Division of distributions by locally definable quasianalytic functions

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Abstract

In this paper we demonstrate that the Łojasiewicz theorem on the division of distributions by analytic functions carries over to the case of division by quasianalytic functions locally definable in an arbitrary polynomially bounded, o-minimal structure which admits smooth cell decomposition. Hence, in particular, the principal ideal generated by a locally definable quasianalytic function is closed in the Fréchet space of smooth functions.

In his famous paper [9], S. Łojasiewicz solved the problem of the division of distributions by analytic functions, posed by L. Schwartz [18]. The first part of his paper was devoted to distributions $\mathcal{D}(U)$ and smooth functions $\mathcal{E}(U)$, i.e. of class C^∞ on U. In the theorem from Section 10, he achieved the division of distributions by a smooth function Φ, provided that its zero locus

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$Z := \{\Phi = 0\}$ admits a certain finite smooth stratification which enjoys some properties of growth and regular separation. In part two, he established an inequality for analytic functions, being a special case of that for subanalytic functions, now called the Łojasiewicz inequality. It was a crucial point in the proof of that every analytic function Φ fulfils the assumptions of the above theorem.

Let us emphasize that the regular separation of any two closed subanalytic subsets is a direct consequence of the Łojasiewicz inequality applied to the distance functions from those two sets. Sometimes the condition of regular separation itself is called Łojasiewicz's inequality too.

We wish to recall Łojasiewicz's theorem under study, and next to prove that its assumptions are fulfilled — after a suitable, generic, linear change of coordinates — by every quasianalytic function definable in a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Note that examples of such structures are the expansions of the real field by restricted quasianalytic functions (including some classical Denjoy–Carleman classes) which satisfy certain natural conditions (cf. [17, 13, 14]).

For a bounded open subset $U \subset \mathbb{R}^n$, let us introduce, after Łojasiewicz, the following notation:

$\mathcal{D}'(U)$, $\mathcal{E}'(U)$ and $\mathcal{P}'(U)$ stand for the spaces of distributions, distributions with compact supports and distributions prolongable onto $\mathbb{R}^n$, respectively.

It is easy to check, by means of a partition of unity, that the division problem is local. Therefore, if the answer to the division problem is affirmative for one of those spaces of distributions, it is so for the remaining two as well.

For a bounded open subset $\Omega \subset \mathbb{R}^k$, let $\mathcal{H}(\Omega)$ denote the set of smooth functions $f : \Omega \longrightarrow \mathbb{R}$ that satisfy the following growth condition:

$$\forall\, \alpha \in \mathbb{N}^k \;\; \exists\, M_\alpha,\, s_\alpha > 0 \quad \left| \frac{\partial^{|\alpha|} f}{\partial x^\alpha}(u) \right| \leq M_\alpha \operatorname{dist}(u, \partial\Omega)^{-s_\alpha} \quad \text{for all} \quad u \in \Omega.$$

Further, $\mathcal{G}(\Omega)$ denotes the set of smooth functions $f : \Omega \longrightarrow \mathbb{R}$ that satisfy the following growth condition:

$$\exists\, \varepsilon, s > 0 \quad |f(u)| > \varepsilon \operatorname{dist}(u, \partial\Omega)^s \quad \text{for all} \quad u \in \Omega.$$

Consider a bounded smooth leaf $\Gamma \subset \mathbb{R}^n$ of the form

$$\Gamma = \{(u, v) \in \Omega \times \mathbb{R}_v^{n-k} : v = \eta(u)\}, \tag{$*$}$$

where Ω is a bounded open subset of $\mathbb{R}^k_u$, $u = (x_1, \ldots, x_k)$, $v = (x_{k+1}, \ldots, x_n)$, $\eta(u) = (\eta_{k+1}(u), \ldots, \eta_n(u))$ and $\eta_i(u) \in \mathcal{H}(\Omega)$ for $i = k+1, \ldots, n$. We say that Γ satisfies condition (R) with respect to a subset $E \subset \mathbb{R}^n$ if the closures $\overline{\Gamma}$ and $\overline{E \setminus \Gamma}$ of Γ and $E \setminus \Gamma$, respectively, are regularly separated.

For the reader's convenience, we recall the definition of regular separation, due to Łojasiewicz (cf. [10], Sect. 18 or [9], Sect. 3). We say that two closed subsets $E, F \subset \mathbb{R}^n$ are regularly separated at a point $a \in E \cap F$, if there are a neighbourhood W of a and $\varepsilon, s > 0$ such that one of the following two equivalent inequalities holds:

$$\operatorname{dist}(x, E) \geq \varepsilon \operatorname{dist}(x, E \cap F)^s \quad \text{for all} \quad x \in F \cap W$$

or

$$\operatorname{dist}(x, E) + \operatorname{dist}(x, E) \geq \varepsilon \operatorname{dist}(x, E \cap F)^s \quad \text{for all} \quad x \in W.$$

The closed subsets E and F are called regularly separated, if they are so at all points $a \in E \cap F$.

Let us mention that condition (R) presented herein differs slightly from the one defined by Łojasiewicz in paper [9], Sect. 4. However, it is also suitable for his proof of the theorem under study (cf. [9], Sect. 10), which can be formulated as follows.

Theorem of Łojasiewicz. *Let U be a bounded open subset of $\mathbb{R}^n$, Φ a smooth function in a neighbourhood of the closure $\overline{U}$ and $Z := \{x \in U : \Phi(x) = 0\}$ be its zero locus. Suppose $\Phi \in \mathcal{G}(U \setminus Z)$ and Z admits a finite smooth stratification*

$$Z = \bigcup_{k=0}^{n-1} \bigcup_i \Gamma_i^k$$

such that each stratum Γ_i^k is a k-dimensional smooth leaf of the form (*) with appropriate functions

$$\eta_{k+1}^{k,i}, \ldots, \eta_n^{k,i} \in \mathcal{H}(\Omega_i^k), \quad \eta^{k,i} = (\eta_{k+1}^{k,i}, \ldots, \eta_n^{k,i}),$$

which satisfies condition (R) with respect to the set

$$\bigcup_{j=0}^{k} \bigcup_i \Gamma_i^j \cup \partial U \cup (\partial \Omega_i^k \times \mathbb{R}^{n-k}).$$

Further, assume that, for each stratum Γ_i^k, there is an integer $l = l_i^k$ such that

$$\Phi = \frac{\partial \Phi}{\partial x_n} = \frac{\partial^2 \Phi}{\partial x_n^2} = \ldots = \frac{\partial^{l-1} \Phi}{\partial x_n^{l-1}} = 0 \quad \text{on} \quad \Gamma_i^k,$$

and

$$\frac{\partial^l \Phi}{\partial x_n^l}(u, \eta^{k,i}(u)) \in \mathcal{G}(\Omega_i^k) \quad \left(\text{a fortiori}, \ \frac{\partial^l \Phi}{\partial x_n^l} \neq 0 \ \text{ on } \ \Gamma_i^k\right).$$

Then the mapping

$$\mathcal{P}'(U) \ni S \longrightarrow \Phi S \in \mathcal{P}'(U)$$

is surjective.

We now fix a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition.

Proposition. *If $f : U \longrightarrow \mathbb{R}$ is a smooth definable function on a bounded open subset $U \subset \mathbb{R}^n$, then*

$$f \in \mathcal{H}(U) \quad \text{and} \quad f \in \mathcal{G}(U \setminus Z(f)),$$

where $Z(f)$ is the zero locus of f.

For the first property, it is sufficient to show that every definable function $g : U \longrightarrow \mathbb{R}$ satisfies the following growth condition:

$$\exists\, M,\, s > 0 \quad g(x) \leq M \operatorname{dist}(x, \partial U)^{-s} \quad \text{for all} \quad x \in U.$$

For any $t > 0$, put

$$\theta(t) := \begin{cases} 0 & \text{if } U \cap \{x : \operatorname{dist}(x, \partial U) = t\} = \emptyset \\ \max\{|g(x)| : \operatorname{dist}(x, \partial U) = t\} & \text{otherwise.} \end{cases}$$

Since the structure $\mathcal{R}$ is polynomially bounded, there are $M,\, s > 0$ such that $\theta(t) \leq M\, t^{-s}$ for all $t > 0$, as desired.

In order to prove the second property, for any $t > 0$, put

$$\zeta(t) := \min\{|f(x)| : x \in U, \ \operatorname{dist}(x, \partial U \cup Z(f)) = t\} > 0.$$

Again, by polynomial boundedness, there are $\varepsilon,\, s > 0$ such that $\zeta(t) > \varepsilon\, t^s$ for all $t > 0$, which completes the proof.

The Łojasiewicz inequality (cf. [10], Sect. 18, [5], Sect. 4.14 or [3], Theorem 6.2) yields immediately the regular separation of any two closed definable subsets. Consider a smooth function Φ definable in a neighbourhood of the closure $\overline{U}$ of a bounded open definable subset $U \subset \mathbb{R}^n$. Then, in view of the foregoing proposition, the assumptions of the theorem in question will be fulfilled for Φ, once we find a finite smooth definable stratification

$$Z = \bigcup_{k=0}^{n-1} \bigcup_{i} \Gamma_i^k$$

such that each stratum Γ_i^k is a k-dimensional smooth leaf of the form (*) with appropriate functions

$$\eta_{k+1}^{k,i}, \dots, \eta_n^{k,i} \in \mathcal{H}(\Omega_i^k), \quad \eta^{k,i} = (\eta_{k+1}^{k,i}, \dots, \eta_n^{k,i}),$$

for which there is an integer $l = l_i^k$ such that

$$\Phi = \frac{\partial \Phi}{\partial x_n} = \frac{\partial^2 \Phi}{\partial x_n^2} = \dots = \frac{\partial^{l-1} \Phi}{\partial x_n^{l-1}} = 0 \quad \text{on} \quad \Gamma_i^k,$$

and

$$\frac{\partial^l \Phi}{\partial x_n^l}(u, \eta^{k,i}(u)) \in \mathcal{G}(\Omega_i^k) \quad \text{(a fortiori, } \frac{\partial^l \Phi}{\partial x_n^l} \neq 0 \text{ on } \Gamma_i^k\text{)}.$$

Our procedure will be to construct a stratification just described. We still need the lemma below, being a generalization of the classical lemma of Koopman–Brown (cf. [8], [10], Sect. 22 or [5], Sect. 4.9).

Good Directions Lemma. *Let E be a definable subset of $\mathbb{R}^n$ of dimension $\leq (n-1)$. Then for a generic line $\lambda \in \mathbb{P}_{n-1}$, i.e. for every line λ outside a nowhere dense, definable subset of the projective space $\mathbb{P}_{n-1}$, we have*

$$\sharp\, (E \cap (a + \lambda)) < \infty \quad \textit{for all} \quad a \in \mathbb{R}^n.$$

As a direct consequence, we obtain the

Corollary. *Let E be a definable subset of $\mathbb{R}^n$ of dimension $\leq (n-k)$. Then for a generic k-dimensional vector subspace $V \in \mathbb{G}_{n,k}$, i.e. for every subspace V outside a nowhere dense, definable subset of the Grassmannian $\mathbb{G}_{n,k}$, we have*

$$\sharp\, (E \cap (a + V)) < \infty \quad \textit{for all} \quad a \in \mathbb{R}^n.$$

Now, we shall demonstrate how to construct a required stratification. First, consider the decreasing sequence of quasianalytic subsets

$$Z := \{x \in U : \Phi(x) = 0\} \supset Z_1 := \{x \in U : \Phi(x) = \frac{\partial \Phi}{\partial x_n} = 0\} \supset \ldots$$

$$\ldots \supset Z_j := \{x \in U : \Phi(x) = \frac{\partial \Phi}{\partial x_n} = \ldots = \frac{\partial^j \Phi}{\partial x_n^j} = 0\} \supset \ldots$$

By topological noetherianity (cf. [5], Sect. 4.17, [3], Theorem 6.1 or [12], Appendix), this sequence stabilizes:

$$Z_l = Z_{l+1} = Z_{l+2} = \ldots \quad \text{for an} \quad l \in \mathbb{N}.$$

Next, choose a line $\lambda_1 \in \mathbb{P}_{n-1}$ according to the above lemma applied to the zero locus $Z := \{\Phi = 0\}$; we may assume that $\lambda_1 = \mathbb{R} \cdot e_n$. Take a smooth cell decomposition $\mathcal{C}$ of $\mathbb{R}^n$ compatible with the sets $Z, Z_1, \ldots, Z_l$. The cells of dimension $(n-1)$ are, of course, smooth definable leaves of the form (*). Project the cells of dimension $< (n-1)$ onto $\mathbb{R}^{n-1}$ in parallel to $\mathbb{R} \cdot e_n$, again choose a line $\lambda_2 \in \mathbb{P}_{n-2}$ according to the above lemma applied to those projections, and take $\lambda_2 = \mathbb{R} \cdot e_{n-1}$. Consequently, the plane $\mathbb{R} \cdot e_n + \mathbb{R} \cdot e_{n-1}$ satisfies the conclusion of the foregoing corollary applied to the initial cells of dimension $(n-2)$ in $\mathbb{R}^n$.

Further, refine the cell decomposition of $\mathbb{R}^{n-2}$ induced by $\mathcal{C}$ so that the cells of dimension $(n-1)$ and $(n-2)$ are smooth definable leaves of the form (*). We continue in this fashion, and eventually attain, after performing a linear change of coordinates, a new, finer, smooth definable cell decomposition such that every cell C is a smooth definable leaf of the form (*).

Finally, we must take a smooth definable stratification compatible with that new cell decomposition. We are thus led to the following quasianalytic generalization of the Łojasiewicz division theorem.

Division Theorem. *Consider a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Let U be a connected open subset of $\mathbb{R}^n$ and $\Phi : U \longrightarrow \mathbb{R}$ a smooth, non-vanishing function locally definable on U with respect to the structure $\mathcal{R}$. Then the mapping*

$$\mathcal{D}'(U) \ni S \longrightarrow \Phi \cdot S \in \mathcal{D}'(U)$$

is surjective.

Hence and by routine arguments from functional analysis, we obtain

Corollary. *Let $\Phi : U \longrightarrow \mathbb{R}$ be a function as in the above theorem. Then $\Phi \cdot \mathcal{E}(U)$ is a closed ideal of the Fréchet algebra $\mathcal{E}(U)$ of smooth functions on U, and the linear mapping*

$$\varphi : \mathcal{E}(U) \ni f \longrightarrow \Phi f \in \Phi \cdot \mathcal{E}(U)$$

is a homeomorphism.

Our proof starts with the observation that, by the Banach open mapping theorem, the ideal $\Phi \cdot \mathcal{E}(U)$ is closed iff the mapping φ is a homeomorphism. Suppose, on the contrary, that φ is not a homeomorphism. Then there is a sequence $(f_\nu) \subset \mathcal{E}(U)$ such that $\Phi f_\nu \to 0$ and $f_\nu \not\to 0$. We may, of course, assume that all the functions f_ν lie outside a neighbourhood W of $0 \in \mathcal{E}(U)$.

For any distribution $T \in \mathcal{E}'(U)$, take a distribution $S \in \mathcal{E}'(U)$ for which $T = \Phi \cdot S$. Then

$$T(f_\nu) = (\Phi \cdot S)(f_\nu) = S(\Phi f_\nu) \to 0,$$

and thus the numerical sequence $(T(f_\nu))$ is bounded. The sequence (f_ν) is therefore bounded in $\mathcal{E}(U)$. Indeed, any subset F of a locally convex topological vector space is bounded iff it is weakly bounded (see e.g. [7], Chap. 6, §8, Theorem 4'). This fact relies on two fundamental results from functional analysis, namely, the Banach–Steinhaus and Hahn–Banach theorems.

Since $\mathcal{E}(U)$ is a Montel space (i.e. it has the Heine–Borel property), the set $\{f_\nu\}$ is relatively compact. Consequently, we can take a subsequence (f_{ν_k}) convergent to an element $f \in \mathcal{E}(U)$. But we must have $f \neq 0$, whence

$$\Phi f_{\nu_k} \to \Phi f \neq 0.$$

This contradicts our assumption, concluding the proof.

Remarks. 1) It follows immediately from the Hahn–Banach theorem that also valid is the converse implication:

If the linear mapping

$$\varphi : \mathcal{E}(U) \ni f \longrightarrow \Phi f \in \Phi \cdot \mathcal{E}(U)$$

is a homeomorphism, then the mapping

$$\mathcal{D}'(U) \ni S \longrightarrow \Phi \cdot S \in \mathcal{D}'(U)$$

is surjective.

2) B. Malgrange generalized the foregoing theorem on closed ideals to the case of ideals generated locally by finitely many analytic functions (cf. [11], Chap. 6, Theorem 1.1). This strengthening seems not to carry over easily to the quasianalytic settings for lack of theorems about good algebraic properties of quasianalytic local rings (such as, for instance, noetherianity, flatness properties or coherence). Actually, the problem whether quasianalytic local rings are noetherian remains open as yet.

3) The above corollary is tantamount to the quasianalytic division theorem to the effect that a smooth function f formally divisible by a quasianalytic function Φ is divisible by Φ. The latter was established by Bierstone–Milman [3], Section 6, who followed Atiyah's proof of the analytic division theorem [1], based on transformation to normal crossings by blowing up.

4) The above results related to division by a quasianalytic function will be applied in our further research (e.g. [15, 16]) on carrying the issues linked with Glaeser's composite function theorem (cf. [6, 4, 2]) over to the quasianalytic settings.

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