A note to Bierstone–Milman–Pawłucki's paper "Composite differentiable functions"

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Abstract

We demonstrate that the composite function theorems of Bierstone–Milman–Pawłucki and Glaeser carry over to any polynomially bounded, o-minimal structure which admits smooth cell decomposition. Also, the assumptions of the o-minimal versions presented here have been considerably relaxed.

Due to the classical theorem of Glaeser [7], every semiproper real analytic mapping φ which is generically a submersion enjoys the composite function property. Bierstone–Milman–Pawłucki [3] introduced a new point of view towards Glaeser's theorem by considering a $\mathcal{C}^k$ composite function property, k being a positive integer, which is fulfilled by every semiproper real analytic mapping. In this paper we demonstrate that the theorem of Bierstone–Milman–Pawłucki and thence that of Glaeser carry over to the case of smooth mappings definable in any polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Moreover, we considerably relax

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the assumptions of the o-minimal versions of those theorems presented in this paper.

Before formulating an o-minimal version of the main theorem from [3], we recall the indispensable terminology. Let $M \subset \mathbb{R}^p$ be a smooth submanifold, $A \subset M$ a locally closed subset and B a closed subset of A. For any $k \in \mathbb{N} \cup \{\infty\}$, $\mathcal{C}^k(A; B)$ denotes the Fréchet algebra of restrictions to A of real functions of class $\mathcal{C}^k$ in the vicinity of A that are k-flat on B. Let $\varphi : M \longrightarrow N$ be a smooth mapping between two real smooth manifolds with closed image $T := \varphi(M) \subset N$. Take a closed subset Z of T and an integer $l \in \mathbb{N}$. Then $(\varphi^* \mathcal{C}^l(T))^\wedge$ denotes the subalgebra of all functions $f \in \mathcal{C}^l(M)$ that are formally composites with φ (i.e., for each $b \in T$, there is $g \in \mathcal{C}^l(T)$ such that the function $f - \varphi^*(g)$ is l-flat on the fibre $\varphi^{-1}(b)$). Put

$$(\varphi^* \mathcal{C}^l(T; Z))^\wedge := (\varphi^* \mathcal{C}^l(T))^\wedge \cap \mathcal{C}^l(M, \varphi^{-1}(Z)).$$

Main Theorem. *Consider a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Let $M \subset \mathbb{R}^p$ and $N \subset \mathbb{R}^q$ be smooth definable submanifolds, $\varphi : M \longrightarrow N$ a smooth definable mapping with closed image $T := \varphi(M) \subset N$, and Z be a closed definable subset of T. Then, for each $k \in \mathbb{N}$, there is an integer $l = l(k) \geq k$ such that*

$$(\varphi^* \mathcal{C}^l(T; Z))^\wedge \subset \varphi^* \mathcal{C}^k(T; Z).$$

Remark. Let us emphasize that the assumptions in the o-minimal version of the $\mathcal{C}^k$ composite function theorem have been considerably reduced in comparison with the classical version. The mapping φ under study does not need to be semiproper, but with closed image $T := \varphi(M) \subset N$ instead. Further, we do not require the image $T := \varphi(M)$ be compact.

The above result along with Whitney's extension theorem immediately yields an o-minimal version of Glaeser's composite function theorem:

Composite Function Theorem. *Under the assumptions of the main theorem, suppose $\varphi : M \longrightarrow N$ is a smooth definable mapping with closed image which is generically a submersion. Then*

$$(\varphi^* \mathcal{C}^\infty(T))^\wedge = \varphi^* \mathcal{C}^\infty(T).$$

Yet another direct consequence of the main theorem is the

Corollary. *We have the equality*

$$(\varphi^* \mathcal{C}^\infty(T;Z))^\wedge = \varphi^* \mathcal{C}^{(\infty)}(T;Z) \quad \textit{where} \quad \mathcal{C}^{(\infty)}(T;Z) := \bigcap_{k\in\mathbb{N}} \mathcal{C}^k(T;Z).$$

In particular, the mapping φ satisfies the composite function property iff $\mathcal{C}^{(\infty)}(T;Z) = \mathcal{C}^\infty(T;Z)$; the latter condition depends only on its image T.

We assume complete familiarity with the paper by Bierstone–Milman–Pawłucki. As we observed after reading [3], the specific arguments from subanalytic geometry involved by them include the following issues: the Łojasiewicz inequality, Whitney regularity and stratifications of subanalytic mappings. By inspection of their proofs, one can conclude that their approach adapts to an o-minimal structure $\mathcal{R}$ whenever the above three issues are valid for $\mathcal{R}$.

Let $\mathcal{R}$ be an o-minimal expansion of the real field. It is well known that the foregoing first two issues hold in the structure $\mathcal{R}$ whenever $\mathcal{R}$ is polynomially bounded (see e.g. [1], Section 6, for the subanalytic version and [6], Section 4, for the o-minimal one). As for the third, a basic tool applied in [3] is the theorem on stratification of a proper continuous subanalytic mapping. Below we state the theorems on trivialization and stratification of a continuous mapping definable in the structure $\mathcal{R}$, which are valid provided that $\mathcal{R}$ admits smooth cell decomposition. In the o-minimal case, however, the assumption of properness is superfluous.

Trivialization Theorem. *Suppose the structure $\mathcal{R}$ admits smooth cell decomposition. Let $S \subset \mathbb{R}^p$ be a definable subset, $\varphi : S \longrightarrow \mathbb{R}^q$ a continuous definable mapping and $T := \varphi(S)$. Further, let $\mathcal{V} = (V_i)$ and $\mathcal{W} = (W_j)$ be finite families of definable subsets of S and T, respectively. Then there exist finite smooth definable cell decompositions $\mathcal{S}$ and $\mathcal{T}$ of S and T, respectively, which enjoy the following properties:*

i) $\varphi(C) \in \mathcal{T}$ for every $C \in \mathcal{T}$;

ii) $\mathcal{S}$ and $\mathcal{T}$ are compatible with $\mathcal{V}$ and $\mathcal{W}$, respectively;

iii) for each cell $D \in \mathcal{T}$, one can find a definable set F and a definable homeomorphism

$$\theta : D \times F \longrightarrow \varphi^{-1}(D)$$

such that $\varphi \circ \theta$ is the canonical projection (obviously, F is homeomorphic to the fibre $\varphi^{-1}(b)$ for every $b \in D$); moreover, there is a finite smooth definable *cell decomposition $\mathcal{F} = (F_k)$ of F such that for each k*

$$\theta(D \times F_k) =: C \in \mathcal{S} \quad \text{and} \quad \operatorname{res}\theta : D \times F_k \longrightarrow C$$

is a diffeomorphism of $D \times F_k$ onto C.

We immediately obtain the

Corollary. (Stratification of a definable mapping) *Under the above assumptions, there exist finite smooth definable stratification $\mathcal{S}$ and $\mathcal{T}$ of S and T, respectively, which enjoy the following properties:*

i) $\varphi(C) \in \mathcal{T}$ for every $C \in \mathcal{T}$;

ii) $\mathcal{S}$ and $\mathcal{T}$ are compatible with $\mathcal{V}$ and $\mathcal{W}$, respectively;

iii) for each cell $D \in \mathcal{T}$, one can find a definable set F and a definable homeomorphism

$$\theta : D \times F \longrightarrow \varphi^{-1}(D)$$

such that $\varphi \circ \theta$ is the canonical projection (obviously, F is homeomorphic to the fibre $\varphi^{-1}(b)$ for every $b \in D$); moreover, there is a finite smooth definable stratification $\mathcal{F} = (F_k)$ of F such that for each k

$$\theta(D \times F_k) =: C \in \mathcal{S} \quad \text{and} \quad \operatorname{res}\theta : D \times F_k \longrightarrow C$$

is a diffeomorphism of $D \times F_k$ onto C.

A pair $(\mathcal{S}, \mathcal{T})$ of definable stratifications as in the above corollary shall be called a stratification of the mapping φ. If we require that $\mathcal{S}$ be only a finite partition into definable leaves (i.e. connected definable subsets of $\mathbb{R}^p$ which are smooth submanifolds), we call the pair $(\mathcal{S}, \mathcal{T})$ a semistratification of φ.

The foregoing trivialization theorem is a strengthening of the classical trivialization, and can be established by combining the latter (cf. [8, 9], [4], Chap. 9, [5], Chap. 9) with the technique of cell decompositions. More precisely, the proof runs via the classical trivialization along with the following two elementary lemmas. We leave the detailed verification to the reader.

Lemma 1. *If $\theta : A \longrightarrow E$ is a definable homeomorphism, then there are finite smooth cell decompositions $\mathcal{A} = (A_i)$ and $\mathcal{E} = (E_i)$ of A and E, respectively, such that each restriction $\operatorname{res}\theta : A_i \longrightarrow E_i$ is a smooth diffeomorphism.*

Lemma 2. *Let $\mathcal{C} = (C_k)$ be a finite smooth cell decomposition of $A = B \times F$ for which all cells C_k lie over the cell B. Then there exist a smooth cell decomposition $\mathcal{F} = (F_k)$ of F and a definable homeomorphism $h : B \times F \longrightarrow B \times F$ such that for each k*

$$h(C_k) = B \times F_k \quad \textit{and} \quad \operatorname{res} h : C_k \longrightarrow B \times F_j$$

is a smooth diffeomorphism.

Finally, we return to the main result of this paper. It is an o-minimal version of the classical $\mathcal{C}^k$ composite function theorem, namely, Theorem 1.2 reduced to Proposition 6.1 from [3]. The reason why we are able to drop the assumption that the mapping φ under study is semiproper and only to require the image $T := \varphi(M)$ be closed is that the integer $l(k)$ constructed in [3] depends in a certain manner merely on some exponents in the Łojasiewicz inequalities and Whitney regularity conditions which occur in the proof. We shall outline how to achieve this result as follows.

Since every definable bounded open subset of $\mathbb{R}^n$ is a finite union of open cells (cf. [15]), it is not difficult to check that a definable submanifold $N \subset \mathbb{R}^q$ of dimension n is a finite union of definable open subsets each of which is diffeomorphic to $\mathbb{R}^n$. Therefore, without loss of generality, we may assume that $N = \mathbb{R}^n$. Now, in view of the following two crucial observations, the proofs of Theorem 1.2 and Proposition 6.1 from [3] can be repeated almost verbatim.

Observation 1. (Łojasiewicz Inequality) *Let $f, g : A \longrightarrow \mathbb{R}$ be two continuous definable functions on a locally closed subset A of $\mathbb{R}^n$ such that*

$$\{x \in A : f(x) = 0\} \subset \{x \in A : g(x) = 0\}.$$

Then there exists a common Łojasiewicz exponent $s > 0$ for all compact definable subsets K of A. More precisely, there exists an exponent $s > 0$ such that, for each compact definable subset K of A, there is a constant $C = C(K) > 0$ such that

$$|g(x)| \leq C\, |f(x)|^s \quad \text{for all} \;\; x \in K.$$

Observation 2. (Whitney Regularity) *Let A be a connected closed definable subset of $\mathbb{R}^n$. Then one can find an exhaustion of A by connected* *compact definable subsets* A_ν *of* $\mathbb{R}^n$, $\nu \in \mathbb{N}$, *i.e.*

$$A = \bigcup_{\nu=1}^{\infty} A_\nu \quad \textit{and} \quad A_1 \subset A_2 \subset A_3 \subset \dots,$$

and a positive integer r *with the following property. For each* A_ν, $\nu \in \mathbb{N}$, *there is a constant* $C = C(A_\nu) > 0$ *such that any two points* $x, y \in A_\nu$ *can be joined by a rectifiable (definable) curve* γ *in* A_ν *of length*

$$|\gamma| \leq C \, \|x - y\|^{1/r}.$$

The former is a special case of the Łojasiewicz inequality with parameter, recalled below:

Łojasiewicz Inequality with Parameter. (cf. [10] and [11], Section 1) *Consider two definable functions* $f, g : A \longrightarrow \mathbb{R}$ *on a set* $A \subset \mathbb{R}^m_u \times \mathbb{R}^n_x$. *Assume that all sections* $A_u := \{x \in \mathbb{R}^n : (u, x) \in A\}$, $u \in \mathbb{R}^m$, *are compact and that all functions*

$$f_u, g_u : A_u \longrightarrow \mathbb{R}, \quad f_u(x) := f(u, x), \quad g_u(x) := g(u, x),$$

are continuous. If $\{f = 0\} \subset \{g = 0\}$, *then there exist an exponent* $s > 0$ *and a definable function* $c : \mathbb{R}^m \longrightarrow (0, \infty)$ *such that*

$$|g(u, x)| \leq c(u) |f(u, x)|^s \quad \text{for all} \quad (u, x) \in A.$$

As an immediate consequence we obtain the following

Hölder Continuity with Parameter. *Under the above assumptions, there exist a positive integer* r *and a definable function* $c : \mathbb{R}^m \longrightarrow (0, \infty)$ *such that*

$$|f(u, x) - f(u, y)| \leq c(u) \|x - y\|^{1/r} \quad \text{for all} \quad (u, x) \in A.$$

The above Hölder continuity, along with the cell triangulation in the sense of Shiota (cf. [14], Chap. II, Theorem II.4.2), can be applied to the proof of Observation 2, outlined below. By a cell complex we mean — after Shiota (*op. cit.*, Chap. I, § 3) — a finite family of semilinear, possibly unbounded cells, which satisfies conditions corresponding to the standard ones for a simplicial complex.

Let (K, τ) be a cell triangulation of the closed definable subset A of $\mathbb{R}^n$; here K is a cell complex and $\tau : |K| \longrightarrow A$ is a definable homeomorphism. It is not difficult to find a constance $C > 0$ and a semilinear mapping

$$\omega : |K| \times |K| \times [0,1] \longrightarrow |K|$$

such that for all points $c, d \in |K|$ the mapping

$$\omega_{c,d} : [0,1] \longrightarrow |K|, \quad \omega_{c,d}(t) := \omega(c, d, t),$$

is an arc from c to d of length $\leq C\|c - d\|$. Define a mapping

$$\gamma : A \times A \times [0,1] \longrightarrow A, \quad \gamma(a, b, t) := \tau(\omega(\tau^{-1}(a), \tau^{-1}(b), t)).$$

Then for all points $a, b \in A$ the mapping

$$\gamma_{a,b} : [0,1] \longrightarrow A, \quad \gamma_{a,b}(t) := \gamma(a, b, t),$$

is an arc from a to b. We may regard the mapping γ as a definable family of curves parametrized by the set $A \times A$. Now, the assertion follows from the Hölder continuity with parameter.

Remarks. 1) Bierstone and Milman ([2], Theorem 1.13) established the equivalence of the composite function property with many other natural, metric, differential and algebro-geometric properties of a closed subanalytic set, including semicoherence, uniform Chevalley estimate as well as the semicontinuity of the Hironaka diagram of initial exponents and that of Hilbert–Samuel function.

2) The quasianalytic version of the composite function theorem plays a crucial role in our subsequent papers [12, 13], which are concerned with quasianalytic multi-parameter perturbation theory and a quasianalytic version of Tamm's theorem on the singular locus of a definable set, respectively.

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