}_{i\in J_\eta}$ of Borel probability measures on $[0,\infty)$ such that $\frac{1}{t_i}=\int_0^\infty\frac{1}{s}\mathrm{d}\mu_i(s)$ for all $i\in J_\eta$, $0<\inf_{i\in J_\eta}\inf\operatorname{supp}(\mu_i)$ and $M:=\sup_{i\in J_\eta}\sup\operatorname{supp}(\mu_i)<\infty$ (again this is always possible, e.g., $\mu_i=\delta_{t_i}$ does the job). Applying Procedure 6.3.1 to $\lambda_{i,1}=x_i$, $i\in J_\eta$, we obtain the required weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$; without loss of generality we can also assume that $S_{\boldsymbol\lambda}$ is extremal in the case when $\kappa\in\mathbb{N}$. Moreover, by (6.3.9), we have $\|S_{\boldsymbol\lambda}\|^2=M$. In particular, if $\mu_i=\delta_{t_i}$ for all $i\in J_\eta$, then $\|S_{\boldsymbol\lambda}\|^2=\sup_{i\in J_\eta}t_i$. If additionally $t_i=\sum_{j\in J_\eta}x_j^2$ for all $i\in J_\eta$, then $\|S_{\boldsymbol\lambda}\|^{-1}S_{\boldsymbol\lambda}$ is an isometry (use Propositions 6.2.4 and 6.2.5).

Let S be a bounded subnormal unilateral classical weighted shift with positive weights $\{\alpha_n\}_{n=1}^{\infty}$. Then $\{\|S^n e_0\|^2\}_{n=0}^{\infty}$ (cf. Remark 3.1.4) is a determinate Stieltjes moment sequence, whose unique representing measure is called a *Berger measure* of S. Given an integer $k\geqslant 1$, we say that S has a *subnormal k-step backward extension* if for some positive scalars $x_1,\ldots,x_k$ the unilateral classical weighted shift with weights $\{x_k,\ldots,x_1,\alpha_1,\alpha_2,\ldots\}$ is subnormal. If S has a subnormal k-step backward extension for all $k\in\mathbb{N}$, then S is said to have a *subnormal ∞-step backward extension*. The following elegant characterization of subnormal k-step backward extendibility is to be found in [**22**, Corollary 6.2] (see also [**20**, Proposition

8] for the case $k=1$).

$$\text{If } k \in \mathbb{N}\cup\{\infty\}, \text{ then } S \text{ has a subnormal } k\text{-step backward extension if and only if } \int_0^\infty \frac{1}{s^n}\mathrm{d}\mu(s) < \infty \text{ for all } n \in J_k, \text{ where } \mu \text{ is the Berger measure of } S. \tag{6.3.10}$$

Note that according to Notation 6.1.9, we have $\mu = \mu_0$. Let us mention that there are subnormal unilateral classical weighted shifts which have no subnormal backward extensions. The famous Bergman shift with weights $\big\{\sqrt{\frac{n}{n+1}}\big\}_{n=1}^\infty$ is among them. On the other hand, the Hardy shift with weights $\{1,1,1,\ldots\}$ does have a subnormal ∞-step backward extension (see [**20, 42, 22**] for more examples; see also [**54**] for subnormal backward extensions of general subnormal operators and [**21, 24, 23**] for subnormal backward extensions of two-variable weighted shifts). We now relate the subnormality of weighted shifts on the directed tree $\mathscr{T}_{\eta,\kappa}$ to that of unilateral classical weighted shifts which have subnormal $(\kappa+1)$-step backward extensions.

Proposition 6.3.4. *Let $\kappa,\eta \in \mathbb{Z}_+ \sqcup \{\infty\}$ and let $\eta \geqslant 2$. If for every $i \in J_\eta$, S_i is a bounded unilateral classical weighted shift with positive weights $\{\alpha_{i,n}\}_{n=1}^\infty$, then the following two conditions are equivalent*:

(i) *there exists a system $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ of positive scalars such that the weighted shift $S_{\boldsymbol\lambda}$ on the directed tree $\mathscr{T}_{\eta,\kappa}$ is bounded and subnormal, and*

$$\alpha_{i,n} = \lambda_{i,n+1}, \quad n\in\mathbb{N},\ i\in J_\eta, \tag{6.3.11}$$

(ii) *S_i has a subnormal $(\kappa+1)$-step backward extension for every $i \in J_\eta$, and $\sup_{i\in J_\eta}\|S_i\| < \infty$.*

Moreover, if $S_{\boldsymbol\lambda}$ is as in (i)*, then $\|S_{\boldsymbol\lambda}\| = \sup_{i\in J_\eta}\|S_i\|$.*

Proof. (i)$\Rightarrow$(ii) It follows from (6.3.11) that $S_{\boldsymbol\lambda_{\to(i,1)}} = S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(i,1))}$ is unitarily equivalent to S_i for all $i\in J_\eta$. Hence $\sup_{i\in J_\eta}\|S_i\| \leqslant \|S_{\boldsymbol\lambda}\|$ and $\|S_{\boldsymbol\lambda}^n e_{i,1}\|^2 = \|S_i^n e_0\|^2$ for all $n\in\mathbb{Z}_+$, which implies that $\mu_{i,1}$ is the Berger measure of S_i. Owing to Corollary 6.2.2, $\int_0^\infty \frac{1}{s^k}\mathrm{d}\mu_{i,1}(s) < \infty$ for all $k\in J_{\kappa+1}$. By (6.3.10), (ii) holds.

(ii)$\Rightarrow$(i) Let μ_i be the Berger measure of S_i. Since $\sup\operatorname{supp}(\mu_i) = \|S_i\|^2$, we deduce that the Borel probability measures μ_i, $i\in J_\eta$, are concentrated on the common finite interval $[0,M]$ with $M := \sup_{j\in J_\eta}\|S_j\|^2$. As S_i has a subnormal $(\kappa+1)$-step backward extension, we infer from (6.3.10) that $\int_0^\infty \frac{1}{s^k}\mathrm{d}\mu_i(s) < \infty$ for all $k\in J_{\kappa+1}$. Applying Lemma 6.3.2 and Procedure 6.3.1, we find a bounded subnormal weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ which satisfy (6.3.2). Hence $\alpha_{i,n} = \lambda_{i,n+1}$ for all $n\in\mathbb{N}$ and $i\in J_\eta$. Employing (6.3.9) completes the proof. $\square$

We now illustrate the question of extendibility discussed in Proposition 6.1.12.

Example 6.3.5. Fix parameters $\eta,\eta',\iota,\kappa,\kappa' \in \mathbb{Z}_+\sqcup\{\infty\}$ such that $2\leqslant \eta < \eta'$ and $1\leqslant \iota\leqslant\kappa\leqslant\kappa'$. It is evident that $\mathscr{T}_{\eta,\kappa}$ can be regarded as a proper subtree of the directed tree $\mathscr{T}_{\eta',\kappa'}$. Denote by $\mathscr{T}_{1,\iota}$ the proper subtree of the directed tree $\mathscr{T}_{\eta,\kappa}$ with $V_{1,\iota} := \big\{-k\colon k\in J_\iota\big\}\sqcup\{0\}\sqcup\big\{(1,j)\colon j\in\mathbb{N}\big\}$. Clearly, the directed tree $\mathscr{T}_{1,\iota}$ can be identified with $\mathbb{Z}_+$ when $\iota<\infty$, or with $\mathbb{Z}$ if $\iota=\infty$. It is easily seen that each pair $(\mathscr{T},\hat{\mathscr{T}}) \in \{(\mathscr{T}_{1,\iota},\mathscr{T}_{\eta,\kappa}),(\mathscr{T}_{\eta,\kappa},\mathscr{T}_{\eta',\kappa'})\}$ satisfies the assumptions of Proposition 6.1.12 with $w=0$. As a consequence, there are no bounded subnormal weighted shifts $S_{\boldsymbol\lambda}$ and $S_{\hat{\boldsymbol\lambda}}$ on $\mathscr{T}$ and $\hat{\mathscr{T}}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_u\}_{u\in V^\circ}$

and $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_u\}_{u\in\hat{V}^\circ}$, respectively, such that $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$. Of course, in the case of the pairs $(\mathscr{T}, \hat{\mathscr{T}}) \in \{(\mathscr{T}_{1,\kappa}, \mathscr{T}_{1,\kappa'}), (\mathscr{T}_{\eta,\kappa}, \mathscr{T}_{\eta,\kappa'})\}$ there always exist bounded subnormal weighted shifts $S_{\boldsymbol{\lambda}}$ and $S_{\hat{\boldsymbol{\lambda}}}$ with nonzero weights such that $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$. Isometric weighted shifts are the simplest examples of such operators (cf. Corollary 3.4.4).

Consider now the pair $(\mathscr{T}, \hat{\mathscr{T}}) = (\mathscr{T}_{\eta,0}, \mathscr{T}_{\eta',0})$. We construct subnormal weighted shifts $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ and $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(\hat{V}))$ on $\mathscr{T}$ and $\hat{\mathscr{T}}$ with positive weights $\boldsymbol{\lambda} = \{\lambda_u\}_{u\in V^\circ}$ and $\hat{\boldsymbol{\lambda}} = \{\hat{\lambda}_u\}_{u\in\hat{V}^\circ}$, respectively, such that $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$. Set $\lambda_{i,j} = \hat{\lambda}_{l,j} = 1$ for $i \in J_\eta$, $l \in J_{\eta'}$ and $j = 2, 3, \ldots$ Let $\{\hat{\lambda}_{i,1}\}_{i\in J_{\eta'}}$ be a system of positive real numbers such that $\sum_{i\in J_{\eta'}} \hat{\lambda}_{i,1}^2 \leqslant 1$. Set $\lambda_{i,1} = \hat{\lambda}_{i,1}$ for $i \in J_\eta$. It follows from Proposition 3.1.8 that $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ and $S_{\hat{\boldsymbol{\lambda}}} \in \boldsymbol{B}(\ell^2(\hat{V}))$. Since $\mu_{i,1}^{\mathscr{T}} = \mu_{l,1}^{\hat{\mathscr{T}}} = \delta_1$ for all $i \in J_\eta$ and $l \in J_{\eta'}$, we see that $S_{\boldsymbol{\lambda}}$ and $S_{\hat{\boldsymbol{\lambda}}}$ satisfy the consistency condition (6.2.12) at $u = 0$. As a consequence of Corollary 6.2.2, $S_{\boldsymbol{\lambda}}$ and $S_{\hat{\boldsymbol{\lambda}}}$ are bounded subnormal weighted shifts with positive weights such that $\boldsymbol{\lambda} \subseteq \hat{\boldsymbol{\lambda}}$. This shows that Proposition 6.1.12 is no longer true when $w = \mathsf{root}(\mathscr{T})$.

Chapter 7. Complete hyperexpansivity

7.1. A general approach. A sequence $\{a_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is said to be *completely alternating* if

$$\sum_{j=0}^{n}(-1)^j \binom{n}{j} a_{m+j} \leqslant 0, \quad m \in \mathbb{Z}_+,\, n \in \mathbb{N}.$$

As an immediate consequence of the definition, we have

(7.1.1) if a sequence $\{a_n\}_{n=0}^\infty$ is completely alternating, then so is $\{a_{n+1}\}_{n=0}^\infty$.

By [**12**, Proposition 4.6.12] (see also [**7**, Remark 1]), a sequence $\{a_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is completely alternating if and only if there exists a positive Borel measure τ on the closed interval $[0,1]$ such that

$$a_n = a_0 + \int_0^1 (1+\ldots+s^{n-1})\,\mathrm{d}\tau(s), \quad n = 1,2,\ldots \tag{7.1.2}$$

(from now on, we abbreviate $\int_{[0,1]}$ to $\int_0^1$). The measure τ is unique (cf. [**46**, Lemma 4.1]) and finite. Call it the *representing measure* of $\{a_n\}_{n=0}^\infty$.

Lemma 7.1.1. *If a sequence $\{a_n\}_{n=0}^\infty \subseteq \mathbb{R}$ is completely alternating and $a_0 = 1$, then $a_n \geqslant 1$ for all $n \in \mathbb{Z}_+$ and the corresponding sequence of quotients $\big\{\frac{a_{n+1}}{a_n}\big\}_{n=0}^\infty$ is monotonically decreasing.*

Proof. Argue as in the proof of [**7**, Proposition 4]. $\square$

The question of backward extendibility of completely alternating sequences has the following solution (compare with Lemma 6.1.2).

Lemma 7.1.2. *Let $\{a_n\}_{n=0}^\infty$ be a completely alternating sequence with the representing measure τ. Set $a_{-1} = 1$. Then the following conditions are equivalent*:

(i) *the sequence $\{a_{n-1}\}_{n=0}^\infty$ is completely alternating,*

(ii) $1 + \int_0^1 \frac{1}{s}\mathrm{d}\tau(s) \leqslant a_0$.

If (i) *holds, then $\tau(\{0\}) = 0$, and the positive Borel measure ϱ on $[0,1]$ defined by*

$$\varrho(\sigma) = \int_\sigma \frac{1}{s}\mathrm{d}\tau(s) + \Big(a_0 - 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau(s)\Big)\delta_0(\sigma), \quad \sigma \in \mathfrak{B}([0,1]), \tag{7.1.3}$$

is the representing measure of $\{a_{n-1}\}_{n=0}^\infty$. Moreover, $\rho(\{0\}) = 0$ if and only if $1 + \int_0^1 \frac{1}{s}\mathrm{d}\tau(s) = a_0$.

Proof. (i)$\Rightarrow$(ii) Let ϱ be a representing measure of $\{a_{n-1}\}_{n=0}^\infty$. Then

$$a_0 = a_{1-1} = a_{-1} + \int_0^1 1\mathrm{d}\varrho(s) = 1 + \varrho([0,1]). \tag{7.1.4}$$

Define the positive Borel measure $\tilde\tau$ on $[0,1]$ by $\mathrm{d}\tilde\tau(s) = s\mathrm{d}\varrho(s)$. Then, we have

$$\begin{aligned} a_n = a_{(n+1)-1} &= a_{-1} + \int_0^1 (1+\ldots+s^n)\mathrm{d}\varrho(s) \\ &= 1 + \varrho([0,1]) + \int_0^1 (1+\ldots+s^{n-1})s\mathrm{d}\varrho(s) \\ &\overset{(7.1.4)}{=} a_0 + \int_0^1 (1+\ldots+s^{n-1})\mathrm{d}\tilde\tau(s), \quad n = 1,2,\ldots \end{aligned}$$

By the uniqueness of the representing measure, we have $\tilde{\tau} = \tau$. This implies that $\tau(\{0\}) = 0$, and consequently

$$a_0 \overset{(7.1.4)}{=} 1 + \rho(\{0\}) + \rho((0,1]) = 1 + \rho(\{0\}) + \int_0^1 \frac{1}{s}\mathrm{d}\tau(s).$$

Hence (ii) holds.

(ii)⇒(i) Define the positive Borel measure ϱ on $[0,1]$ by (7.1.3). Then

$$\begin{aligned} a_{-1} &+ \int_0^1 (1 + \ldots + s^{n-1})\mathrm{d}\varrho(s) \\ &= 1 + \int_0^1 \frac{1 + \ldots + s^{n-1}}{s}\mathrm{d}\tau(s) + \Big(a_0 - 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau(s)\Big) \\ &= 1 + \int_0^1 \frac{1}{s}\mathrm{d}\tau(s) + \int_0^1 (1 + \ldots + s^{n-2})\mathrm{d}\tau(s) + \Big(a_0 - 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau(s)\Big) \\ &= a_0 + \int_0^1 (1 + \ldots + s^{n-2})\mathrm{d}\tau(s) = a_{n-1}, \quad n = 2, 3, \ldots \end{aligned}$$

Since

$$a_{-1} + \int_0^1 1\mathrm{d}\rho(s) \overset{(7.1.3)}{=} 1 + \int_0^1 \frac{1}{s}\mathrm{d}\tau(s) + \Big(a_0 - 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau(s)\Big) = a_0 = a_{1-1},$$

we deduce that the sequence $\{a_{n-1}\}_{n=0}^\infty$ is completely alternating with the representing measure ϱ. This completes the proof. □

We are now ready to recall the definition of our present object of study from [**7**]. Let $\mathcal{H}$ be a complex Hilbert space. An operator $A \in \boldsymbol{B}(\mathcal{H})$ is said to be *completely hyperexpansive* if the sequence $\{\|A^n h\|^2\}_{n=0}^\infty$ is completely alternating for every $h \in \mathcal{H}$. In view of the above discussion, A is completely hyperexpansive if and only if (substitute $A^m h$ in place of h)

$$\sum_{j=0}^n (-1)^j \binom{n}{j} \|A^j h\|^2 \leqslant 0, \quad n \in \mathbb{N},\, h \in \mathcal{H}, \tag{7.1.5}$$

Completely hyperexpansive operators are antithetical to contractive subnormal operators in the sense that their defining properties and behavior are related to the theory of completely alternating functions on abelian semigroups (subnormality is connected with positive definiteness). This is their great advantage and one of the reasons why they attract attention of researchers (see e.g., [**4, 1, 2, 3, 7, 11, 72, 9, 10, 46, 8, 47, 27**]).

Note that if A is a completely hyperexpansive operator, then $\|Ah\| \geqslant \|h\|$ for all $h \in \mathcal{H}$ (apply (7.1.5) to $n = 1$), which means that A is injective. In view of this, we can deduce from Proposition 3.1.7 that a directed tree which admits a completely hyperexpansive weighted shift must be leafless.

Proposition 7.1.3. *If $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a completely hyperexpansive weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, then $\mathscr{T}$ is leafless and $\sum_{v \in \mathsf{Chi}(u)} |\lambda_v|^2 > 0$ for all $u \in V$.*

Complete hyperexpansivity of weighted shifts on directed trees can be characterized as follows.

THEOREM 7.1.4. *If $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ is a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the following conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is completely hyperexpansive,*
(ii) *$\Big\{ \sum_{v\in \mathsf{Chi}^{\langle n\rangle}(u)} |\lambda_{u|v}|^2\Big\}_{n=0}^\infty$ is a completely alternating sequence for all $u \in V$,*
(iii) *$\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is a completely alternating sequence for all $u \in V$,*
(iv) *$\sum_{j=0}^n (-1)^j \binom{n}{j} \|S_{\boldsymbol\lambda}^j e_u\|^2 \leqslant 0$ for all $n \in \mathbb{N}$ and $u \in V$.*

PROOF. The implications (i)⇒(iii) and (iii)⇒(iv) are evident. By Lemma 6.1.1, the conditions (ii) and (iii) are equivalent.

(iv)⇒(i) Take $f \in \ell^2(V)$. It follows from (6.1.8) that

$$\begin{aligned}\sum_{j=0}^n (-1)^j \binom{n}{j} \|S_{\boldsymbol\lambda}^j f\|^2 &= \sum_{j=0}^n (-1)^j \binom{n}{j} \sum_{u\in V} |f(u)|^2 \|S_{\boldsymbol\lambda}^j e_u\|^2 \\ &= \sum_{u\in V} |f(u)|^2 \Big(\sum_{j=0}^n (-1)^j \binom{n}{j} \|S_{\boldsymbol\lambda}^j e_u\|^2 \Big) \overset{\text{(iv)}}{\leqslant} 0\end{aligned}$$

for all $n \in \mathbb{N}$. This, together with (7.1.5), completes the proof. □

As shown in Corollary 7.1.5 below, the study of complete hyperexpansivity of weighted shifts on directed trees reduces to the case of trees with root. This is very similar to what happens in the case of subnormality. The proof of Corollary 7.1.5 is essentially the same as that of Corollary 6.1.4 (use Theorem 7.1.4 instead of Theorem 6.1.3).

COROLLARY 7.1.5. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Suppose that X is a subset of V such that $V = \bigcup_{x\in X} \mathsf{Des}(x)$. Then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if the operator $S_{\boldsymbol\lambda_{\to(x)}}$ is completely hyperexpansive for every $x \in X$ (cf. Notation* 3.1.5).

We now formulate the counterpart of the small lemma (cf. Lemma 6.1.5) for completely hyperexpansive weighted shifts on directed trees.

LEMMA 7.1.6. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$ and let $u_0, u_1 \in V$ be such that $\mathsf{Chi}(u_0) = \{u_1\}$. If the sequence $\{\|S_{\boldsymbol\lambda}^n e_{u_0}\|^2\}_{n=0}^\infty$ is completely alternating and $\lambda_{u_1} \neq 0$, then the sequence $\{\|S_{\boldsymbol\lambda}^n e_{u_1}\|^2\}_{n=0}^\infty$ is completely alternating.*

PROOF. Noticing that

$$\|S_{\boldsymbol\lambda}^n e_{u_1}\|^2 \overset{(3.1.4)}{=} \frac{1}{|\lambda_{u_1}|^2} \|S_{\boldsymbol\lambda}^{n+1} e_{u_0}\|^2, \quad n \in \mathbb{Z}_+,$$

and employing (7.1.1), we complete the proof. □

It turns out that Lemma 7.1.6 is no longer true if $\mathrm{card}(\mathsf{Chi}(u_0)) \geqslant 2$. Indeed, the weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{2,0}$ defined in Example 6.1.6 has the property that the sequence $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^\infty = \{1, 1, \ldots\}$ is completely alternating (with the representing measure $\tau = 0$, cf. (7.1.2)), and neither of the sequences $\{\|S_{\boldsymbol\lambda}^n e_{1,1}\|^2\}_{n=0}^\infty = \{1, (\frac{b}{a})^2, 1, 1, \ldots\}$ and $\{\|S_{\boldsymbol\lambda}^n e_{2,1}\|^2\}_{n=0}^\infty = \{1, (\frac{a}{b})^2, 1, 1, \ldots\}$ is completely alternating (as neither of them is monotonically increasing).

Our next goal is to find relationships between representing measures of completely alternating sequences $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$, $u \in V$. Let us first fix the notation that is used throughout (compare with Notation 6.1.9).

NOTATION 7.1.7. Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$. If for some $u \in V$, the sequence $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is completely alternating, then its unique representing measure which is concentrated on $[0,1]$ will be denoted by τ_u (or by $\tau_u^{\mathscr{T}}$ if we wish to make clear the dependence of τ_u on $\mathscr{T}$).

The result which follows is a counterpart of the big lemma (cf. Lemma 6.1.10) for completely hyperexpansive weighted shifts on directed trees.

LEMMA 7.1.8. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, and let $u \in V'$ be such that the sequence $\{\|S_{\boldsymbol\lambda}^n e_v\|^2\}_{n=0}^\infty$ is completely alternating for every $v \in \mathsf{Chi}(u)$. Then the following conditions are equivalent*:

(i) *the sequence $\{\|S_{\boldsymbol\lambda}^n e_u\|^2\}_{n=0}^\infty$ is completely alternating,*

(ii) *$S_{\boldsymbol\lambda}$ satisfies the consistency condition at u, i.e.,*

$$\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \geqslant 1 + \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v(s). \tag{7.1.6}$$

If (i) *holds, then $\tau_v(\{0\}) = 0$ for every $v \in \mathsf{Chi}(u)$ such that $\lambda_v \neq 0$, and*

$$\begin{aligned}\tau_u(\sigma) = &\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_\sigma \frac{1}{s}\mathrm{d}\tau_v(s)\\ &+ \Big(\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 - 1 - \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v(s)\Big)\delta_0(\sigma)\end{aligned} \tag{7.1.7}$$

for every $\sigma \in \mathfrak{B}([0,1])$. Moreover, $\tau_u(\{0\}) = 0$ if and only if $S_{\boldsymbol\lambda}$ satisfies the strong consistency condition at u, i.e.,

$$\sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 = 1 + \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v(s). \tag{7.1.8}$$

PROOF. Define the positive Borel measure τ on $[0,1]$ by

$$\tau(\sigma) = \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \tau_v(\sigma), \quad \sigma \in \mathfrak{B}([0,1]).$$

Applying a version of (6.1.15) (with $\mu = \tau$ and $\mu_v = \tau_v$), we see that

$$\begin{aligned}\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2 &\overset{(6.1.18)}{=} \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \|S_{\boldsymbol\lambda}^n e_v\|^2\\ &= \sum_{v\in\mathsf{Chi}(u)} |\lambda_v|^2 \Big(1 + \int_0^1 (1 + \ldots + s^{n-1})\mathrm{d}\tau_v(s)\Big)\\ &\overset{(3.1.4)}{=} \|S_{\boldsymbol\lambda} e_u\|^2 + \int_0^1 (1 + \ldots + s^{n-1})\mathrm{d}\tau(s), \quad n = 1, 2, \ldots,\end{aligned}$$

which means that the sequence $\{\|S_{\boldsymbol\lambda}^{n+1} e_u\|^2\}_{n=0}^\infty$ is completely alternating with the representing measure τ. Employing now a version of (6.1.16) and Lemma 7.1.2 with $a_n = \|S_{\boldsymbol\lambda}^{n+1} e_u\|^2$, we see that the conditions (i) and (ii) are equivalent. The formula (7.1.7) can be inferred from (7.1.3) by applying a version of (6.1.15). The rest of the conclusion is clearly true. □

Because of uniqueness of representing measures τ_v, there is strong hope that Lemma 7.1.8 holds for unbounded completely hyperexpansive weighted shifts on directed trees (see [**49, 46, 47**] for an invitation to unbounded completely hyperexpansive operators).

It is interesting to note that the direct counterpart of Proposition 6.1.12 for completely hyperexpansive weighted shifts on directed trees is no longer true (cf. Example 7.5.2). However, its weaker version remains valid (cf. Proposition 7.5.1).

7.2. Complete hyperexpansivity on $\mathscr{T}_{\eta,\kappa}$. Now we confine our attention to discussing the question of complete hyperexpansivity of weighted shifts on the directed trees $\mathscr{T}_{\eta,\kappa}$ (cf. (6.2.10)). Before formulating the counterpart of Theorem 6.2.1 for completely hyperexpansive weighted shifts, we first recall the definition of completely hyperexpansive k-step backward extendibility of unilateral classical weighted shifts (cf. [**48**, Definition 4.1]). Given an integer $k \geqslant 1$, we say that a unilateral classical weighted shift with positive weights $\{\lambda_n\}_{n=1}^\infty$ (cf. Remark 3.1.4) has a *completely hyperexpansive k-step backward extension* if for some positive scalars $\lambda_{-k+1}, \ldots, \lambda_0$, the unilateral classical weighted shift with weights $\{\lambda_{-k+n}\}_{n=1}^\infty$ is completely hyperexpansive (note that completely hyperexpansive unilateral classical weighted shifts are automatically bounded, cf. [**49**, Proposition 6.2 (i)]). Each unilateral classical weighted shift which has a completely hyperexpansive k-step backward extension is automatically completely hyperexpansive (cf. [**48**, Section 4])).

Recall that a unilateral classical weighted shift S with positive weights $\{\lambda_n\}_{n=1}^\infty$ is completely hyperexpansive if and only if the sequence $\{\|S^n e_0\|^2\}_{n=0}^\infty$ is completely alternating (cf. [**7**, Proposition 3]); the representing measure of $\{\|S^n e_0\|^2\}_{n=0}^\infty$ will be called the *representing measure* of S. The following characterization of completely hyperexpansive k-step backward extendibility was given in [**48**, Theorem 4.2].

(7.2.1) If $k \in \mathbb{N}$, then S has a completely hyperexpansive k-step backward extension if and only if S is completely hyperexpansive and $\int_0^1 \sum_{l=1}^k \frac{1}{s^l}\mathrm{d}\tau(s) < 1$, where τ is the representing measure of S.

Below we adhere to Notation 7.1.7.

THEOREM 7.2.1. *Suppose that $\mathscr{T}$ is a directed tree for which there exists $\omega \in V$ such that* $\mathrm{card}(\mathsf{Chi}(\omega)) \geqslant 2$ *and* $\mathrm{card}(\mathsf{Chi}(v)) = 1$ *for every $v \in V \setminus \{\omega\}$. Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on the directed tree $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V^\circ}$. Then the following assertions hold.*

(i) *If $\omega = \mathsf{root}$, then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if the sequence $\{\|S_{\boldsymbol\lambda}^n e_v\|^2\}_{n=0}^\infty$ is completely alternating for every $v \in \mathsf{Chi}(\omega)$, and $S_{\boldsymbol\lambda}$ satisfies the consistency condition at ω, i.e.,* (7.1.6) *is valid for $u = \omega$.*

(ii) *If $\mathscr{T}$ has a root and $\omega \neq \mathsf{root}$, then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if one of the following two equivalent conditions holds*:

(ii-a) *$S_{\boldsymbol\lambda_{\to(\omega)}}$ is completely hyperexpansive,* (7.1.8) *is valid for $u = \omega$,*

$$|\lambda_{\mathsf{par}^{k-1}(\omega)}|^2 = 1 + \Big|\prod_{j=0}^{k-1} \lambda_{\mathsf{par}^j(\omega)}\Big|^2 \sum_{v \in \mathsf{Chi}(\omega)} |\lambda_v|^2 \int_0^1 \frac{1}{s^{k+1}}\mathrm{d}\tau_v(s)$$

for all $k \in J_{\kappa-1}$, *and*

$$|\lambda_{\mathsf{par}^{\kappa-1}(\omega)}|^2 \geqslant 1 + \Big|\prod_{j=0}^{\kappa-1} \lambda_{\mathsf{par}^j(\omega)}\Big|^2 \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 \int_0^1 \frac{1}{s^{\kappa+1}} \mathrm{d}\tau_v(s),$$

where κ *is a unique positive integer such that* $\mathsf{par}^\kappa(\omega) = \mathsf{root}$,

(ii-b) *the sequences* $\{\|S^n_{\boldsymbol\lambda} e_{\mathsf{root}}\|^2\}_{n=0}^\infty$ *and* $\{\|S^n_{\boldsymbol\lambda} e_v\|^2\}_{n=0}^\infty$ *are completely alternating for all* $v \in \mathsf{Chi}(\omega)$.

(iii) *If* $\mathscr{T}$ *is rootless, then* $S_{\boldsymbol\lambda}$ *is completely hyperexpansive if and only if* $S_{\boldsymbol\lambda}$ *is an isometry.*

Proof. The proofs of (i) and (ii) are essentially the same as the corresponding parts of the proof of Theorem 6.2.1. We only have to use Lemmas 7.1.6 and 7.1.8 and Theorem 7.1.4 in place of Lemmas 6.1.5 and 6.1.10 and Theorem 6.1.3, respectively. Moreover, in proving (ii) we need to exploit the explicit formulas for representing measures $\tau_{\mathsf{par}^k(\omega)}$, $k \in J_{\kappa-1}$, which are given by

$$\frac{\tau_{\mathsf{par}^k(\omega)}(\sigma)}{|\prod_{j=0}^{k-1} \lambda_{\mathsf{par}^j(\omega)}|^2} = \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 \int_\sigma \frac{1}{s^{k+1}} \mathrm{d}\tau_v(s), \quad \sigma \in \mathfrak{B}(\mathbb{R})$$

(the measure τ_ω is given by $\tau_\omega(\sigma) = \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 \int_\sigma \frac{1}{s}\mathrm{d}\tau_v(s)$ for $\sigma \in \mathfrak{B}([0,1])$).

(iii) Suppose that $\mathscr{T}$ is rootless and $S_{\boldsymbol\lambda}$ is completely hyperexpansive. Then the sequences $\{\|S^n_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|^2\}_{n=0}^\infty$ and $\{\|S^n_{\boldsymbol\lambda} e_v\|^2\}_{n=0}^\infty$ are completely alternating for all $k \in \mathbb{Z}_+$ and $v \in \mathsf{Chi}(\omega)$. Fix $k \in \mathbb{Z}_+$. Consider the unilateral classical weighted shift W_k with weights $\big\{\|S^n_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\| \cdot \|S^{n-1}_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|^{-1}\big\}_{n=1}^\infty$. By [**7**, Proposition 3], W_k is completely hyperexpansive. Since, by (3.1.4),

$$S^l_{\boldsymbol\lambda} e_{\mathsf{par}^{k+l}(\omega)} = \lambda_{\mathsf{par}^{k+l-1}(\omega)} \cdots \lambda_{\mathsf{par}^k(\omega)} e_{\mathsf{par}^k(\omega)}, \quad l \in \mathbb{N},$$

we deduce that

$$\frac{\|S^{n+l}_{\boldsymbol\lambda} e_{\mathsf{par}^{k+l}(\omega)}\|}{\|S^{n-1+l}_{\boldsymbol\lambda} e_{\mathsf{par}^{k+l}(\omega)}\|} = \frac{\|S^n_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|}{\|S^{n-1}_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|}, \quad n, l \geqslant 1.$$

As W_{k+l} is completely hyperexpansive, we see that the unilateral classical weighted shift W_k has a completely hyperexpansive l-step backward extension for all integers $l \geqslant 1$. Hence, by [**48**, Corollary 4.6 (i)], the weights of W_k are equal to 1, which, together with (3.1.4), implies that

$$1 = \|S_{\boldsymbol\lambda} e_{\mathsf{par}^k(\omega)}\|^2 = \begin{cases} |\lambda_{\mathsf{par}^{k-1}(\omega)}|^2 & \text{for } k \in \mathbb{N}, \\ \sum_{v\in\mathsf{Chi}(\omega)} |\lambda_v|^2 & \text{for } k = 0. \end{cases} \tag{7.2.2}$$

Applying the consistency condition (7.1.6) at $u = \omega$ and (7.2.2), we deduce that $\tau_v = 0$ for all $v \in \mathsf{Chi}(\omega)$. Employing the integral representation (7.1.2), we see that $\|S^n_{\boldsymbol\lambda} e_v\| = 1$ for all $n \in \mathbb{Z}_+$ and $v \in \mathsf{Chi}(\omega)$. Next, using an induction argument and (2.1.10) (with $u = \omega$), we infer that $\|S_{\boldsymbol\lambda} e_u\| = 1$ for all $u \in \mathsf{Des}(\omega)$. Finally, applying Proposition 2.1.6 (iv) to $u = \omega$ and Corollary 3.4.4, we conclude that $S_{\boldsymbol\lambda}$ is an isometry. The reverse implication is obvious. This completes the proof. □

Remark 7.2.2. A careful look at the proof of Theorem 7.2.1 reveals that the characterization (ii-a) of complete hyperexpansivity of bounded weighted shifts on $\mathscr{T}$ with nonzero weights remains valid even if ω has only one child.

As an immediate application of Theorem 7.2.1 we have the following result, being a counterpart of Corollary 6.2.2 for completely hyperexpansive weighted shifts. As before, we adhere to notation $\lambda_{i,j}$ instead of a more formal expression $\lambda_{(i,j)}$. Recall also that $\eta, \kappa \in \mathbb{Z}_+ \sqcup \{\infty\}$ and $\eta \geqslant 2$.

Corollary 7.2.3. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ be a weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$. Then the following assertions hold.*

(i) *If $\kappa = 0$, then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if there exist positive Borel measures $\{\tau_i\}_{i=1}^\eta$ on $[0,1]$ such that*

$$1 + \int_0^1 (1 + \ldots + s^{n-1})\,\mathrm{d}\tau_i(s) = \Big|\prod_{j=2}^{n+1} \lambda_{i,j}\Big|^2, \quad n \in \mathbb{N},\ i \in J_\eta, \tag{7.2.3}$$

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 \geqslant 1 + \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_i(s). \tag{7.2.4}$$

(ii) *If $0 < \kappa < \infty$, then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if one of the following two equivalent conditions holds*:

(ii-a) *there exist positive Borel measures $\{\tau_i\}_{i=1}^\eta$ on $[0,1]$ which satisfy* (7.2.3) *and the following requirements*:

$$\sum_{i=1}^\eta |\lambda_{i,1}|^2 = 1 + \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_i(s). \tag{7.2.5}$$

$$|\lambda_{-(k-1)}|^2 = 1 + \Big|\prod_{j=0}^{k-1} \lambda_{-j}\Big|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^1 \frac{1}{s^{k+1}}\mathrm{d}\tau_i(s), \quad k \in J_{\kappa-1}, \tag{7.2.6}$$

$$|\lambda_{-(\kappa-1)}|^2 \geqslant 1 + \Big|\prod_{j=0}^{\kappa-1} \lambda_{-j}\Big|^2 \sum_{i=1}^\eta |\lambda_{i,1}|^2 \int_0^1 \frac{1}{s^{\kappa+1}}\mathrm{d}\tau_i(s); \tag{7.2.7}$$

(ii-b) *there exist positive Borel measures $\{\tau_i\}_{i=1}^\eta$ and ν on $[0,1]$ which satisfy* (7.2.3) *and the equations below*

$$1 + \int_0^1 (1 + \ldots + s^{n-1})\mathrm{d}\nu(s) = \begin{cases} |\prod_{j=\kappa-n}^{\kappa-1} \lambda_{-j}|^2 & \text{if } n \in J_\kappa, \\ |\prod_{j=0}^{\kappa-1} \lambda_{-j}|^2 \big(\sum_{i=1}^\eta |\prod_{j=1}^{n-\kappa} \lambda_{i,j}|^2\big) & \text{if } n \in \mathbb{N} \setminus J_\kappa. \end{cases}$$

(iii) *If $\kappa = \infty$, then $S_{\boldsymbol\lambda}$ is completely hyperexpansive if and only if $S_{\boldsymbol\lambda}$ is an isometry.*

Moreover, if $S_{\boldsymbol\lambda}$ is completely hyperexpansive and $\{\tau_i\}_{i=1}^\eta$ are positive Borel measures on $[0,1]$ satisfying (7.2.3), *then $\tau_i = \tau_{i,1}$ for all $i \in J_\eta$.*

7.3. Modelling complete hyperexpansivity on $\mathscr{T}_{\eta,\kappa}$. Our aim in this section is to find a model for completely hyperexpansive weighted shifts (with nonzero weights) on $\mathscr{T}_{\eta,\kappa}$ (cf. (6.2.10)). In view of Theorem 3.2.1 and Corollary 7.2.3, we can confine ourselves to discussing the case when κ is finite and the weights of weighted shifts under consideration are positive. We begin by formulating a simple necessary condition which has to be satisfied by representing measures $\tau_{i,1}$ (see Notation 7.1.7).

Lemma 7.3.1. *If $\kappa \in \mathbb{Z}_+$ and $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights, then $\sup_{i \in J_\eta} \tau_{i,1}([0,1]) < \infty$.*

Proof. By (7.1.2), we have $\tau_{i,1}([0,1]) = \|S_{\boldsymbol\lambda} e_{i,1}\|^2 - 1 \leqslant \|S_{\boldsymbol\lambda}\|^2 - 1$ for all $i \in J_\eta$, which completes the proof. □

We now show that a completely hyperexpansive weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$ is determined, in a sense, by its weights which correspond to $\mathsf{Chi}(0)$.

Lemma 7.3.2. *Let $\eta \in \{2,3,\ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. Suppose that $\boldsymbol\tau = \{\tau_i\}_{i=1}^\eta$ is a sequence of positive Borel measures on $[0,1]$ such that $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$. Then the following assertions hold.*

(i) *If $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$, then the system $\boldsymbol{t} := \{t_i\}_{i=1}^\eta$ with $t_i := \lambda_{i,1}$ satisfies the following conditions:*

$$\sum_{i=1}^{\eta} t_i^2 < \infty, \tag{7.3.1}$$

$$\begin{cases} \displaystyle\sum_{i=1}^{\eta} t_i^2 \geqslant 1 + \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) & \text{if } \kappa = 0, \\ \displaystyle\sum_{i=1}^{\eta} t_i^2 = 1 + \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) & \text{if } \kappa > 0, \end{cases} \tag{7.3.2}$$

$$\sum_{i=1}^{\eta} t_i^2 > \sum_{i=1}^{\eta} t_i^2 \int_0^1 \Big(\frac{1}{s} + \ldots + \frac{1}{s^{\kappa+1}}\Big)\mathrm{d}\tau_i(s). \tag{7.3.3}$$

(ii) *Let $\boldsymbol{t} = \{t_i\}_{i=1}^\eta \subseteq (0,\infty)$ satisfy (7.3.1), (7.3.2) and (7.3.3). Then there exists a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\lambda_{i,1} = t_i$ and $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$. If $\kappa = 0$, $S_{\boldsymbol\lambda}$ is unique. If $\kappa \geqslant 1$, all the weights of $S_{\boldsymbol\lambda}$, except for $\lambda_{-\kappa+1}$, are uniquely determined by $\boldsymbol{t}$ and $\boldsymbol\tau$; the weight $\lambda_{-\kappa+1}$ can be chosen arbitrarily within the interval $\Big[\sqrt{\frac{\zeta_\kappa}{\zeta_{\kappa+1}}}, \infty\Big)$, where*

$$\zeta_k = \sum_{i=1}^{\eta} t_i^2 \Big(1 - \int_0^1 \sum_{j=1}^{k} \frac{1}{s^j}\,\mathrm{d}\tau_i(s)\Big), \quad k \in J_{\kappa+1}. \tag{7.3.4}$$

Moreover, the norm of $S_{\boldsymbol\lambda}$ is given by

$$\|S_{\boldsymbol\lambda}\|^2 = \begin{cases} \max\big\{\sum_{i=1}^{\eta} t_i^2, \sup_{i\in J_\eta}(1+\tau_i([0,1]))\big\} & \text{for } \kappa = 0, \\ \max\big\{\lambda_{-(\kappa-1)}^2, \sup_{i\in J_\eta}(1+\tau_i([0,1]))\big\} & \text{for } \kappa \geqslant 1. \end{cases} \tag{7.3.5}$$

It is worth noting that if $\kappa = 0$, then (7.3.2) implies (7.3.3). Observe also that the quantities ζ_k are defined only in the case when $\kappa \geqslant 1$, and that $\zeta_1 = 1$ (use (7.3.2)).

Proof of Lemma 7.3.2. (i) In view of Corollary 7.2.3, $\int_0^1 \frac{1}{s^{k+1}}\mathrm{d}\tau_i(s) < \infty$ for all $k \in J_\kappa$ and $i \in J_\eta$. By (3.1.4), we have $\sum_{i=1}^\eta t_i^2 = \|S_{\boldsymbol\lambda} e_0\|^2 < \infty$, which gives (7.3.1). The condition (7.3.2) follows from (7.2.4) and (7.2.5). Thus, it remains to prove (7.3.3).

If $\kappa = 0$, then, as noted just above, (7.3.2) implies (7.3.3).

If $\kappa = 1$, then by applying the inequality (7.2.7) we get

$$
\begin{aligned}
1 &\leqslant \lambda_0^2\Big(1 - \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^2}\mathrm{d}\tau_i(s)\Big) \\
&= \lambda_0^2\Big(\big(1 - \sum_{i=1}^{\eta} t_i^2\big) + \sum_{i=1}^{\eta} t_i^2\Big(1 - \int_0^1 \frac{1}{s^2}\mathrm{d}\tau_i(s)\Big)\Big) \\
&\overset{(7.2.5)}{=} \lambda_0^2 \sum_{i=1}^{\eta} t_i^2\Big(1 - \int_0^1 \Big(\frac{1}{s} + \frac{1}{s^2}\Big)\mathrm{d}\tau_i(s)\Big) = \lambda_0^2\zeta_2,
\end{aligned} \tag{7.3.6}
$$

where ζ_2 is as in (7.3.4). Hence, (7.3.3) follows from (7.3.6).

Assume now that $\kappa = 2$. Arguing as in (7.3.6) and using (7.2.6) in place of (7.2.7), we obtain

$$
1 = \lambda_0^2\zeta_2. \tag{7.3.7}
$$

It follows from (7.2.7) that

$$
\begin{aligned}
1 &\leqslant \lambda_{-1}^2\Big(1 - \lambda_0^2 \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^3}\mathrm{d}\tau_i(s)\Big) \\
&\overset{(7.3.7)}{=} \lambda_{-1}^2\left(1 - \frac{\sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^3}\mathrm{d}\tau_i(s)}{\sum_{i=1}^{\eta} t_i^2\Big(1 - \int_0^1 \sum_{j=1}^{2} \frac{1}{s^j}\,\mathrm{d}\tau_i(s)\Big)}\right) \\
&= \lambda_{-1}^2\frac{\zeta_3}{\zeta_2},
\end{aligned} \tag{7.3.8}
$$

which together with (7.3.7) implies (7.3.3).

Suppose now that $\kappa \geqslant 3$. We claim that the following two conditions hold for all $k \in \{2, \ldots, \kappa - 1\}$:

$$
\sum_{i=1}^{\eta} t_i^2 > \sum_{i=1}^{\eta} t_i^2 \int_0^1 \sum_{j=1}^{k+1} \frac{1}{s^j}\,\mathrm{d}\tau_i(s), \tag{7.3.9}
$$

$$
1 = \lambda_{-(k-1)}^2\frac{\zeta_{k+1}}{\zeta_k}. \tag{7.3.10}
$$

Arguing as in (7.3.6) and (7.3.8), and using (7.2.6) with $k = 1, 2$ in place of (7.2.7), we get (7.3.7) and the equality

$$
1 = \lambda_{-1}^2\frac{\zeta_3}{\zeta_2},
$$

which implies that (7.3.9) and (7.3.10) hold for $k = 2$. This proves our claim for $\kappa = 3$. If $\kappa \geqslant 4$, we proceed by induction. Fix an integer n such that $2 \leqslant n < \kappa - 1$ and assume that (7.3.9) and (7.3.10) hold for all $k = 2, \ldots, n$. By (7.2.6), applied to $k = n + 1$, we obtain (note that (7.3.7) is valid for $\kappa \geqslant 2$)

$$
\begin{aligned}
1 &= \lambda_{-n}^2\Big(1 - \lambda_0^2\lambda_{-1}^2 \cdots \lambda_{-(n-1)}^2 \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^{n+2}}\mathrm{d}\tau_i(s)\Big) \\
&\overset{(7.3.7)\&(7.3.10)}{=} \lambda_{-n}^2\Big(1 - \frac{1}{\zeta_2}\frac{\zeta_2}{\zeta_3} \cdots \frac{\zeta_n}{\zeta_{n+1}} \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^{n+2}}\mathrm{d}\tau_i(s)\Big)
\end{aligned} \tag{7.3.11}
$$

$$= \lambda_{-n}^2\Big(1 - \frac{1}{\zeta_{n+1}}\sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s^{n+2}}\mathrm{d}\tau_i(s)\Big) = \lambda_{-n}^2\frac{\zeta_{n+2}}{\zeta_{n+1}},$$

which shows that (7.3.9) and (7.3.10) hold for $k = n+1$. This proves our claim. Arguing as in the proof of (7.3.11) with $n = \kappa - 1$ and using (7.2.7) in place of (7.2.6), we get

$$1 \leqslant \lambda_{-(\kappa-1)}^2 \frac{\zeta_{\kappa+1}}{\zeta_\kappa}, \tag{7.3.12}$$

which, when combined with (7.3.9) applied to $k = \kappa - 1$, implies (7.3.3). Hence (i) is proved.

(ii) Assume that $\boldsymbol{t} := \{t_i\}_{i=1}^{\eta} \subseteq (0,\infty)$ satisfies (7.3.1), (7.3.2) and (7.3.3). Our aim now is to define the system $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ} \subseteq (0,\infty)$. For $i \in J_\eta$, we set

$$\lambda_{i,j} = \begin{cases} t_i & \text{for } j = 1,\\ \sqrt{1+\tau_i([0,1])} & \text{for } j = 2,\\ \sqrt{\dfrac{1+\int_0^1(1+\ldots+s^{j-2})\mathrm{d}\tau_i(s)}{1+\int_0^1(1+\ldots+s^{j-3})\mathrm{d}\tau_i(s)}} & \text{for } j \geqslant 3. \end{cases} \tag{7.3.13}$$

If $\kappa = 0$, then the weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v\in V_{\eta,0}^\circ}$ just defined satisfy (7.2.3) and (7.2.4) (the latter because of (7.3.2)).

If $\kappa = 1$, then λ_0 can be considered as any number from the interval $[1/\sqrt{\zeta_2}, \infty)$. Clearly, (7.2.3) is valid. It follows from (7.3.2) with $\kappa = 1$ that (7.2.5) holds. Hence, we can reverse the reasoning in (7.3.6) and verify that (7.2.7) is valid for $\kappa = 1$.

If $\kappa \geqslant 2$ and $\vartheta \in \Big[\sqrt{\frac{\zeta_\kappa}{\zeta_{\kappa+1}}}, \infty\Big)$, then we define the weights $\{\lambda_{-k}\}_{k=0}^{\kappa-1}$ by

$$\lambda_{-k} = \begin{cases} \frac{1}{\sqrt{\zeta_2}} & \text{for } k = 0,\\ \sqrt{\frac{\zeta_{k+1}}{\zeta_{k+2}}} & \text{for } k \in J_{\kappa-2},\\ \vartheta & \text{for } k = \kappa - 1. \end{cases} \tag{7.3.14}$$

(Of course, if $\kappa = 2$, then the middle expression in (7.3.14) does not appear.) According to (7.3.1) and (7.3.3), the above definition is correct. Reversing the reasonings in (7.3.7) and (7.3.11), we deduce that (7.2.6) holds. Arguing as in (7.3.11) with $n = \kappa - 1$, we see that (7.2.7) holds. As above, we conclude that (7.2.3) and (7.2.5) are valid as well.

Thus, it remains to show that the weighted shift $S_{\boldsymbol{\lambda}}$ is bounded. It follows from (7.3.13) and Lemma 7.1.1 that for every $i \in J_\eta$ the sequence $\{\lambda_{i,j}\}_{j=2}^\infty$ is monotonically decreasing. As a consequence, we have

$$\sup_{i\in J_\eta}\sup_{j\geqslant 2}\lambda_{i,j}^2 = \sup_{i\in J_\eta}(1+\tau_i([0,1])) < \infty. \tag{7.3.15}$$

Combining this with (7.3.1) and $\kappa < \infty$, we see that $\sup_{u\in V_{\eta,\kappa}}\sum_{v\in \mathsf{Chi}(u)}\lambda_v^2 < \infty$. Hence, by Proposition 3.1.8, $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$. Applying Corollary 7.2.3, we conclude that the weighted shift $S_{\boldsymbol{\lambda}}$ is completely hyperexpansive and $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$. The uniqueness assertion in (ii) can be deduced from (7.3.13), (7.3.7) and (7.3.10).

We now prove the "moreover" part of (ii). If $\kappa = 0$, then the top equality in (7.3.5) follows from (3.1.7) and (7.3.15). Assume that $\kappa \geqslant 1$. Since the sequence $\{\|S_{\boldsymbol\lambda}^n e_{-\kappa}\|^2\}_{n=0}^\infty$ is completely alternating, we infer from Lemma 7.1.1 that the corresponding sequence of quotients

$$\text{(7.3.16)} \qquad \Big\{\lambda_{-(\kappa-1)}^2, \ldots, \lambda_0^2, \sum_{i=1}^{\eta} t_i^2, \frac{\sum_{i=1}^{\eta} t_i^2(1+\tau_i([0,1]))}{\sum_{i=1}^{\eta} t_i^2}, \ldots\Big\}$$

is monotonically decreasing. In particular, we have $\lambda_{-(\kappa-1)}^2 \geqslant \ldots \geqslant \lambda_0^2 \geqslant \sum_{i=1}^{\eta} t_i^2$. This, combined with (3.1.7) and (7.3.15), yields the bottom equality in (7.3.5). $\square$

Remark 7.3.3. As in the subnormal case, we can single out a class of extremal completely hyperexpansive weighted shifts on $\mathscr{T}_{\eta,\kappa}$ (cf. Remark 6.2.3). Suppose that $\kappa \in \mathbb{N}$. We say that a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ is *extremal* if

$$\|S_{\boldsymbol\lambda} e_{\mathsf{root}}\| = \min \|S_{\tilde{\boldsymbol\lambda}} e_{\mathsf{root}}\|,$$

where the minimum is taken over all completely hyperexpansive weighted shifts $S_{\tilde{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\tilde{\boldsymbol\lambda} = \{\tilde\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $S_{\boldsymbol\lambda_{\to(-\kappa+1)}} = S_{\tilde{\boldsymbol\lambda}_{\to(-\kappa+1)}}$, or equivalently that $\lambda_v = \tilde\lambda_v$ for all $v \neq -\kappa+1$. It follows from Corollary 7.2.3 that a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ is extremal if and only if $S_{\boldsymbol\lambda}$ satisfies the condition (ii-a) with the inequality in (7.2.7) replaced by equality; in other words, $S_{\boldsymbol\lambda}$ is extremal if and only if $S_{\boldsymbol\lambda}$ satisfies the strong consistency condition at each vertex $u \in V_{\eta,\kappa}$ (cf. (7.1.8)).

As stated in Theorem 7.3.4 below, extremal completely hyperexpansive weighted shifts on $\mathscr{T}_{\eta,\kappa}$ with a fixed system of representing measures $\{\tau_{i,1}\}_{i=1}^\eta$ are in one-to-one and onto correspondence with sequences $\{t_i\}_{i=1}^\eta \subseteq (0,\infty)$ satisfying the conditions (7.3.1), (7.3.2) and (7.3.3).

THEOREM 7.3.4. *Let $\eta \in \{2,3,\ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. Assume $\boldsymbol\tau = \{\tau_i\}_{i=1}^\eta$ is a sequence of positive Borel measures on $[0,1]$ such that $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$. Let $\mathscr{U}_{\eta,\kappa}^{\boldsymbol\tau}$ be the set of all sequences $\boldsymbol{t} = \{t_i\}_{i=1}^\eta \subseteq (0,\infty)$ satisfying (7.3.1), (7.3.2) and (7.3.3), and let $\mathscr{V}_{\eta,\kappa}^{\boldsymbol\tau}$ be the set of all completely hyperexpansive weighted shifts $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$. Denote by $\mathscr{W}_{\eta,\kappa}^{\boldsymbol\tau}$ the set of all completely hyperexpansive weighted shifts $S_{\boldsymbol\lambda} \in \mathscr{V}_{\eta,\kappa}^{\boldsymbol\tau}$ which are extremal. If $\kappa = 0$, then the mapping*

$$\varPhi_{\eta,0}\colon \mathscr{U}_{\eta,0}^{\boldsymbol\tau} \ni \boldsymbol{t} \mapsto S_{\boldsymbol\lambda} \in \mathscr{V}_{\eta,0}^{\boldsymbol\tau}$$

defined by (7.3.13) is a bijection. If $\kappa \geqslant 1$, then the mapping

$$\varPhi_{\eta,\kappa}\colon \mathscr{U}_{\eta,\kappa}^{\boldsymbol\tau} \ni \boldsymbol{t} \mapsto S_{\boldsymbol\lambda} \in \mathscr{W}_{\eta,\kappa}^{\boldsymbol\tau}$$

defined by (7.3.13) and

$$\text{(7.3.17)} \qquad \lambda_{-k} = \begin{cases} \frac{1}{\sqrt{\zeta_2}} & \text{for } k = 0, \\ \sqrt{\frac{\zeta_{k+1}}{\zeta_{k+2}}} & \text{for } k \in J_{\kappa-1}, \end{cases}$$

is a bijection (see (7.3.4) for the definition of ζ_k). Moreover, if $S_{\boldsymbol\lambda} \in \mathscr{V}_{\eta,\kappa}^{\boldsymbol\tau}$, then $\boldsymbol{t} = \{t_i\}_{i=1}^\eta \in \mathscr{U}_{\eta,\kappa}^{\boldsymbol\tau}$ with $t_i := \lambda_{i,1}$, $\lambda_{-\kappa+1} \in \Big[\sqrt{\frac{\zeta_\kappa}{\zeta_{\kappa+1}}}, \infty\Big)$ and $\lambda_v = \tilde\lambda_v$ for all

$v \neq -\kappa+1$, where $S_{\tilde{\boldsymbol\lambda}} = \varPhi_{\eta,\kappa}(\boldsymbol t)$. Conversely, if $\boldsymbol t \in \mathscr{U}^{\boldsymbol\tau}_{\eta,\kappa}$ and $S_{\boldsymbol\lambda} = \varPhi_{\eta,\kappa}(\boldsymbol t)$, then for every $\vartheta \in \Big[\sqrt{\frac{\zeta_\kappa}{\zeta_{\kappa+1}}}, \infty\Big)$, the weighted shift $S_{\tilde{\boldsymbol\lambda}}$ with weights $\tilde{\boldsymbol\lambda} = \{\tilde\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$ given by $\tilde\lambda_v = \lambda_v$ for all $v \neq -\kappa+1$, and $\tilde\lambda_{-\kappa+1} = \vartheta$ is a member of $\mathscr{V}^{\boldsymbol\tau}_{\eta,\kappa}$.

PROOF. Apply Lemma 7.3.2 as well as its proof. □

Our next aim is to find necessary and sufficient conditions for the parameterizing set $\mathscr{U}^{\boldsymbol\tau}_{\eta,\kappa}$ to be nonempty.

PROPOSITION 7.3.5. *If η, κ, $\boldsymbol\tau = \{\tau_i\}_{i=1}^\eta$ and $\mathscr{U}^{\boldsymbol\tau}_{\eta,\kappa}$ are as in Theorem 7.3.4, then the following two conditions are equivalent:*

(i) $\mathscr{U}^{\boldsymbol\tau}_{\eta,\kappa} \neq \varnothing$,

(ii) *$\int_0^1 \frac{1}{s^{\kappa+1}} \mathrm{d}\tau_i(s) < \infty$ for all $i \in J_\eta$, and $\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0}(s) < 1$ for some $i_0 \in J_\eta$.*

PROOF. Note that if (i) or (ii) holds, then $\int_0^1 \frac{1}{s^j}\mathrm{d}\tau_i(s) < \infty$ for all $i \in J_\eta$ and $j \in J_{\kappa+1}$.

First, we consider the case when $\kappa = 0$.

(i)⇒(ii) Suppose that, contrary to our claim, $\int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \geqslant 1$ for all $i \in J_\eta$. Take $\{t_i\}_{i=1}^\eta \in \mathscr{U}^{\boldsymbol\tau}_{\eta,\kappa}$. Then $\sum_{i=1}^\eta t_i^2 \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \geqslant \sum_{i=1}^\eta t_i^2$, which contradicts (7.3.3).

(ii)⇒(i) Set

$$J_\eta^+ = \Big\{i \in J_\eta \colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) > 0\Big\}, \quad J_\eta^- = \Big\{i \in J_\eta \colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \leqslant 0\Big\}.$$

Then $J_\eta = J_\eta^+ \sqcup J_\eta^-$ and $i_0 \in J_\eta^+$. Let $\{\tilde t_i\}_{i\in J_\eta^+}$ and $\{\tilde t_i\}_{i\in J_\eta^-}$ be systems of positive real numbers such that (see the convention preceding Proposition 3.1.3)

$$\sum_{i\in J_\eta^+} \tilde t_i^2 < \infty, \ \sum_{i\in J_\eta^-} \tilde t_i^2 < \infty \text{ and } \beta := \sum_{i\in J_\eta^-} \tilde t_i^2 \Big(\int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) - 1\Big) < \infty.$$

Then clearly $\alpha := \sum_{i\in J_\eta^+} \tilde t_i^2 (1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s)) < \infty$. If we consider the sequence $\boldsymbol t = \{t_i\}_{i=1}^\eta$ given by

$$t_i = \begin{cases} r_1 \tilde t_i & \text{for } i \in J_\eta^+, \\ r_2 \tilde t_i & \text{for } i \in J_\eta^-, \end{cases}$$

where $r_1, r_2 \in \mathbb{R}$, then the inequality in (7.3.2) takes the form $1 \leqslant r_1^2\alpha - r_2^2\beta$. Since $\alpha \in (0,\infty)$, we deduce that this inequality has a solution in positive reals r_1 and r_2. Hence the sequence $\{t_i\}_{i=1}^\eta$ satisfies (7.3.1) and (7.3.2), and consequently (7.3.3). In fact, the sequence $\boldsymbol t$ can be chosen so as to satisfy (7.3.1) and the equality $\sum_{i=1}^\eta t_i^2 = 1 + \sum_{i=1}^\eta t_i^2 \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s)$.

Consider now the case when $\kappa \geqslant 1$.

(i)⇒(ii) Repeat the argument used in the proof of the case $\kappa = 0$.

(ii)⇒(i) Set

$$J_\eta^{++} = \Big\{i \in J_\eta \colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) > 0 \ \ \& \ \ 1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) > 0\Big\},$$

$$J_\eta^{+-} = \Big\{i \in J_\eta \colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) > 0 \ \ \& \ \ 1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) \leqslant 0\Big\},$$

$$J_\eta^{-+} = \Big\{i \in J_\eta\colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \leqslant 0 \;\;\&\;\; 1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) > 0\Big\},$$

$$J_\eta^{--} = \Big\{i \in J_\eta\colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \leqslant 0 \;\;\&\;\; 1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) \leqslant 0\Big\}.$$

It is clear that $J_\eta^{++} = \big\{i \in J_\eta\colon 1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) > 0\big\} \neq \varnothing$, $J_\eta^{-+} = \varnothing$ and $J_\eta^{--} = \big\{i \in J_\eta\colon 1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \leqslant 0\big\}$. Moreover, the sets J_η^{++}, J_η^{+-} and J_η^{--} are pairwise disjoint and $J_\eta = J_\eta^{++} \cup J_\eta^{+-} \cup J_\eta^{--}$. Arguing as in Section 6.3, we find square summable systems $\{\tilde t_i\}_{i\in J_\eta^{++}}, \{\tilde t_i\}_{i\in J_\eta^{+-}}, \{\tilde t_i\}_{i\in J_\eta^{--}} \subseteq (0,\infty)$ such that

$$\begin{aligned}
1 \;&=\; \sum_{i\in J_\eta^{++}} \tilde t_i^2\Big(1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s)\Big),\\
\vartheta_1 &:= \sum_{i\in J_\eta^{+-}} \tilde t_i^2\Big(1 - \int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s)\Big) < \infty,\\
\vartheta_2 &:= \sum_{i\in J_\eta^{--}} \tilde t_i^2\Big(\int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) - 1\Big) < \infty,\\
\alpha_1 &:= \sum_{i\in J_\eta^{+-}} \tilde t_i^2\Big(\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) - 1\Big) < \infty,\\
\alpha_2 &:= \sum_{i\in J_\eta^{--}} \tilde t_i^2\Big(\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) - 1\Big) < \infty.
\end{aligned}$$

Since the system $\{\tilde t_i\}_{i\in J_\eta^{++}}$ is square summable, we deduce that

$$0 < \alpha_0 := \sum_{i\in J_\eta^{++}} \tilde t_i^2\Big(1 - \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s)\Big) < \infty.$$

If we consider the sequence $\boldsymbol{t} = \{t_i\}_{i=1}^\eta$ given by

$$t_i = \begin{cases} r_0\tilde t_i & \text{for } i \in J_\eta^{++},\\ r_1\tilde t_i & \text{for } i \in J_\eta^{+-},\\ r_2\tilde t_i & \text{for } i \in J_\eta^{--},\end{cases}$$

where $r_0, r_1, r_2 \in \mathbb{R}$, then the conditions (7.3.2) and (7.3.3) take the following form:

$$1 = r_0^2 + r_1^2\vartheta_1 - r_2^2\vartheta_2, \quad r_0^2\alpha_0 - r_1^2\alpha_1 - r_2^2\alpha_2 > 0. \tag{7.3.18}$$

Since $\alpha_0 \in (0,\infty)$, we easily verify that there exist positive real numbers r_0, r_1 and r_2, which satisfy (7.3.18). This completes the proof. $\square$

It may be worth noting that the proof of Proposition 7.3.5 also works without assuming that $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$.

We are now ready to give a method of constructing all possible completely hyperexpansive bounded weighted shifts with nonzero weights on the directed tree $\mathscr{T}_{\eta,\kappa}$ with $\kappa < \infty$. The reader is asked to compare this method with that for subnormal weighted shifts described in Procedure 6.3.1. In particular, Procedure

6.3.1 enables us to construct bounded subnormal weighted shifts on $\mathscr{T}_{\eta,\infty}$ with nonzero weights which are not isometric. In view of Corollary 7.2.3 (iii), this never happens in the case of completely hyperexpansive weighted shifts, because such operators are isometric. Isometric weighted shifts are discussed in Propositions 6.2.4 and 6.2.5 (the case of the directed tree $\mathscr{T}_{\eta,\kappa}$) and in Corollary 3.4.4 (the general situation).

PROCEDURE 7.3.6. Fix $\eta \in \{2,3,\ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. Let $\{\tau_i\}_{i=1}^{\eta}$ be a sequence of positive Borel measures on $[0,1]$ such that $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$, $\int_0^1 \frac{1}{s^{\kappa+1}} \mathrm{d}\tau_i(s) < \infty$ for all $i \in J_\eta$ and $\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0}(s) < 1$ for some $i_0 \in J_\eta$. Using Proposition 7.3.5, we get a sequence $\boldsymbol{t} = \{t_i\}_{i=1}^{\eta}$ of positive real numbers satisfying the conditions (7.3.1), (7.3.2) and (7.3.3). Next, applying Theorem 7.3.4, we get a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\lambda_{i,1} = t_i$ and $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$.

7.4. Completion of weights on $\mathscr{T}_{\eta,\kappa}$. Using the modelling procedure described in Section 7.3, we give a deeper insight into complete hyperexpansivity of weighted shifts on $\mathscr{T}_{\eta,\kappa}$. We begin with writing some estimates (from above and from below) for $\sum_{i=1}^{\eta} \lambda_{i,1}^2$ and $\tau_{j,1}([0,1])$, $j \in J_\eta$. Under some circumstances, this enables us to simplify the formula (7.3.5) for the norm of a completely hyperexpansive weighted shift on the directed tree $\mathscr{T}_{\eta,\kappa}$.

PROPOSITION 7.4.1. *Let $\eta \in \{2,3,\ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. Assume that $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ is a completely hyperexpansive weighted shift on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$. Set $\tau_i = \tau_{i,1}$ and $t_i = \lambda_{i,1}$ for $i \in J_\eta$. Then the following assertions hold.*

(i) *There exists $i_0 \in J_\eta$ such that $\tau_{i_0}([0,1]) < \frac{1}{\kappa+1}$.*

(ii) *$\sum_{i=1}^{\eta} t_i^2 \geqslant 1$; moreover, $\sum_{i=1}^{\eta} t_i^2 = 1$ if and only if either $\kappa = 0$ and $S_{\boldsymbol\lambda}$ is an isometry or $\kappa \geqslant 1$ and $S_{\boldsymbol\lambda_{\to(-\kappa+1)}}$ is an isometry (if $S_{\boldsymbol\lambda}$ is extremal, then $\sum_{i=1}^{\eta} t_i^2 = 1$ if and only if $S_{\boldsymbol\lambda}$ is an isometry).*

(iii) *If $\kappa \geqslant 1$, then $\sum_{i=1}^{\eta} t_i^2 < \frac{\kappa+1}{\kappa}$.*

(iv) *$\sum_{i=1}^{\eta} t_i^2 \geqslant 1 + \inf\{\tau_i([0,1])\colon i \in J_\eta\}$.*

(v) *If $\tau_i([0,1]) = \tau_1([0,1])$ for all $i \in J_\eta$, then*

$$\|S_{\boldsymbol\lambda}\|^2 = \begin{cases} \sum_{i=1}^{\eta} t_i^2 & \text{for } \kappa = 0, \\ \lambda_{-(\kappa-1)}^2 & \text{for } \kappa \geqslant 1. \end{cases} \tag{7.4.1}$$

PROOF. (i) It follows from Theorem 7.3.4 and Proposition 7.3.5 that the exists $i_0 \in J_\eta$ such that $\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0}(s) < 1$. As a consequence, we have

$$(\kappa+1)\tau_{i_0}([0,1]) \leqslant \int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0}(s) < 1.$$

(ii) The inequality $\sum_{i=1}^{\eta} t_i^2 \geqslant 1$ follows from (7.3.2). Suppose that $\sum_{i=1}^{\eta} t_i^2 = 1$. Using (7.3.2) again, we see that $\tau_i = 0$ for all $i \in J_\eta$, which in view of Theorem 7.3.4 implies that $\sum_{v\in\mathsf{Chi}(u)} \lambda_v^2 = 1$ for all $u \in V_{\eta,\kappa}$ when $\kappa = 0$, and for all $u \in V_{\eta,\kappa} \setminus \{-\kappa\}$ when $\kappa \geqslant 1$. Thus, by Corollary 3.4.4, $S_{\boldsymbol\lambda}$ is an isometry when $\kappa = 0$, and $S_{\boldsymbol\lambda_{\to(-\kappa+1)}}$ is an isometry when $\kappa \geqslant 1$. The reverse implication is obvious. A similar reasoning applies to the case when $S_{\boldsymbol\lambda}$ is extremal (cf. Remark 7.3.3).

(iii) It follows from Lemma 7.3.2 (i) that

$$\sum_{i=1}^{\eta} t_i^2 \overset{(7.3.3)}{>} \sum_{i=1}^{\eta} t_i^2 \int_0^1 \Big(\frac{1}{s} + \ldots + \frac{1}{s^{\kappa+1}}\Big) \mathrm{d}\tau_i(s) \quad \geqslant (\kappa+1) \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s)$$
$$\overset{(7.3.2)}{=} (\kappa+1)\Big(\sum_{i=1}^{\eta} t_i^2 - 1\Big),$$

which implies that $\sum_{i=1}^{\eta} t_i^2 < \frac{\kappa+1}{\kappa}$.

(iv) Since the sequence $\{\|S_{\boldsymbol\lambda}^n e_0\|^2\}_{n=0}^{\infty}$ is completely alternating, we infer from (7.3.13) and Lemma 7.1.1 that the corresponding sequence of quotients

$$\Big\{ \sum_{i=1}^{\eta} t_i^2, \frac{\sum_{i=1}^{\eta} t_i^2 (1+\tau_i([0,1]))}{\sum_{i=1}^{\eta} t_i^2}, \ldots \Big\}$$

is monotonically decreasing. This implies that $\sum_{i=1}^{\eta} t_i^2 \geqslant 1 + \inf_{i \in J_\eta} \tau_i([0,1])$.

(v) Apply (iv), (7.3.5) and (7.3.16). □

Regarding parts (ii) and (iii) of Proposition 7.4.1, it is worth noting that if $\kappa = 0$, then there is no upper bound for $\sum_{i=1}^{\eta} t_i^2$. Moreover, for each $\Theta \in [1, \frac{\kappa+1}{\kappa})$ (with the usual convention that $\frac{1}{0} = \infty$) there exists a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that $\Theta = \sum_{i=1}^{\eta} \lambda_{i,1}^2$. In fact, we can prove a more general result (see Example 6.3.3 for the discussion of the case of subnormality).

Proposition 7.4.2. *Let $\eta \in \{2,3,\ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. If $\{t_i\}_{i=1}^{\eta}$ is a sequence of positive real numbers such that $1 \leqslant \sum_{i=1}^{\eta} t_i^2 < \frac{\kappa+1}{\kappa}$, then there exists a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v \in V_{\eta,\kappa}^\circ}$ such that $\lambda_{i,1} = t_i$ for all $i \in J_\eta$.*

Proof. Set $\Theta = \sum_{i=1}^{\eta} t_i^2$. According to our assumptions, we have $\Theta \in [1, \frac{\kappa+1}{\kappa})$. Define the sequence $\{\tau_i\}_{i=1}^{\eta}$ of positive Borel measures on $[0,1]$ by $\tau_i = \frac{\Theta-1}{\Theta}\delta_1$ for $i \in J_\eta$. It is easily seen that

$$\sum_{i=1}^{\eta} t_i^2 = 1 + \sum_{i=1}^{\eta} t_i^2 \int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s).$$

Hence, if $\kappa = 0$, then by applying Lemma 7.3.2 (ii) we get the required weighted shift $S_{\boldsymbol\lambda}$. If $\kappa \geqslant 1$, then $1 \leqslant \Theta < \frac{\kappa+1}{\kappa}$ implies that $1 - (\kappa+1)\frac{\Theta-1}{\Theta} > 0$. Thus

$$\sum_{i=1}^{\eta} t_i^2 \Big(1 - \int_0^1 \Big(\frac{1}{s} + \ldots + \frac{1}{s^{\kappa+1}}\Big) \mathrm{d}\tau_i(s)\Big) = \sum_{i=1}^{\eta} t_i^2 \Big(1 - (\kappa+1)\frac{\Theta - 1}{\Theta}\Big) > 0,$$

which enables us once more to employ Lemma 7.3.2 (ii). □

If total masses of representing measures τ_i are not identical, then the formula (7.4.1) for the norm of $S_{\boldsymbol\lambda}$ is no longer true.

Example 7.4.3. Consider the case when $\kappa = 1$, $\eta = 2$, $\tau_1 = 0$ and $\tau_2 = \delta_1$. Set $t_1 = 1$ and take any $t_2 \in (0,1)$. We easily verify that the conditions (7.3.1)-(7.3.3) are satisfied. By Theorem 7.3.4, there exists a unique extremal completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{2,1}))$ on $\mathscr{T}_{2,1}$ with positive weights $\boldsymbol\lambda =$

$\{\lambda_v\}_{v\in V_{2,1}^\circ}$ such that $\lambda_{i,1} = t_i$ and $\tau_{i,1} = \tau_i$ for $i = 1, 2$. Applying (7.3.17), we see that $\lambda_0 = \frac{1}{\sqrt{1-t_2^2}}$.

If $t_2 \in (0, \frac{1}{\sqrt{2}})$, then $\lambda_0 \in (1, \sqrt{2})$ and so $\lambda_0^2 < 2 = \max\{1 + \tau_i([0,1])\colon i = 1, 2\}$. Hence, by (7.3.5), $\|S_{\boldsymbol\lambda}\|^2 = \max\{1 + \tau_i([0,1])\colon i = 1, 2\}$, which means that (7.4.1) does not hold.

If $t_2 = \frac{1}{\sqrt{2}}$, then $\lambda_0 = \sqrt{2}$ and $\|S_{\boldsymbol\lambda}\|^2 = \lambda_0^2 = \max\{1+\tau_i([0,1])\colon i = 1, 2\}$. Thus (7.4.1) holds.

Finally, if $t_2 \in (\frac{1}{\sqrt{2}}, 1)$, then $\lambda_0 \in (\sqrt{2}, \infty)$, which implies that $\|S_{\boldsymbol\lambda}\|^2 = \lambda_0^2 > \max\{1 + \tau_i([0,1])\colon i = 1, 2\}$. Therefore (7.4.1) also holds.

The following result is a counterpart of Proposition 6.3.4 for completely hyperexpansive weighted shifts (see Section 7.2 for the definition of k-step backward extendibility). The reader should be aware of the difference between the condition (ii) of Proposition 6.3.4 and its counterpart in Proposition 7.4.4 below.

PROPOSITION 7.4.4. *Let $\eta \in \{2, 3, \ldots\} \sqcup \{\infty\}$ and $\kappa \in \mathbb{Z}_+$. If for every $i \in J_\eta$, S_i is a bounded unilateral classical weighted shift with positive weights $\{\alpha_{i,n}\}_{n=1}^\infty$, then the following two conditions are equivalent*:

(i) *there exists a system $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ of positive scalars such that the weighted shift $S_{\boldsymbol\lambda}$ on the directed tree $\mathscr{T}_{\eta,\kappa}$ is bounded and completely hyperexpansive, and*

$$\alpha_{i,n} = \lambda_{i,n+1}, \quad n \in \mathbb{N}, i \in J_\eta, \tag{7.4.2}$$

(ii) *the operator S_i is completely hyperexpansive and $\int_0^1 \frac{1}{s^{\kappa+1}} \mathrm{d}\tau_i(s) < \infty$ for every $i \in J_\eta$ (τ_i is the representing measure of S_i), S_{i_0} has a completely hyperexpansive $(\kappa + 1)$-step backward extension for some $i_0 \in J_\eta$, and $\sup_{i\in J_\eta} \|S_i\| < \infty$.*

PROOF. We argue essentially as in the proof of Proposition 6.3.4.

(i)⇒(ii) By (7.4.2), the operator $S_{\boldsymbol\lambda}|_{\ell^2(\mathsf{Des}(i,1))}$ is unitarily equivalent to S_i, and hence $\sup_{i\in J_\eta} \|S_i\| \leqslant \|S_{\boldsymbol\lambda}\|$. Since $\|S_{\boldsymbol\lambda}^n e_{i,1}\|^2 = \|S_i^n e_0\|^2$ for all $n \in \mathbb{Z}_+$, we see that S_i is completely hyperexpansive and $\tau_{i,1}$ is the representing measure of S_i. Owing to Theorem 7.3.4 and Proposition 7.3.5, $\int_0^1 \frac{1}{s^{\kappa+1}} \mathrm{d}\tau_{i,1}(s) < \infty$ for all $i \in J_\eta$, and $\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0,1}(s) < 1$ for some $i_0 \in J_\eta$. Applying (7.2.1) to S_{i_0}, we get (ii).

(ii)⇒(i) Since the weights of a completely hyperexpansive unilateral classical weighted shift are monotonically decreasing (use (7.1.2) and Lemma 7.1.1), we deduce that $\|S_i\|^2 = 1 + \tau_i([0,1])$. This yields $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$. It follows from (7.2.1), applied to S_{i_0}, that $\int_0^1 \sum_{l=1}^{\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_{i_0}(s) < 1$. Employing Procedure 7.3.6, we get a bounded completely hyperexpansive weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\tau_{i,1} = \tau_i$ for all $i \in J_\eta$. Hence $\|S_{\boldsymbol\lambda}^n e_{i,1}\|^2 = \|S_i^n e_0\|^2$ for all $n \in \mathbb{Z}_+$ and $i \in J_\eta$, which implies (7.4.2). □

COROLLARY 7.4.5. *Let η, κ, $\{\alpha_{i,n}\}_{n=1}^\infty$ and S_i be as in Proposition* 7.4.4. *If S_i has a completely hyperexpansive $(\kappa+1)$-step backward extension for every $i \in J_\eta$ and $\sup_{i\in J_\eta} \|S_i\| < \infty$, then there exists a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,\kappa}^\circ}$ such that $\alpha_{i,n} = \lambda_{i,n+1}$ for all $n \in \mathbb{N}$ and $i \in J_\eta$.*

PROOF. Apply (7.2.1) and Proposition 7.4.4. □

The converse of Corollary 7.4.5 does not hold. In fact, one can construct a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with positive weights such that the set of all $i \in J_\eta$ for which S_i has a completely hyperexpansive $(\kappa+1)$-step backward extension consists of one point (e.g., the required weighted shift can be obtained by applying Procedure 7.3.6 to the measures $\{\tau_i\}_{i\in J_\eta}$ given by $\tau_1 = 0$ and $\tau_i = \delta_1$ for $i \neq 1$).

7.5. Graph extensions. It turns out that the direct counterpart of Proposition 6.1.12 for complete hyperexpansivity is no longer true (cf. Example 7.5.2). In fact, the situation is now more complicated, and so we have to make stronger assumptions.

PROPOSITION 7.5.1. *Let $\mathscr{T} = (V, E)$ be a subtree of a directed tree $\hat{\mathscr{T}} = (\hat V, \hat E)$ such that, for some $w \in V \setminus \mathsf{Root}(\mathscr{T})$, $\mathsf{Chi}_{\mathscr{T}}(w) \neq \mathsf{Chi}_{\hat{\mathscr{T}}}(w)$, $\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w)) = \mathsf{Chi}_{\hat{\mathscr{T}}}(\mathsf{par}(w))$, and $\mathsf{Des}_{\mathscr{T}}(v) = \mathsf{Des}_{\hat{\mathscr{T}}}(v)$ for all $v \in \mathsf{Chi}_{\mathscr{T}}(w) \cup \big(\mathsf{Chi}_{\mathscr{T}}(\mathsf{par}(w)) \setminus \{w\}\big)$. Assume that $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ is a completely hyperexpansive weighted shift on $\mathscr{T}$ with nonzero weights $\boldsymbol\lambda = \{\lambda_u\}_{u\in V^\circ}$. If $S_{\boldsymbol\lambda}$ satisfies one of the following conditions:*

- (i) *$w \in V \setminus \big(\mathsf{Root}(\mathscr{T}) \cup \mathsf{Chi}(\mathsf{Root}(\mathscr{T}))\big)$,*
- (ii) *$S_{\boldsymbol\lambda}$ satisfies the strong consistency condition at $u = \mathsf{par}(w)$, i.e.,* (7.1.8) *is valid for $u = \mathsf{par}(w)$,*

then there exists no completely hyperexpansive weighted shift $S_{\hat{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(\hat V))$ on $\hat{\mathscr{T}}$ with nonzero weights $\hat{\boldsymbol\lambda} = \{\hat\lambda_u\}_{u\in\hat V^\circ}$ such that $\boldsymbol\lambda \subseteq \hat{\boldsymbol\lambda}$, i.e., $\lambda_u = \hat\lambda_u$ for all $u \in V^\circ$.

PROOF. Applying Lemma 7.1.8 to $u = \mathsf{par}^2(w)$, we see that (i) implies (ii). Assume that (ii) holds. Suppose that, contrary to our claim, such an $S_{\hat{\boldsymbol\lambda}}$ exists. It follows from Proposition 7.1.3 that $\mathscr{T}$ and $\hat{\mathscr{T}}$ are leafless. Hence $\varnothing \neq \mathsf{Chi}_{\mathscr{T}}(w) \subsetneq \mathsf{Chi}_{\hat{\mathscr{T}}}(w)$. Applying Lemma 7.1.8 to $u = \mathsf{par}(w)$, we deduce that $\tau_w^{\mathscr{T}}(\{0\}) = 0$, which, again by Lemma 7.1.8 applied now to $u = w$, yields

$$1 = \sum_{v\in\mathsf{Chi}_{\mathscr{T}}(w)} |\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\mathscr{T}}(s)\Big). \tag{7.5.1}$$

The same is true for $S_{\hat{\boldsymbol\lambda}}$. Since $\boldsymbol\lambda \subseteq \hat{\boldsymbol\lambda}$ and $\tau_v^{\mathscr{T}} = \tau_v^{\hat{\mathscr{T}}}$ for all $v \in \mathsf{Chi}_{\mathscr{T}}(w)$ (see the proof of Proposition 6.1.12), we have

$$\begin{aligned} 1 &= \sum_{v\in\mathsf{Chi}_{\hat{\mathscr{T}}}(w)} |\hat\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\hat{\mathscr{T}}}(s)\Big) \\ &= \sum_{v\in\mathsf{Chi}_{\mathscr{T}}(w)} |\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\mathscr{T}}(s)\Big) \\ &\quad + \sum_{v\in\mathsf{Chi}_{\hat{\mathscr{T}}}(w)\setminus\mathsf{Chi}_{\mathscr{T}}(w)} |\hat\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\hat{\mathscr{T}}}(s)\Big) \\ &\overset{(7.5.1)}{=} 1 + \sum_{v\in\mathsf{Chi}_{\hat{\mathscr{T}}}(w)\setminus\mathsf{Chi}_{\mathscr{T}}(w)} |\hat\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\hat{\mathscr{T}}}(s)\Big), \end{aligned}$$

which implies that

$$\sum_{v\in\mathsf{Chi}_{\hat{\mathscr{T}}}(w)\setminus\mathsf{Chi}_{\mathscr{T}}(w)} |\hat\lambda_v|^2\Big(1 - \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\hat{\mathscr{T}}}(s)\Big) = 0. \tag{7.5.2}$$

We now turn to the second part of the proof. Applying Lemma 7.1.8 to $u = \mathsf{par}(w)$ (as well as to both operators $S_{\boldsymbol\lambda}$ and $S_{\hat{\boldsymbol\lambda}}$), and using the assumption $\mathsf{Chi}_{\mathscr T}(\mathsf{par}(w)) = \mathsf{Chi}_{\hat{\mathscr T}}(\mathsf{par}(w))$, we deduce that

$$\begin{aligned} (7.5.3)\qquad 1 + \sum_{v\in\mathsf{Chi}_{\mathscr T}(\mathsf{par}(w))} |\lambda_v|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\mathscr T}(s) &\overset{\text{(ii)}}{=} \sum_{v\in\mathsf{Chi}_{\mathscr T}(\mathsf{par}(w))} |\lambda_v|^2 \\ \overset{\boldsymbol\lambda\subseteq\hat{\boldsymbol\lambda}}{=} \sum_{v\in\mathsf{Chi}_{\hat{\mathscr T}}(\mathsf{par}(w))} |\hat\lambda_v|^2 &\geqslant 1 + \sum_{v\in\mathsf{Chi}_{\hat{\mathscr T}}(\mathsf{par}(w))} |\hat\lambda_v|^2 \int_0^1 \frac{1}{s}\,\mathrm{d}\tau_v^{\hat{\mathscr T}}(s). \end{aligned}$$

It follows from our assumptions that $\tau_v^{\mathscr T} = \tau_v^{\hat{\mathscr T}}$ for all $v \in \mathsf{Chi}_{\mathscr T}(\mathsf{par}(w)) \setminus \{w\}$. Hence, by (7.5.3) and $\boldsymbol\lambda \subseteq \hat{\boldsymbol\lambda}$, we have

$$(7.5.4)\qquad \int_0^1 \frac{1}{s}\mathrm{d}\tau_w^{\mathscr T}(s) \geqslant \int_0^1 \frac{1}{s}\mathrm{d}\tau_w^{\hat{\mathscr T}}(s).$$

Applying Lemma 7.1.8 to $u = w$ (recall that $\tau_w^{\mathscr T}(\{0\}) = \tau_w^{\hat{\mathscr T}}(\{0\}) = 0$), we deduce from (7.1.7) that

$$\begin{aligned} \sum_{v\in\mathsf{Chi}_{\mathscr T}(w)} |\lambda_v|^2 \int_0^1 \frac{1}{s^2}\,\mathrm{d}\tau_v^{\mathscr T}(s) &= \int_0^1 \frac{1}{s}\mathrm{d}\tau_w^{\mathscr T}(s) \\ &\overset{(7.5.4)}{\geqslant} \int_0^1 \frac{1}{s}\mathrm{d}\tau_w^{\hat{\mathscr T}}(s) = \sum_{v\in\mathsf{Chi}_{\hat{\mathscr T}}(w)} |\hat\lambda_v|^2 \int_0^1 \frac{1}{s^2}\,\mathrm{d}\tau_v^{\hat{\mathscr T}}(s) \\ &\overset{\boldsymbol\lambda\subseteq\hat{\boldsymbol\lambda}}{=} \sum_{v\in\mathsf{Chi}_{\mathscr T}(w)} |\lambda_v|^2 \int_0^1 \frac{1}{s^2}\,\mathrm{d}\tau_v^{\mathscr T}(s) + \sum_{v\in\mathsf{Chi}_{\hat{\mathscr T}}(w)\setminus\mathsf{Chi}_{\mathscr T}(w)} |\hat\lambda_v|^2 \int_0^1 \frac{1}{s^2}\,\mathrm{d}\tau_v^{\hat{\mathscr T}}(s), \end{aligned}$$

which implies that

$$\sum_{v\in\mathsf{Chi}_{\hat{\mathscr T}}(w)\setminus\mathsf{Chi}_{\mathscr T}(w)} |\hat\lambda_v|^2 \int_0^1 \frac{1}{s^2}\,\mathrm{d}\tau_v^{\hat{\mathscr T}}(s) = 0.$$

Since $\mathsf{Chi}_{\hat{\mathscr T}}(w) \setminus \mathsf{Chi}_{\mathscr T}(w) \neq \varnothing$ and all the weights $\hat\lambda_v$ are nonzero, we conclude that $\tau_v^{\hat{\mathscr T}} = 0$ for all $v \in \mathsf{Chi}_{\hat{\mathscr T}}(w) \setminus \mathsf{Chi}_{\mathscr T}(w)$, which contradicts (7.5.2). This completes the proof. □

Regarding Proposition 7.5.1, note that if $\mathscr T = \mathscr T_{\eta,1}$ and $w = 0$, then (ii) is equivalent to assuming that $S_{\boldsymbol\lambda}$ is extremal.

We now show by example that the conclusion of Proposition 7.5.1 can fail if one of the assumptions (i) or (ii) is not satisfied. In fact, we give a method of constructing such examples. The reader who is interested in a simple example may consider the measures $\{\tau_i\}_{i\in J_{\hat\eta}}$ given by $\tau_i = 0$ for $i \in J_\eta$ and $\tau_i = \delta_1$ for $i \in J_{\hat\eta} \setminus J_\eta$.

Example 7.5.2. Let $\eta, \hat\eta \in \{2, 3, \ldots\} \sqcup \{\infty\}$ be such that $\eta < \hat\eta$ and let $\hat\kappa \in \mathbb Z_+$. Set $\mathscr T = \mathscr T_{\eta,0}$, $\tilde{\mathscr T} = \mathscr T_{\eta,\hat\kappa}$ and $\hat{\mathscr T} = \mathscr T_{\hat\eta,\hat\kappa}$. Take a (finite) sequence $\{\tau_i\}_{i=1}^{\eta}$ of positive Borel measures on $[0,1]$ such that $\sup_{i\in J_\eta} \tau_i([0,1]) < \infty$, $\int_0^1 \frac{1}{s^{\hat\kappa+1}}\mathrm{d}\tau_i(s) < \infty$ for all $i \in J_\eta$ and $\int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_{i_0}(s) < 1$ for some $i_0 \in J_\eta$. Applying Procedure 7.3.6 to $\tilde{\mathscr T}$, we deduce that there exists a completely hyperexpansive weighted shift

$S_{\tilde{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(V_{\eta,\hat\kappa}))$ on $\tilde{\mathscr T}$ with positive weights $\tilde{\boldsymbol\lambda} = \{\tilde\lambda_v\}_{v\in V_{\eta,\hat\kappa}^\circ}$ such that $\tau_{i,1}^{\tilde{\mathscr T}} = \tau_i$ for all $i \in J_\eta$. It follows from (7.3.3), applied to $S_{\tilde{\boldsymbol\lambda}}$, that

$$\sum_{i=1}^{\eta} \tilde\lambda_{i,1}^2 \Big(1 - \int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_i(s)\Big) > 0. \tag{7.5.5}$$

Consider now a supplementary sequence $\{\tau_i\}_{i\in J_{\hat\eta}\setminus J_\eta}$ of positive Borel measures on $[0,1]$ such that

$$\int_0^1 \frac{1}{s^{\hat\kappa+1}} \mathrm{d}\tau_i(s) < \infty, \quad i \in J_{\hat\eta} \setminus J_\eta, \tag{7.5.6}$$

$$\int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s) = 1, \quad i \in J_{\hat\eta} \setminus J_\eta. \tag{7.5.7}$$

Since $\int_0^1 \frac{1}{s}\mathrm{d}\tau_i(s) \geqslant \tau_i([0,1])$, we deduce from (7.5.7) that $\sup_{i\in J_{\hat\eta}} \tau_i([0,1]) < \infty$. By (7.5.5) and (7.5.6), there exists a square summable sequence $\{t_i\}_{i\in J_{\hat\eta}\setminus J_\eta}$ of positive real numbers such that $\sum_{i=\eta+1}^{\hat\eta} t_i^2 \big|1 - \int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s)\big| < \infty$ and

$$\sum_{i=1}^{\eta} \tilde\lambda_{i,1}^2 \Big(1 - \int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_i(s)\Big) + \sum_{i=\eta+1}^{\hat\eta} t_i^2 \Big(1 - \int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l} \mathrm{d}\tau_i(s)\Big) > 0. \tag{7.5.8}$$

Note that the sequence $\hat{\boldsymbol t} := \{\hat t_i\}_{i=1}^{\hat\eta}$ given by $\hat t_i = \tilde\lambda_{i,1}$ for $i \in J_\eta$ and $\hat t_i = t_i$ for $i \in J_{\hat\eta} \setminus J_\eta$ is square summable and (apply (7.3.2) to $S_{\tilde{\boldsymbol\lambda}}$)

$$\sum_{i=1}^{\hat\eta} \hat t_i^2 \Big(1 - \int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s)\Big) \overset{(7.5.7)}{=} \sum_{i=1}^{\eta} \tilde\lambda_{i,1}^2 \Big(1 - \int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s)\Big) \begin{cases} \geqslant 1 & \text{if } \hat\kappa = 0, \\ = 1 & \text{if } \hat\kappa \geqslant 1. \end{cases} \tag{7.5.9}$$

In view of[12] (7.5.8) and (7.5.9), we can apply Theorem 7.3.4 to the directed tree $\hat{\mathscr T}$ and to the sequences $\{\tau_i\}_{i\in J_{\hat\eta}}$ and $\hat{\boldsymbol t}$. In this way we obtain a completely hyperexpansive weighted shift $S_{\hat{\boldsymbol\lambda}} \in \boldsymbol{B}(\ell^2(V_{\hat\eta,\hat\kappa}))$ on $\hat{\mathscr T}$ (which is extremal when $\hat\kappa \geqslant 1$) with positive weights $\hat{\boldsymbol\lambda} = \{\hat\lambda_v\}_{v\in V_{\hat\eta,\hat\kappa}^\circ}$ such that

$$\hat\lambda_{i,1} = \hat t_i \text{ and } \tau_{i,1}^{\hat{\mathscr T}} = \tau_i \text{ for all } i \in J_{\hat\eta}. \tag{7.5.10}$$

We are now ready to show that the conclusion of Proposition 7.5.1 may not hold if $w \in \mathsf{Root}(\mathscr T)$. Indeed, applying Theorem 7.3.4 to $\mathscr T_{\eta,0}$, $\{\tau_i\}_{i=1}^\eta$ and $\boldsymbol t = \{\tilde\lambda_{i,1}\}_{i=1}^\eta$, we get a completely hyperexpansive weighted shift $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{\eta,0}))$ on $\mathscr T$ with positive weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V_{\eta,0}^\circ}$ such that $\tau_{i,1}^{\mathscr T} = \tau_i$ and $\lambda_{i,1} = \tilde\lambda_{i,1}$ for all $i \in J_\eta$. Since, by (7.5.10), $\lambda_{i,1} = \hat\lambda_{i,1}$ and $\tau_{i,1}^{\mathscr T} = \tau_{i,1}^{\hat{\mathscr T}}$ for all $i \in J_\eta$, we deduce from (7.3.13) that $\boldsymbol\lambda \subseteq \hat{\boldsymbol\lambda}$.

Next example shows that the direct counterpart of Proposition 6.1.12 for complete hyperexpansivity breaks down when $w \in \mathsf{Chi}(\mathsf{Root}(\mathscr T))$. Moreover, it exhibits that Proposition 7.5.1 is no longer true if $S_{\boldsymbol\lambda}$ does not satisfy the strong consistency condition at $u = \mathsf{par}(w)$ (even though $S_{\hat{\boldsymbol\lambda}}$ satisfies the strong consistency condition at $u = \mathsf{par}(w)$). For this purpose, we consider the pair $(\tilde{\mathscr T}, \hat{\mathscr T})$ with $\hat\kappa = 1$, i.e., $\tilde{\mathscr T} = \mathscr T_{\eta,1}$ and $\hat{\mathscr T} = \mathscr T_{\hat\eta,1}$. Let $S_{\tilde{\boldsymbol\lambda}}$ and $S_{\hat{\boldsymbol\lambda}}$ be as in the penultimate paragraph. Define the new system $\boldsymbol\lambda^\flat = \{\lambda_v^\flat\}_{v\in V_{\eta,1}^\circ}$ of positive weights by modifying the old

[12] The reader should be aware of the fact that $\sum_{i=1}^{\hat\eta} \hat t_i^2 \int_0^1 \sum_{l=1}^{\hat\kappa+1} \frac{1}{s^l}\mathrm{d}\tau_i(s) < \infty$.

one $\tilde{\boldsymbol\lambda}$ as follows: $\lambda^\flat_v = \tilde\lambda_v$ for all $v \neq 0$ and $\lambda^\flat_0 = \hat\lambda_0$. Since $S_{\hat{\boldsymbol\lambda}}$ is extremal, we infer from (7.3.17) and (7.5.10) that

$$\begin{aligned}
\lambda^\flat_0 &= \frac{1}{\sqrt{\sum_{i=1}^{\hat\eta} \hat\lambda_{i,1}^2 \Big(1 - \int_0^1 \sum_{l=1}^2 \frac{1}{s^l} \mathrm{d}\tau_i(s)\Big)}} \\
&= \frac{1}{\sqrt{\sum_{i=1}^{\eta} \tilde\lambda_{i,1}^2 \Big(1 - \int_0^1 \sum_{l=1}^2 \frac{1}{s^l} \mathrm{d}\tau_i(s)\Big) - \sum_{i=\eta+1}^{\hat\eta} t_i^2 \Big(\int_0^1 \sum_{l=1}^2 \frac{1}{s^l} \mathrm{d}\tau_i(s) - 1\Big)}} \\
&> \frac{1}{\sqrt{\sum_{i=1}^{\eta} \tilde\lambda_{i,1}^2 \Big(1 - \int_0^1 \sum_{l=1}^2 \frac{1}{s^l} \mathrm{d}\tau^{\tilde{\mathscr T}}_{i,1}(s)\Big)}},
\end{aligned}$$

where the last inequality is a consequence of the following estimate

$$1 \overset{(7.5.7)}{=} \int_0^1 \frac{1}{s} \mathrm{d}\tau_i(s) < \int_0^1 \sum_{l=1}^2 \frac{1}{s^l} \mathrm{d}\tau_i(s), \quad i \in J_{\hat\eta} \setminus J_\eta.$$

Hence, by Theorem 7.3.4, $S_{\boldsymbol\lambda^\flat} \in \boldsymbol{B}(\ell^2(V_{\eta,1}))$ is a completely hyperexpansive weighted shift on $\tilde{\mathscr T}$ with positive weights $\boldsymbol\lambda^\flat$ such that $\boldsymbol\lambda^\flat \subseteq \hat{\boldsymbol\lambda}$. Certainly, $S_{\boldsymbol\lambda^\flat}$ is not extremal, or equivalently $S_{\boldsymbol\lambda^\flat}$ does not satisfy the strong consistency condition at $u = \mathsf{par}(w) = -1$. However, $S_{\hat{\boldsymbol\lambda}}$ does satisfy the strong consistency condition at $u = \mathsf{par}(w)$.

Chapter 8. Miscellanea

8.1. Admissibility of assorted weighted shifts. In this section we characterize directed trees admitting weighted shifts with assorted properties. To be more precise, a directed tree $\mathscr{T}$ is said to *admit* a weighted shift with a property $\mathcal{P}$ if there exists a weighted shift on $\mathscr{T}$ with this property. First, we describe directed trees admitting weighted shifts with dense range. For this, we prove the following lemma (see Remarks 3.1.4 and 3.4.2 for the definitions of directed trees $\mathbb{Z}$ and $\mathbb{Z}_-$).

LEMMA 8.1.1. *If $S_{\boldsymbol\lambda}$ is a densely defined weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the following two conditions are equivalent*:

(i) *$\mathscr{R}(S_{\boldsymbol\lambda})$ is dense in $\ell^2(V)$,*
(ii) *the directed tree $\mathscr{T}$ is isomorphic either to $\mathbb{Z}_-$ or to $\mathbb{Z}$, and $\lambda_u \neq 0$ for all $u \in V$.*

PROOF. (i)$\Rightarrow$(ii) It follows from Proposition 3.5.1 (ii) that the directed tree $\mathscr{T}$ is rootless and $\dim\big(\ell^2(\mathsf{Chi}(u)) \ominus \langle\boldsymbol\lambda^u\rangle\big) = 0$ for all $u \in V'$. It is a matter of routine to verify that the latter implies that

$$\operatorname{card}(\mathsf{Chi}(u)) = 1, \quad u \in V', \tag{8.1.1}$$

and $\boldsymbol\lambda^u \neq 0$ for all $u \in V'$. Since $\mathscr{T}$ is rootless, we deduce that $\lambda_v \neq 0$ for all $v \in V$. Fix $w \in V$. By (8.1.1) and Proposition 2.1.6 (i) and (iv), the entries of the sequence $\{\mathsf{par}^k(w)\}_{k=1}^\infty$ are distinct and $V = \{\mathsf{par}^k(w)\}_{k=1}^\infty \sqcup \mathsf{Des}(w)$. This, together with (2.1.10) and (8.1.1), implies that either $\mathsf{Chi}^{\langle n\rangle}(w) \neq \varnothing$ for all $n \in \mathbb{N}$ and consequently $\mathscr{T}$ is isomorphic to $\mathbb{Z}$, or there exists a smallest positive integer n such that $\mathsf{Chi}^{\langle n\rangle}(w) = \varnothing$ and consequently $\mathscr{T}$ is isomorphic to $\mathbb{Z}_-$.

(ii)$\Rightarrow$(i) Evident due to (3.1.4). □

By Lemma 8.1.1, the only densely defined weighted shifts on directed trees with dense range are either bilateral classical weighted shifts with nonzero weights or the adjoints of unilateral classical weighted shifts with nonzero weights.

PROPOSITION 8.1.2. *If $\mathscr{T}$ is a directed tree, then the following conditions are equivalent*:

(i) *$\mathscr{T}$ admits a densely defined weighted shift with dense range,*
(ii) *the directed tree $\mathscr{T}$ is isomorphic either to $\mathbb{Z}_-$ or to $\mathbb{Z}$.*

Moreover, if (i) holds, then each densely defined weighted shift on $\mathscr{T}$ with nonzero weights has dense range.

PROOF. (i)$\Rightarrow$(ii) Apply Lemma 8.1.1.

(ii)$\Rightarrow$(i) Consider the bounded weighted shift on $\mathbb{Z}_-$ (or on $\mathbb{Z}$) with $\lambda_u \equiv 1$. □

The question of when a directed tree admits a weighted shift which is respectively hyponormal, subnormal and completely hyperexpansive, has a simple answer.

PROPOSITION 8.1.3. *If $\mathscr{T}$ is a directed tree with $V^\circ \neq \varnothing$, then the following conditions are equivalent*:

(i) *$\mathscr{T}$ admits a bounded hyponormal weighted shift with nonzero weights,*
(ii) *$\mathscr{T}$ admits a bounded subnormal weighted shift with nonzero weights,*
(iii) *$\mathscr{T}$ admits an isometric weighted shift with nonzero weights,*
(iv) *$\mathscr{T}$ admits a bounded completely hyperexpansive weighted shift with nonzero weights,*

(v) *$\mathscr{T}$ is leafless and* $\operatorname{card}(V) = \aleph_0$,
(vi) *$\mathscr{T}$ admits a bounded injective weighted shift with nonzero weights.*

Proof. (i)$\Rightarrow$(v) Apply Proposition 5.1.1.

(v)$\Rightarrow$(iii)&(v)$\Rightarrow$(vi) Argue as in the proof of Proposition 3.1.10 and use Corollary 3.4.4.

(iv)$\Rightarrow$(v) Apply Propositions 3.1.10 and 7.1.3.

Since the implications (iii)$\Rightarrow$(iv), (iii)$\Rightarrow$(ii) and (ii)$\Rightarrow$(i) are obvious, and the implication (vi)$\Rightarrow$(v) is a consequence of Propositions 3.1.7 and 3.1.10, the proof is complete. □

Proposition 8.1.3 fails to hold if the requirement of nonzero weights is dropped. Indeed, if $\mathscr{T}$ is a directed tree which comes from $\mathbb{Z}_+$ by gluing a leaf to the directed tree $\mathbb{Z}_+$ at its root, and $S_{\boldsymbol\lambda}$ is the weighted shifts on $\mathscr{T}$ with 0 weight attached to the glued leaf, the remaining weights being equal to 1, then $S_{\boldsymbol\lambda}$ is subnormal, but $\mathscr{T}$ is not leafless.

As stated below, admissibility of adjoints of isometric (in short: *coisometric*) weighted shifts is much more restrictive.

Proposition 8.1.4. *If $\mathscr{T}$ is a directed tree, then the following assertions hold.*

(i) *$\mathscr{T}$ admits a coisometric weighted shift if and only if the directed tree $\mathscr{T}$ is isomorphic either to $\mathbb{Z}_-$ or to $\mathbb{Z}$; moreover, if $S_{\boldsymbol\lambda}$ is a coisometric weighted shift on $\mathscr{T}$, then all its weights are nonzero.*
(ii) *$\mathscr{T}$ admits a unitary weighted shift if and only if $\mathscr{T}$ is isomorphic to $\mathbb{Z}$.*

Proof. The assertion (i) is a direct consequence of Lemma 8.1.1 because each coisometry is surjective (consult also Remark 3.4.2). In turn, the assertion (ii) follows from (i) because any bounded weighted shift on $\mathbb{Z}_-$ is not injective. □

Using Theorem 5.2.2, one can construct bounded cohyponormal weighted shifts on directed trees with nonzero weights which are non-injective and non-coisometric. Most of the model trees appearing in Theorem 5.2.2 (ii) are far from being isomorphic to the directed trees $\mathbb{Z}_-$ and $\mathbb{Z}$. Hence, by Lemma 8.1.1, weighted shifts on these model trees (except for $\mathbb{Z}_-$ and $\mathbb{Z}$) do not have dense range.

We now discuss the question of when a given directed tree admits a bounded normal weighted shift with nonzero weights.

Lemma 8.1.5. *If $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ is a nonzero weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$, then the following two conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is normal,*
(ii) *there exists a sequence $\{u_n\}_{n=-\infty}^{\infty} \subseteq V$ such that*

$$u_{n-1} = \mathsf{par}(u_n) \text{ and } |\lambda_{u_{n-1}}| = |\lambda_{u_n}|$$

for all $n \in \mathbb{Z}$, and $\lambda_v = 0$ for all $v \in V \setminus \{u_n \colon n \in \mathbb{Z}\}$.

Proof. (i)$\Rightarrow$(ii) Note first that if $u \in V^\circ$, then by (3.1.4) and (3.4.1) we have

$$\|S_{\boldsymbol\lambda} e_u\|^2 e_u = S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda} e_u = S_{\boldsymbol\lambda} S_{\boldsymbol\lambda}^* e_u = |\lambda_u|^2 e_u + \sum_{v \in \mathsf{Chi}(\mathsf{par}(u))\setminus\{u\}} \lambda_v \overline{\lambda_u} e_v. \tag{8.1.2}$$

Hence, if $\|S_{\boldsymbol\lambda} e_u\| = 0$ for some $u \in V^\circ$, then $\lambda_u = 0$. This, combined with Theorem 5.2.2 and (8.1.2), establishes the implication (i)$\Rightarrow$(ii) (observe that the situation described in Theorem 5.2.2 (ii) is excluded because it forces $S_{\boldsymbol\lambda}$ to be the zero operator).

(ii)⇒(i) Argue as in (8.1.2). □

PROPOSITION 8.1.6. *If $\mathscr{T}$ is a directed tree with $V^\circ \neq \varnothing$, then the following two conditions are equivalent*:

(i) *$\mathscr{T}$ admits a bounded normal weighted shift with nonzero weights,*
(ii) *the directed tree $\mathscr{T}$ is isomorphic to $\mathbb{Z}$.*

PROOF. (i)⇒(ii) Apply Lemma 8.1.5.

(ii)⇒(i) Obvious. □

It turns out that quasinormal weighted shifts on directed trees with nonzero weights are scalar multiplies of isometric operators. Recall that an operator $A \in \boldsymbol{B}(\mathcal{H})$ acting on a complex Hilbert space $\mathcal{H}$ is said to be *quasinormal* if $A|A| = |A|A$, or equivalently if $AA^*A = A^*AA$. It is well known that normal operators are quasinormal and quasinormal operators are subnormal, but neither of these implications is reversible in general (cf. [**19**]).

PROPOSITION 8.1.7. *Let $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol\lambda = \{\lambda_v\}_{v\in V^\circ}$. Then the following conditions are equivalent*:

(i) *$S_{\boldsymbol\lambda}$ is quasinormal,*
(ii) *$\|S_{\boldsymbol\lambda} e_u\| = \|S_{\boldsymbol\lambda} e_v\|$ for all $u \in V$ and $v \in \mathsf{Chi}(u)$ such that $\lambda_v \neq 0$.*

Moreover, if $V^\circ \neq \varnothing$ and $\lambda_v \neq 0$ for all $v \in V^\circ$, then $S_{\boldsymbol\lambda}$ is quasinormal if and only if $\|S_{\boldsymbol\lambda}\|^{-1} S_{\boldsymbol\lambda}$ is an isometry.

PROOF. It follows from Proposition 3.4.3 that

$$S_{\boldsymbol\lambda}(S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda})e_u = \|S_{\boldsymbol\lambda} e_u\|^2 S_{\boldsymbol\lambda} e_u \overset{(3.1.4)}{=} \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_u\|^2 \lambda_v e_v, \quad u \in V,$$

and

$$(S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}) S_{\boldsymbol\lambda} e_u = \sum_{v\in\mathsf{Chi}(u)} \lambda_v (S_{\boldsymbol\lambda}^* S_{\boldsymbol\lambda}) e_v = \sum_{v\in\mathsf{Chi}(u)} \|S_{\boldsymbol\lambda} e_v\|^2 \lambda_v e_v, \quad u \in V.$$

Putting this all together completes the proof of the equivalence (i)⇔(ii).

Suppose now that $S_{\boldsymbol\lambda}$ is quasinormal, $V^\circ \neq \varnothing$ and $\lambda_v \neq 0$ for all $v \in V^\circ$. First, we claim that $\|S_{\boldsymbol\lambda} e_u\| = \text{const}$. For this, take $u \in V$. Using an induction argument and the implication (i)⇒(ii), we see that $\|S_{\boldsymbol\lambda} e_v\| = \|S_{\boldsymbol\lambda} e_u\|$ for all $v \in \mathsf{Chi}^{\langle n\rangle}(u)$ and $n \in \mathbb{Z}_+$. In view of (2.1.10), this implies that $\|S_{\boldsymbol\lambda} e_v\| = \|S_{\boldsymbol\lambda} e_u\|$ for all $v \in \mathsf{Des}(u)$. An application of Proposition 2.1.4 proves our claim. Hence, by (3.1.4) and Corollary 3.4.4, the operator $\|S_{\boldsymbol\lambda}\|^{-1} S_{\boldsymbol\lambda}$ is an isometry. The reverse implication is obvious. □

Note that if $\mathscr{T}$ is a directed tree (with or without root) such that

$$1 \leqslant \operatorname{card}(\mathsf{Chi}(u)) = \operatorname{card}(\mathsf{Chi}(v)) < \infty, \quad u, v \in V,$$

then the weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}$ with weights $\lambda_v = \text{const}$ is bounded and quasinormal.

8.2. p-hyponormality. Recall that an operator $A \in \boldsymbol{B}(\mathcal{H})$ acting on a complex Hilbert space $\mathcal{H}$ is said to be *p-hyponormal*, where p is a positive real number, if $|A^*|^{2p} \leqslant |A|^{2p}$. By the Löwner-Heinz inequality, for all positive real numbers p, q such that $p < q$, if $A \in \boldsymbol{B}(\mathcal{H})$ is q-hyponormal, then A is p-hyponormal (see [**81**] and [**35**] for more information on the subject). Clearly, the notions of 1-hyponormality and hyponormality coincide. This means that the following characterization of p-hyponormality can be thought of as a generalization of Theorem 5.1.2.

THEOREM 8.2.1. *Let $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ be a weighted shift on a directed tree $\mathscr{T}$ with weights $\boldsymbol{\lambda} = \{\lambda_v\}_{v \in V^\circ}$, and let p be a positive real number. Then the following assertions are equivalent*:

(i) *$S_{\boldsymbol{\lambda}}$ is p-hyponormal,*
(ii) *the following two conditions hold*:

$$\text{for every } u \in V, \text{ if } v \in \mathsf{Chi}(u) \text{ and } \|S_{\boldsymbol{\lambda}} e_v\| = 0, \text{ then } \lambda_v = 0, \tag{8.2.1}$$

$$\|S_{\boldsymbol{\lambda}} e_u\|^{2(p-1)} \sum_{v \in \mathsf{Chi}_{\boldsymbol{\lambda}}^+(u)} \frac{|\lambda_v|^2}{\|S_{\boldsymbol{\lambda}} e_v\|^{2p}} \leqslant 1, \quad u \in V_{\boldsymbol{\lambda}}^+. \tag{8.2.2}$$

Note that if $S_{\boldsymbol{\lambda}} \in \boldsymbol{B}(\ell^2(V))$ is a p-hyponormal weighted shift on a directed tree $\mathscr{T}$, then for every $u \in V_{\boldsymbol{\lambda}}^+$, the left-hand side of the inequality (8.2.2) never vanishes.

PROOF OF THEOREM 8.2.1. We make use of some ideas from the proof of Theorem 5.1.2. Let $S_{\boldsymbol{\lambda}} = U|S_{\boldsymbol{\lambda}}|$ be the polar decomposition of $S_{\boldsymbol{\lambda}}$. It follows from Propositions 3.4.1 (iii) and 3.5.1 that for every $f \in \ell^2(V)$,

$$(U^* f)(u) = \begin{cases} \frac{1}{\|S_{\boldsymbol{\lambda}} e_u\|} \sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v) & \text{for } u \in V_{\boldsymbol{\lambda}}^+, \\ 0 & \text{for } u \in V \setminus V_{\boldsymbol{\lambda}}^+. \end{cases} \tag{8.2.3}$$

Since $|S_{\boldsymbol{\lambda}}^*|^{2p} = U|S_{\boldsymbol{\lambda}}|^{2p}U^*$ (cf. [**35**, Theorem 4 in §2.2.2]), we deduce from Proposition 3.4.3 (iv) that

$$\begin{aligned} \langle |S_{\boldsymbol{\lambda}}^*|^{2p} f, f\rangle &= \langle |S_{\boldsymbol{\lambda}}|^{2p} U^* f, U^* f\rangle \\ &= \sum_{u \in V} (|S_{\boldsymbol{\lambda}}|^{2p} U^* f)(u) \overline{(U^* f)(u)} \\ &= \sum_{u \in V} \|S_{\boldsymbol{\lambda}} e_u\|^{2p} |(U^* f)(u)|^2 \\ &\overset{(8.2.3)}{=} \sum_{u \in V_{\boldsymbol{\lambda}}^+} \|S_{\boldsymbol{\lambda}} e_u\|^{2(p-1)} \Big| \sum_{v \in \mathsf{Chi}(u)} \overline{\lambda_v} f(v) \Big|^2, \quad f \in \ell^2(V). \end{aligned} \tag{8.2.4}$$

Similar reasoning leads to

$$\begin{aligned} \langle |S_{\boldsymbol{\lambda}}|^{2p} f, f\rangle &= \sum_{u \in V} \|S_{\boldsymbol{\lambda}} e_u\|^{2p} |f(u)|^2 \\ &\overset{(2.1.3)}{=} \|S_{\boldsymbol{\lambda}} e_{\mathsf{root}}\|^{2p} |f(\mathsf{root})|^2 \\ &\qquad + \sum_{u \in V} \sum_{v \in \mathsf{Chi}_{\boldsymbol{\lambda}}^+(u)} \|S_{\boldsymbol{\lambda}} e_v\|^{2p} |f(v)|^2, \quad f \in \ell^2(V), \end{aligned} \tag{8.2.5}$$

where the term $\|S_{\boldsymbol\lambda} e_{\mathsf{root}}\|^{2p}|f(\mathsf{root})|^2$ appears in (8.2.5) only if $\mathscr{T}$ has a root. Combining (8.2.4) with (8.2.5) (see also the proof of Theorem 4.1.1), we deduce that $S_{\boldsymbol\lambda}$ is p-hyponormal if and only if

$$\begin{aligned}\sum_{u\in V_{\boldsymbol\lambda}^+} \|S_{\boldsymbol\lambda} e_u\|^{2(p-1)}\Big|\sum_{v\in\mathsf{Chi}(u)} \overline{\lambda_v} f(v)\Big|^2 \\ \leqslant \sum_{u\in V}\sum_{v\in\mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \|S_{\boldsymbol\lambda} e_v\|^{2p}|f(v)|^2, \quad f\in\ell^2(V).\end{aligned} \tag{8.2.6}$$

Suppose that $S_{\boldsymbol\lambda}$ is p-hyponormal. If $v\in\mathsf{Chi}(u)$ is such that $\|S_{\boldsymbol\lambda} e_v\|=0$, then by substituting $f=e_v$ into (8.2.6) we obtain $\lambda_v=0$, which proves (8.2.1) (note that if $u\in V\setminus V_{\boldsymbol\lambda}^+$, then automatically $\lambda_v=0$). In view of (8.2.1) and (8.2.6), we see that for every $u\in V_{\boldsymbol\lambda}^+$,

$$\|S_{\boldsymbol\lambda} e_u\|^{2(p-1)}\Big|\sum_{v\in\mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \overline{\lambda_v} f(v)\Big|^2 \leqslant \sum_{v\in\mathsf{Chi}^+_{\boldsymbol\lambda}(u)} \|S_{\boldsymbol\lambda} e_v\|^{2p}|f(v)|^2, \quad f\in\ell^2(\mathsf{Chi}^+_{\boldsymbol\lambda}(u)).$$

This implies (8.2.2) (consult the part of the proof of Theorem 4.1.1 which comes after the inequality (4.1.4)). It is a simple matter to verify that the above reasoning can be reversed. This completes the proof. □

The following well known fact is a direct consequence of Theorem 8.2.1.

Corollary 8.2.2. *Let $p\in(0,\infty)$. A bounded unilateral or bilateral classical weighted shift S with nonzero weights is p-hyponormal if and only if it is hyponormal.*

Proof. The inequalities (8.2.2) are easily seen to be equivalent to the fact that the moduli of weights of S form a monotonically increasing sequence, which in turn is equivalent to the hyponormality of S. □

Theorem 8.2.1 provides us with a handy characterization of the p-hyponormality of weighted shifts on the directed tree $\mathscr{T}_{\eta,\kappa}$ defined in (6.2.10).

Corollary 8.2.3. *Let $\eta\in\{2,3,\ldots\}\sqcup\{\infty\}$, $\kappa\in\mathbb{Z}_+\sqcup\{\infty\}$ and $p\in(0,\infty)$. A weighted shift $S_{\boldsymbol\lambda}\in\boldsymbol{B}(\ell^2(V_{\eta,\kappa}))$ on $\mathscr{T}_{\eta,\kappa}$ with nonzero weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ_{\eta,\kappa}}$ is p-hyponormal if and only if $S_{\boldsymbol\lambda}$ satisfies the following conditions*:

$$\begin{aligned}&|\lambda_{i,j}|\leqslant|\lambda_{i,j+1}| \text{ for all } i\in J_\eta \text{ and } j\geqslant 2,\\ &\Big(\sum_{i=1}^{\eta}|\lambda_{i,1}|^2\Big)^{p-1}\Big(\sum_{i=1}^{\eta}\frac{|\lambda_{i,1}|^2}{|\lambda_{i,2}|^{2p}}\Big)\leqslant 1,\\ &|\lambda_0|^2\leqslant\sum_{i=1}^{\eta}|\lambda_{i,1}|^2, \text{ provided } \kappa\geqslant 1,\\ &|\lambda_{-(k+1)}|\leqslant|\lambda_{-k}| \text{ for } k=0,\ldots,\kappa-2, \text{ provided } \kappa\geqslant 2.\end{aligned}$$

We now show how to separate p-hyponormality classes with weighted shifts on the directed tree $\mathscr{T}_{2,1}$ (see [**55, 56**] and [**17**] for analogous results for weighted shifts with special matrix weights and composition operators, respectively).

Example 8.2.4. Let a,b be positive real numbers. Consider a weighted shift $S_{\boldsymbol\lambda}$ on $\mathscr{T}_{2,1}$ with weights $\boldsymbol\lambda=\{\lambda_v\}_{v\in V^\circ_{2,1}}$ such that $\lambda_0\in(0,\infty)$, $\lambda_{1,1}=\lambda_{2,1}=1/\sqrt{2}$

and $\lambda_{1,j} = 1/a$, $\lambda_{2,j} = 1/b$ for $j = 2, 3, \ldots$ By Corollary 3.1.9, $S_{\boldsymbol\lambda} \in \boldsymbol{B}(\ell^2(V_{2,1}))$. It follows from Corollary 8.2.3 that

$$S_{\boldsymbol\lambda} \text{ is } p\text{-hyponormal if and only if } \lambda_0 \leqslant 1 \text{ and } (a, b) \in \varDelta_p, \tag{8.2.7}$$

where $\varDelta_p = \{(x, y) \in \mathbb{R}^2 \colon x, y > 0,\ x^{2p} + y^{2p} \leqslant 2\}$. Observe that the set $\varDelta_p$ consists of all points of the first open quarter of the plane which lie on or below the graph of the function $x \mapsto \sqrt[2p]{2 - x^{2p}}$ (see Figure 7). By more or less elementary calculations, one can verify that $\varDelta_q \subsetneq \varDelta_p$ for all $p, q \in (0, \infty)$ such that $p < q$. What is more, if $0 < p < q$, then $(1, 1)$ is the only point of $\varDelta_q$ which is in the topological boundary of $\varDelta_p$. One can also check that

$$\begin{aligned} \varDelta_\infty &:= \bigcap_{p>0} \varDelta_p = \{(x, y) \in \mathbb{R}^2 \colon 0 < x \leqslant 1,\ 0 < y \leqslant 1\}, \\ \varDelta_0 &:= \bigcup_{p>0} \varDelta_p = \{(x, y) \in \mathbb{R}^2 \colon x, y > 0,\ xy < 1\} \cup \{(1, 1)\}. \end{aligned} \tag{8.2.8}$$

The sets $\varDelta_p$ are plotted in Figure 7 for some choices of p; the most external one corresponds to $p = 0$, while the most internal to $p = \infty$.

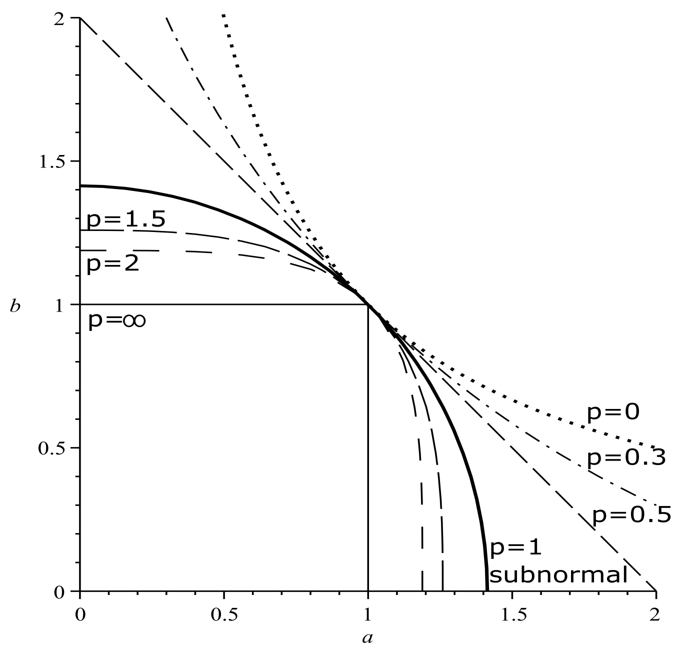

Figure 7

By (8.2.7) and (8.2.8), the operator $S_{\boldsymbol\lambda}$ is ∞-hyponormal (i.e., p-hyponormal for all $p \in (0, \infty)$) if and only if $\lambda_0, a, b \leqslant 1$. Owing to Proposition 6.2.5, $S_{\boldsymbol\lambda}$ is an isometry if and only if $\lambda_0 = a = b = 1$. Subnormality of $S_{\boldsymbol\lambda}$ can also be described in terms of the parameters a, b and λ_0. Namely, applying Corollary 6.2.2 (ii) to $\mu_1 = \delta_{1/a^2}$ and $\mu_2 = \delta_{1/b^2}$, we deduce that

$$S_{\boldsymbol\lambda} \text{ is subnormal if and only if } \frac{a^4 + b^4}{2} \leqslant \frac{1}{\lambda_0^2} \text{ and } a^2 + b^2 = 2. \tag{8.2.9}$$

Fix now any real λ_0 such that $0 < \lambda_0 \leqslant \frac{1}{\sqrt{2}}$. Since $x^2 + y^2 = 2$ implies $\frac{x^4+y^4}{2} < \frac{1}{\lambda_0^2}$ whenever $x, y > 0$, we deduce from (8.2.7) and (8.2.9) that $S_{\boldsymbol{\lambda}}$ is p-hyponormal if and only if $(a, b) \in \varDelta_p$, and $S_{\boldsymbol{\lambda}}$ is subnormal if and only if $a^2 + b^2 = 2$. In view of the above discussion, if $(a, b) \neq (1, 1)$, then the operator $S_{\boldsymbol{\lambda}}$ is simultaneously subnormal and p-hyponormal if and only if $0 < p \leqslant 1$ and $a^2 + b^2 = 2$. What is more, if $(a, b) \in \varDelta_\infty \setminus \{(1, 1)\}$, then $S_{\boldsymbol{\lambda}}$ is ∞-hyponormal but not subnormal. On the other hand, if $a^2 + b^2 = 2$ and $(a, b) \neq (1, 1)$, then $S_{\boldsymbol{\lambda}}$ is subnormal but not ∞-hyponormal (see [**17**, Examples 3.2 and 3.3] for the case of composition operators).

Acknowledgement. A substantial part of this paper was written while the first and the third authors visited Kyungpook National University during the springs and the autumns of 2008 and 2009; they wish to thank the faculty and administration of this unit for their warm hospitality.

References

[1] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. I, *Integral Equations Operator Theory* **21** (1995), 383-429.

[2] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. II, *Integral Equations Operator Theory* **23** (1995), 1-48.

[3] J. Agler, M. Stankus, m-isometric transformations of Hilbert space. III, *Integral Equations Operator Theory* **24** (1996), 379-421.

[4] A. Aleman, *The multiplication operators on Hilbert spaces of analytic functions*, Habilitationsschrift, Fernuversität Hagen, 1993.

[5] K. Aomoto, Selfadjointness and limit pointness for adjacency operators on a tree, *J. Analyse Math.* **53** (1989), 219-232.

[6] W. Arveson, D. W. Hadwin, T. B. Hoover, E. E. Kymala, Circular operators, *Indiana Univ. Math. J.* **33** (1984), 583-595.

[7] A. Athavale, On completely hyperexpansive operators, *Proc. Amer. Math. Soc.* **124** (1996), 3745-3752.

[8] A. Athavale, The complete hyperexpansivity analog of the Embry conditions, *Studia Math.* **154** (2003), 233-242.

[9] A. Athavale, A. Ranjekar, Bernstein functions, complete hyperexpansivity and subnormality. I. *Integral Equations Operator Theory* **43** (2002), 253-263.

[10] A. Athavale, A. Ranjekar, Bernstein functions, complete hyperexpansivity and subnormality, II, *Integral Equations Operator Theory* **44** (2002), 1-9.

[11] A. Athavale, V. M. Sholapurkar, Completely hyperexpansive operator tuples, *Positivity* **3** (1999), 245-257.

[12] C. Berg, J. P. R. Christensen, P. Ressel, *Harmonic Analysis on Semigroups*, Springer, Berlin, 1984.

[13] N. L. Biggs, B. Mohar, J. Shawe-Taylor, The spectral radius of infinite graphs, *Bull. London Math. Soc.* **20** (1988), 116-120.

[14] M. Sh. Birman, M. Z. Solomjak, *Spectral theory of selfadjoint operators in Hilbert space*, D. Reidel Publishing Co., Dordrecht, 1987.

[15] A. Brown, C. Pearcy, *Introduction to operator theory I, Elements of functional analysis*, Graduate Texts in Mathematics, No. **55**, Springer-Verlag, New York-Heidelberg, 1977.

[16] C. Burnap, I. B. Jung, Composition operators with weak hyponormality, *J. Math. Anal. Appl.* **337** (2008), 686-694.

[17] C. Burnap, I. B. Jung, A. Lambert, Separating partial normality classes with composition operators, *J. Operator Theory* **53** (2005), 381-397.

[18] J. B. Conway, *A course in functional analysis*, Graduate Texts in Mathematics **96**, Springer-Verlag, New York, 1990.

[19] J. B. Conway, *The theory of subnormal operators*, Mathematical Surveys and Monographs, Providence, Rhode Island, 1991.

[20] R. E. Curto, Quadratically hyponormal weighted shifts, *Integral Equations Operator Theory* **13** (1990), 49-66.

[21] R. E. Curto, S. H. Lee, J. Yoon, k-hyponormality of multivariable weighted shifts, *J. Funct. Anal.* **229** (2005), 462-480.

[22] R. E. Curto, W. Y. Lee, k-hyponormality of finite rank perturbations of unilateral weighted shifts, *Trans. Amer. Math. Soc.* **357** (2005), 4719-4737.

[23] R. E. Curto, J. Yoon, Jointly hyponormal pairs of commuting subnormal operators need not be jointly subnormal, *Trans. Amer. Math. Soc.* **358** (2006), 5139-5159.

[24] R. E. Curto, J. Yoon, Disintegration-of-measure techniques for commuting multivariable weighted shifts, *Proc. London Math. Soc.* **92** (2006), 381-402.

[25] D. M. Cvetković, M. Doob, H. Sachs, *Spectra of Graphs*, Johann Ambrosius Barth, Heidelberg, 1995.

[26] P. Dibrell, J. T. Campbell, Hyponormal powers of composition operators, *Proc. Amer. Math. Soc.* **102** (1988), 914-918.

[27] G. Exner, I. B. Jung, C. Li, On k-hyperexpansive operators, *J. Math. Anal. Appl.* **323** (2006), 569-582.

[28] C. Foias, I. B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators, *J. Funct. Anal.* **253** (2007), 628-646.

[29] C. Foias, I. B. Jung, E. Ko, C. Pearcy, On rank-one perturbations of normal operators, II, *Indiana Univ. Math. J.* **57** (2008), 2745-2760.

[30] B. Fuglede, The multidimensional moment problem, *Expo. Math.* **1** (1983), 47-65.

[31] J. Fujii, M. Fujii, H. Sasaoka, Y. Watatani, The spectrum of an infinite directed graph. *Math. Japon.* **36** (1991), 607-625.

[32] M. Fujii, H. Sasaoka, Y. Watatani, Adjacency operators of infinite directed graphs, *Math. Japon.* **34** (1989), 727-735.

[33] M. Fujii, H. Sasaoka, Y. Watatani, The numerical radius of an infinite directed graph, *Math. Japon.* **35** (1990), 577-582.

[34] T. Furuta, On the class of paranormal operators, *Proc. Japan. Acad.* **43** (1967), 594-598.

[35] T. Furuta, *Invitation to linear operators*, Taylor & Francis, Ltd., London, 2001.

[36] R. Gellar, L. J. Wallen, Subnormal weighted shifts and the Halmos-Bram criterion, *Proc. Japan Acad.* **46** (1970), 375-378.

[37] I. Gohberg, S. Goldberg, M. A. Kaashoek, *Classes of linear operators*, Vol. I, Operator Theory: Advances and Applications, **49**, Birkhaüser Verlag, Basel, 1990.

[38] S. Goldberg, *Unbounded linear operators. Theory and applications*, Reprint of the 1966 edition, Dover Publications, Inc., New York, 1985.

[39] P. Halmos, Normal dilations and extensions of operators, *Summa Bras. Math.* **2**(1950),124-134.

[40] P. R. Halmos, Ten problems in Hilbert space, *Bull. Amer. Math. Soc.* **76** (1970), 887-933.

[41] P. R. Halmos, *A Hilbert space problem book*, New York, Springer-Verlag, 1982.

[42] T. Hoover, I. B. Jung, A. Lambert, Moment sequences and backward extensions of subnormal weighted shifts, *J. Austral. Math. Soc.* **73** (2002), 27-36.

[43] E. J. Ionascu, Rank-one perturbations of diagonal operators, *Integral Equations Operator Theory* **39** (2001), 421-440.

[44] V. I. Istrăţescu, On some hyponormal operators, *Pacific J. Math.* **22** (1967), 413-417.

[45] T. Ito, T. K. Wong, Subnormality and quasinormality of Toeplitz operators, *Proc. Amer. Math. Soc.* **34** (1972), 157-164.

[46] Z. J. Jabłoński, Complete hyperexpansivity, subnormality and inverted boundedness conditions, *Integral Equations Operator Theory* **44** (2002), 316-336.

[47] Z. J. Jabłoński, Hyperexpansive composition operators, *Math. Proc. Camb. Phil. Soc.* **135** (2003), 513-526.

[48] Z. J. Jabłoński, I. B. Jung, J. Stochel, Backward extensions of hyperexpansive operators, *Studia Math.* **173** (2006), 233-257.

[49] Z. J. Jabłoński, J. Stochel, Unbounded 2-hyperexpansive operators, *Proc. Edinburgh Math. Soc.* **44** (2001), 613-629.

[50] J. Janas, *On unbounded hyponormal operators*, Ark. Mat. **27** (1989), 273–281.

[51] J. Janas, *On unbounded hyponormal operators.* II, Integr. Equat. Oper. Th. **15** (1992), 470–478.

[52] J. Janas, *On unbounded hyponormal operators.* III, Studia Math. **112** (1994), 75–82.

[53] W. B. Jones, W. J. Thron, H. Waadeland, A strong Stieltjes moment problem, *Trans. Amer. Math. Soc.* **261** (1980), 503-528.

[54] I. B. Jung, A. Lambert, J. Stochel, Backward extensions of subnormal operators, *Proc. Amer. Math. Soc.*, **132** (2004), 2291-2302.

[55] I. B. Jung, P. S. Lim, S. S. Park, Gaps of operators, *J. Math. Anal. Appl.* **304** (2005), 87-95.

[56] I. B. Jung, M. R. Lee, P. S. Lim, Gaps of operators, II, *Glasg. Math. J.* **47** (2005), 461-469.

[57] A. Lambert, Subnormality and weighted shifts, *J. London Math. Soc.* **14** (1976), 476-480.

[58] W. Mlak, The Schrödinger type couples related to weighted shifts, *Univ. Iagel. Acta Math.* **27** (1988), 297-301.

[59] W. Mlak, Note on unitary circular operators, *Univ. Iagel. Acta Math.* **27** (1988), 303-307.

[60] W. Mlak, Notes on circular operators I, II, III, IV, *Univ. Iagel. Acta Math.* **29** (1992), 145-152, 153-162, 163-170, 171-175.

[61] W. Mlak, M. Słociński, Quantum phase and circular operators, *Univ. Iagel. Acta Math.* **24** (1992), 133-144.

[62] B. Mohar, The spectrum of an infinite graph, *Linear Algebra Appl.* **48** (1982), 245-256.

[63] B. Mohar, W. Woess, A survey on spectra of infinite graphs, *Bull. London Math. Soc.* **21** (1989), 209-234.

[64] V. Müller, On the spectrum of an infinite graph, *Linear Algebra Appl.* **93** (1987), 187-189.

[65] O. Ore, *Theory of graphs*, American Mathematical Society Colloquium Publications, Vol. XXXVIII, American Mathematical Society, Providence, R.I. 1962.
[66] S. Ôta, K. Schmüdgen, *On some classes of unbounded operators*, Integr. Equat. Oper. Th. **12** (1989), 211–226.
[67] J. R. Ringrose, *Compact non-self-adjoint operators*, van Nostrad, Rein. Math. Stud. **35**, (1971).
[68] W. Rudin, *Real and Complex Analysis*, McGraw-Hill, New York 1987.
[69] R. Schatten, *Norm Ideals of completely continuous operators*, Berlin, Springer-Verlag, 1960.
[70] A. L. Shields, Weighted shift operators and analytic function theory, *Topics in operator theory*, pp. 49-128. Math. Surveys, No. 13, Amer. Math. Soc., Providence, R.I., 1974.
[71] J. A. Shohat, J. D. Tamarkin, *The problem of moments*, Math. Surveys **1**, Amer. Math. Soc., Providence, Rhode Island, 1943.
[72] V. M. Sholapurkar, ; A. Athavale, Completely and alternatingly hyperexpansive operators, *J. Operator Theory* **43** (2000), 43-68.
[73] B. Simon, Spectral analysis of rank one perturbations and applications, *Mathematical quantum theory. II, Schrödinger operators* (Vancouver, BC, 1993), 109–149, CRM Proc. Lecture Notes, 8, Amer. Math. Soc., Providence, RI, 1995.
[74] J. Stochel, An asymmetric Putnam-Fuglede theorem for unbounded operators, *Proc. Amer. Math. Soc.* **129** (2001), 2261-2271.
[75] J. Stochel and F. H. Szafraniec, On normal extensions of unbounded operators, II, *Acta Sci. Math.* (*Szeged*) **53** (1989), 153-177.
[76] M. H. Stone, *Linear transformations in Hilbert space and their applications to analysis*, Amer. Math. Soc. Colloq. Publ. 15, Amer. Math. Soc., Providence, R.I. **1932**.
[77] F. H. Szafraniec, Boundedness of the shift operator related to positive definite forms: an application to moment problems, *Ark. Mat.* **19** (1981), 251-259.
[78] F. H. Szafraniec, On extending backwards positive definite sequences, *Numer. Algorithms* **3** (1992), 419-426.
[79] M. Takamura, H. Takemoto, The numerical radius of an infinite directed regular graph, *Math. Japon.* **45** (1997), 337-343.
[80] J. Weidmann, *Linear operators in Hilbert spaces*, Springer-Verlag, Berlin, Heidelberg, New York, **1980**.
[81] D. Xia, *Spectral theory of hyponormal operators,* Birkhäuser Verlag, Basel, Boston, Stuttgart, 1983.

List of symbols

Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków

E-mail address: Zenon.Jablonski@im.uj.edu.pl

Department of Mathematics, College of Natural Sciences, Kyungpook National University, Daegu 702-701 Korea

E-mail address: ibjung@knu.ac.kr

Instytut Matematyki, Uniwersytet Jagielloński, ul. Łojasiewicza 6, PL-30348 Kraków

E-mail address: Jan.Stochel@im.uj.edu.pl