On the singular locus of sets definable in a quasianalytic structure

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Abstract

Given a quasianalytic structure, we prove that the singular locus of a quasi-subanalytic set E is a closed quasi-subanalytic subset of E. We rely on some stabilization effects linked to Gateaux differentiability and formally composite functions. An essential ingredient of the proof is a quasianalytic version of Glaeser's composite function theorem, presented in our previous paper.

1. Introduction. We are concerned with a quasianalytic structure $\mathcal{R}$ on the real field with restricted quasianalytic functions. The sets definable (with parameters) in the structure $\mathcal{R}$ are precisely the subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, that are globally quasi-subanalytic (including infinity). We say that $a \in E$

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is a smooth point of a subset E of $\mathbb{R}^n$ if E is near a a smooth (i.e. of class $\mathcal{C}^\infty$) submanifold of $\mathbb{R}^n$. Denote by $\operatorname{Sing} E$ the set of all $a \in E$ which are not smooth points of E. The main purpose of this paper is to prove the following theorems:

Theorem 1. *If $E \subset \mathbb{R}^n$ is a definable subset, then the singular locus* $\operatorname{Sing} E$ *is a closed definable subset of E.*

Theorem 2. *Let U be an open subset of $\mathbb{R}^n$ and $f : U \longrightarrow \mathbb{R}$ a continuous definable function. Denote by* $\operatorname{Reg}(f) = \mathcal{C}_\infty(f)$ *the set of those points $a \in U$ at which f is a smooth function. Then* $\operatorname{Reg}(f)$ *is an open definable subset of U.*

The above results are quasianalytic generalizations of those of Tamm [24] for subanalytic sets and functions. As in the subanalytic case, the former comes down to the latter (cf. [10], Sect. 7, or [1], Sect. 7), via an argument due to Poly–Raby [21]. Several proofs of the subanalytic versions have been provided by Tamm [24], Bierstone–Milman [1] and Kurdyka [10]; see also paper [4] by van den Dries–Miller. However, they cannot be applied to the above theorems in the case of quasianalytic settings, as explained below.

The proofs by Tamm and by Bierstone–Milman rely on Malgrange's idea of "graphic points", and more precisely, on the fact that, for analytic functions, a formally graphic point is graphic (cf. [12]). This implication may be seen as a corollary, in a very particular case, to Gabrielov's rank theorem [8], which was also proved by Eakin–Harris [6]. Quasianalytic geometry has no such theorem at its disposal. It follows from a current paper by Elkhadiri [7], who proved that every quasianalytic system with the Eakin–Harris property is a subsystem of that of convergent power series.

Kurdyka [10], in turn, uses the effect of stabilization of the descending sequence of subsets $\mathcal{G}_k(f)$ where a given subanalytic function f has the j-th Gateaux differentials for all $j \le k$, and the Bochnak–Siciak analyticity theorem. Unfortunately, the latter is unavailable in the quasianalytic settings.

Our approach is based on some stabilization effects linked to Gateaux differentiability and to formally composite functions, which are presented in Sections 2 and 3, respectively. Section 4 provides a proof of Theorem 2. Its essential ingredient is a quasianalytic version of Glaeser's composite function theorem, presented in our paper [19].

We now recall the precise definitions. As in our previous papers [16, 17, 18], fix a quasianalytic system $\mathcal{Q} = (\mathcal{Q}_n)_{n\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^n$, fulfilling conditions 1–6 below. For each open subset $U \subset \mathbb{R}^n$, $\mathcal{Q}(U) = \mathcal{Q}_n(U)$ is thus a subalgebra of the algebra $\mathcal{C}^\infty(U)$ of real smooth functions on U. By a Q-analytic function (or a Q-function, for short), we mean any function $f \in \mathcal{Q}(U)$. Similarly,

$$f = (f_1, \dots, f_k) : U \longrightarrow \mathbb{R}^k$$

is called Q-analytic (or a Q-mapping) if so are its components $f_1, \dots, f_k$. The following six conditions are imposed:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;
2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);
3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \longrightarrow V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^n$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \longrightarrow U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;
4. $\mathcal{Q}$ is closed under differentiation;
5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_n) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;
6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f_a} =$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

Q-mappings give rise, in the ordinary manner, to the category Q of Q-manifolds and Q-mappings, which is a subcategory of that of smooth manifolds and smooth mappings. Similarly, Q-analytic, Q-semianalytic and Q-subanalytic sets can be defined.

These conditions ensure some resolution of singularities in the category Q, including transformation to normal crossings by blowing up (cf. [2, 22]), whereon the geometry of quasianalytic structures relies (especially, in the absence of their good algebraic properties).

The examples of such categories (*op. cit.*) come from quasianalytic Denjoy–Carleman classes Q_M, where $M = (M_j)_{j\in\mathbb{N}}$ are logarithmically convex sequences. The class Q_M consists of smooth functions $f(x) = f(x_1, \ldots, x_n)$ in n variables, $n \in \mathbb{N}$, which satisfy locally the following growth condition

$$|\partial^{|\alpha|}/\partial x^{\alpha}(x)| \leq C \cdot R^{|\alpha|} \cdot |\alpha|! \cdot M_{|\alpha|} \quad \text{for all} \quad \alpha \in \mathbb{N}^n,$$

with some constants $C, R > 0$ depending only on the vicinity of a given point. Obviously, the class Q_M contains the real analytic functions. It is quasianalytic iff

$$\sum_{j=0}^{\infty} \frac{M_j}{(j+1)M_{j+1}} = \infty$$

(the Denjoy–Carleman theorem). It is closed under composition (Roumieu [23]), under inverse (Komatsu [9]), and is closed under differentiation and under division by a coordinate iff

$$\sup_j \sqrt[j]{\left(\frac{M_{j+1}}{M_j}\right)} < \infty$$

(cf. [13, 25]). On the other hand, every polynomially bounded, o-minimal structure $\mathcal{R}$ determines a quasianalytic system of sheaves of germs of smooth functions that are locally definable in $\mathcal{R}$ (cf. [14] for the quasianalyticity of these sheaves).

Denote by $\mathcal{R} = \mathcal{R}_Q$ the expansion of the real field $\mathbb{R}$ by restricted Q-analytic functions, i.e. functions of the form

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^n \\ 0, & \text{otherwise} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^n$. The structure $\mathcal{R}_Q$ is model complete and o-minimal (cf. [22, 16]). The definable subsets in $\mathcal{R}_Q$ coincide with those subsets of $\mathbb{R}^n$, $n \in \mathbb{N}$, that are Q-subanalytic in a semialgebraic compactification of $\mathbb{R}^n$. From now on, the word "definable" means "definable with parameters" in the structure $\mathcal{R}$.

2. Gateaux differentiability. This section is devoted to a study of stabilization properties linked to Gateaux differentiability. For the sake of simplicity, however, it is more suitable to weaken slightly the concept of Gateaux differentials by taking into account only the right-hand side directional derivatives. In the sequel, we shall consider formal power series built from the (weak) k-th Gateaux differentials, $k \in \mathbb{N}$, which will play a key role in our approach to apply formally composite functions.

For a definable function $f : U \longrightarrow \mathbb{R}$ on an open subset $U \subset \mathbb{R}^n$, consider definable functions

$$F : U \times \mathbb{R}^n_u \times \mathbb{R}_t \longrightarrow \mathbb{R}, \quad F(x, u, t) := f(x + tu),$$

and $\delta_k : U \times \mathbb{R}^n_u \longrightarrow \mathbb{R}$ by putting

$$\delta_k(x, u) := \begin{cases} \partial^k F/\partial t^k (x, u, 0^+), & \text{if this derivative exists for all } u \in \mathbb{R}^n; \\ 1, & \text{otherwise;} \end{cases}$$

here $\partial^k F/\partial t^k (x, u, 0^+)$ denotes the k-th right-hand side derivative at $t = 0$.

If $\delta_k(x, u)$ is a homogeneous polynomial of degree k in the variables u, this polynomial

$$\delta^k_x f(u) := \delta_k(x, u) \in \mathcal{P}_k$$

is called the (weak) k-th Gateaux differential of f at x; here, $\mathcal{P}_k$ stands for the real vector space of homogeneous polynomials of degree k in $\mathbb{R}^n_u$. We say then that f is (weakly) k times Gateaux differentiable at x.

For $k \in \mathbb{N}$, let $\mathcal{G}_k\,(f)$ be the set of those points x at which f has (weak) j-th Gateaux differentials $\delta^j_x f$ for all $j \le k$; let $\mathcal{G}_\infty\,(f)$ denote the set of those points x at which f has (weak) j-th Gateaux differentials $\delta^j_x f$ for all $j \in \mathbb{N}$. The main result of this section is the following

Theorem 3. (on Gateaux Differentiability) *If $f : U \longrightarrow \mathbb{R}$ is a definable function on an open subset $U \subset \mathbb{R}^n$, then*

- *the sets $\mathcal{G}_k\,(f))$, $k \in \mathbb{N}$, are definable;*
- *the descending sequence $(\mathcal{G}_k\,(f))_{k \in \mathbb{N}}$ stabilizes, whence $\mathcal{G}_\infty\,(f) = \mathcal{G}_k\,(f)$ for $k \in \mathbb{N}$ large enough;*
- *furthermore, there exists a finite, definable, Q-analytic cell decomposition $\mathcal{B}$ of $\mathcal{G}_\infty\,(f)$ such that the restriction to each cell $B \in \mathcal{B}$ of every (weak) Gateaux differential*

$$\delta^k f : B \ni x \longrightarrow \delta^k_x f \in \mathcal{P}_k$$

is a definable Q-analytic mapping.

The above is a quasianalytic counterpart of a theorem by Kurdyka [10] on subanalytic functions, strengthened by adding the conclusion about the existence of a definable, Q-analytic cell decomposition $\mathcal{B}$ on each cell of which all the Gateaux differentials under study are Q-analytic. This strengthening allows us to achieve (in Section 3) some stabilization effects linked to formally composite functions. An important role in the proof of Theorem 3 is played by a quasianalytic version of Puiseux's theorem with parameter, presented below (cf. [20] for a classical analytic version).

Puiseux's Theorem with Parameter. *Let $E \subset \mathbb{R}^n_x$ be a definable subset and*

$$f : E \times (0,1) \longrightarrow \mathbb{R}$$

be a definable function. Then one can find a definable cell decomposition of E into finitely many Q-analytic cells $E_1, \dots, E_s$ and open definable neighbourhoods Ω_i of $E_i \times \{0\}$ in $E_i \times \mathbb{R}_t$, $i = 1, \dots, s$, for which

- *either the function f vanishes on $\Omega_i \cap (E_i \times (0,1))$;*
- *or there exist $r \in \mathbb{N}$, $p \in \mathbb{Q}$ and a definable function $F(x,t)$, Q-analytic on Ω_i, such that*

$$f(x,t) = t^p \cdot F(x, t^{1/r}) \quad \text{for all} \quad (x,t) \in \Omega_i \cap (E_i \times (0,1))$$

and

$$F(x,0) \neq 0 \quad \text{for all} \quad x \in E_i.$$

Observe first that we may assume that the function f is bounded. Indeed, put

$$A := \{(x,t) \in E \times (0,1) : |f(x,t)| \leq 1\}$$

and

$$B := \{(x,t) \in E \times (0,1) : |f(x,t)| > 1\}.$$

Further, consider a finite, definable, Q-analytic cell decomposition $\mathcal{D}$ of $E \times (0,1)$ which is compatible with the subsets A and B, and denote by $D_1, \dots, D_j$ the lowest cells from $\mathcal{D}$. It is easy to see that they are cells of the layer type lying over some cells $E_1, \dots, E_j$, which form a cell decomposition of E. Then $(E_i \times \{0\}) \cup D_i$ is a neighbourhood of $E_i \times \{0\}$ in $E_i \times [0,1)$. Clearly, it suffices to analyse the restrictions of the function f to the cells D_i, $i = 1, \dots, j$. If $D_i \subset A$, the restriction of f to D_i is bounded. Otherwise, we can first analyse the function $1/f$, which is bounded on D_i, and next deduce the conclusion of the theorem for the restriction of f to D_i.

We shall continue the proof by induction with respect to the dimension k of the set E. Via cell decomposition, we may, of course, assume that $E = (0,1)^k$ is an open cube in $\mathbb{R}^k$. Then the set

$$\Gamma := \text{closure}\,(\text{graph}\, f) \subset \mathbb{R}^k_x \times \mathbb{R}_t \times \mathbb{R}$$

is a compact definable subset of dimension $k+1$. By the uniformization theorem (cf. [2], Sect. 5), there exist a compact definable Q-analytic manifold M of dimension $k+1$ and a Q-analytic mapping

$$(\varphi, g) : M \longrightarrow \mathbb{R}^k_x \times \mathbb{R}_t \times \mathbb{R}, \quad \varphi = (\varphi_1, \ldots, \varphi_{k+1}),$$

which is generically a submersion and such that $(\varphi, g)(M) = \Gamma$. Actually, we can take M to be a finite number of $(k+1)$-dimensional spheres; this follows immediately from the theorem on decomposition into immersion cubes from [16].

Put $H := (0,1)^k \times \{0\}$; then $\widetilde{H} := \varphi^{-1}(H)$ is a definable subset of M of dimension k, and φ_{k+1} vanishes on $\widetilde{H}$. By means of a finite, definable, Q-analytic cell decomposition and the induction hypothesis, we can assume that $\widetilde{H}$ is a definable Q-analytic hypersurface of M. Further, the same argument allows us to assume that the restriction of φ to $\widetilde{H}$ is of constant rank k, and that φ_{k+1} is of a constant order $r \in \mathbb{N}$ along $\widetilde{H}$. The hypersurface $\widetilde{H}$ can be given in suitable local coordinates (v, w), $v = (v_1, \ldots, v_k)$, on M by the equation $w = 0$. Then

$$\varphi_{k+1}(v, w) = w^r \cdot \psi_{k+1}(v, w)$$

for a Q-analytic function ψ_{k+1} with $\psi_{k+1}(v, 0) > 0$. Therefore, the mapping

$$\psi(v, w) := (\varphi_1(v, w), \ldots, \varphi_k(v, w), w \cdot \psi_{k+1}^{1/r}(v, w)) : M \longrightarrow \mathbb{R}^k_x \times \mathbb{R}_t$$

is a local Q-diffeomorphism along $\widetilde{H}$. Again, via cell decomposition, we can assume that ψ is a definable Q-diffeomorphism of a neighbourhood of $\widetilde{H}$ in M onto a neighbourhood of H in $(0,1)^k \times \mathbb{R}_t$.

Putting $\alpha(x, t) := (x, t^r)$, we get $\alpha \circ \psi = \varphi$ whence $\varphi \circ \psi^{-1} = \alpha$, and consequently,

$$f \circ \alpha = f \circ \varphi \circ \psi^{-1} = g \circ \psi^{-1} =: F$$

is a definable function Q-analytic on a neighborhood Ω of H in $(0,1)^k \times \mathbb{R}_t$.

To complete the proof, it remains only to partition $(0,1)^k$ into a finite number of definable, Q-analytic cells E_i, so that the function F has a constant t-order ν_i on each set $E_i \times \{0\}$. Then the conclusion of the theorem holds with Q-analytic cells E_i and their neighborhoods

$$\Omega_i := \Omega \cap (E_i \times \mathbb{R}_t).$$

Moreover, one disjunction of the conclusion follows if $\nu_i = \infty$, and the other with $p = \nu_i / r$ follows if $\nu_i < \infty$.

We now turn to the

Proof of Theorem 3. We keep the foregoing notation. Let $\mu(k)$ be the dimension of the real vector space $\mathcal{P}_k$ of homogeneous polynomials of degree k. It is clear that generic systems of points $p_{k,1}, \dots, p_{k,\mu(k)} \in \mathbb{R}^n$ determine the polynomials from $\mathcal{P}_k$, i.e. for any $q_1, \dots, q_{\mu(k)} \in \mathbb{R}$ there is a unique

$$P \in \mathcal{P}_k, \quad P(u) = \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} a_\alpha u^\alpha,$$

such that $P(p_{k,i}) = q_i$ for all $i = 1, \dots, \mu(k)$. Moreover, the coefficients a_α of the polynomial P depend linearly on the values $q_1, \dots, q_{\mu(k)}$:

$$a_\alpha = a_\alpha(q_1, \dots, q_{\mu(k)}), \quad \alpha \in \mathbb{N}^n, \ |\alpha| = k.$$

Therefore, f has the (weak) k-th Gateaux differential $\delta_x^k(u) = \delta_k(x,u)$ at x iff

$$\delta_k(x,u) = v_k(x,u) := \sum_{\alpha \in \mathbb{N}^n, |\alpha| = k} a_\alpha(\delta_k(x, p_{k,1}), \dots, \delta_k(x, p_{k,\mu(k)})) \cdot u^\alpha$$

for all $u \in \mathbb{R}^n$. Put

$$w_k(x,u) := \delta_k(x,u) - v_k(x,u), \quad x \in U, \ u \in \mathbb{R}^n.$$

We thus attain the following ***characterization*** of Gateaux differentiability:

f has a (weak) k-th Gateaux differential at x iff $w_k(x,u) = 0$ *for all* $u \in \mathbb{R}^n$.

We now apply Puiseux's theorem with parameter to the function $F(x,u,t)$. Since, by definition, it is bounded with respect to the variable t, one can find a definable cell decomposition of $U \times \mathbb{R}^n$ into finitely many Q-analytic cells $E_1, \ldots, E_s$ for which

• either the function $F(x, u, t)$ in the variable t vanishes near zero for all $(x, u) \in E_i$;

• or there exist $r \in \mathbb{N}$ and a definable function $G(x, u, t)$, Q-analytic in a neighbourhood Ω_i of $E_i \times \{0\}$ in $E_i \times \mathbb{R}_t$, such that

$$F(x, u, t) = G(x, u, t^{1/r}) \quad \text{for all} \quad (x, u, t) \in \Omega_i \cap (E_i \times (0, \infty)).$$

We shall show that there exists a finite, definable, Q-analytic cell decomposition $\mathcal{C}$ of $U \times \mathbb{R}^n$ such that the restriction to each cell $C \in \mathcal{C}$ of every function $\delta_k(x, u)$, $k \in \mathbb{N}$, is Q-analytic. Indeed, for a given cell E_i as above, and $l \in \mathbb{N}$, put

$$E_{i,l} := \{(x, u) \in E_i : \partial^k G/\partial t^k(x, u, 0) = 0 \quad \text{for all} \quad k < lr,\ k \notin r\mathbb{N}\}.$$

The set $E_{i,l}$ is the set of those points (x, u) in E_i such that the derivatives $\partial^k F/\partial t^k(x, u, 0^+)$ exist for all $k \leq l$.

The descending sequence $(E_{i,l})_{l \in \mathbb{N}}$ consists of the zero sets of some families of definable Q-analytic functions on E_i. Hence and by topological noetherianity (cf. [5, 2, 15]), this sequence stabilizes, i.e. there is an $L = L(i) \in \mathbb{N}$ such that $E_{i,L} = E_{i,l}$ for all $l \geq L$.

It follows immediately that if $\mathcal{C}_i$ is a finite, definable, Q-analytic cell decomposition of E_i compatible with the sets $E_{i,1}, \ldots, E_{i,L}$, then the restriction to each cell $C \in \mathcal{C}_i$ of every function $\delta_k(x, u)$, $k \in \mathbb{N}$, is Q-analytic. Therefore, it remains to take as $\mathcal{C}$ any finite definable Q-analytic cell decomposition of $U \times \mathbb{R}^n$ compatible with the decompositions $\mathcal{C}_i$, $i = 1, \ldots, s$.

We still need the following elementary

Lemma. *Let $\mathcal{C}$ be a finite definable Q-analytic cell decomposition of $U \times \mathbb{R}^n$ and $\mathcal{D}$ the induced cell decomposition of U. Then there exists a refinement $\mathcal{D}'$ of $\mathcal{D}$ for which, over each cell $D \in \mathcal{D}'$, one can find a cell $C \in \mathcal{C}$ and an open subset $W \subset \mathbb{R}^n$ such that $D \times W \subset C$.*

Observe first that it suffices to prove the lemma for a cell decomposition $\mathcal{C}$ of $U \times (0, 1)^n$. Proceeding with induction with respect to n, we can easily reduce the proof to the case $n = 1$. Thus suppose $\mathcal{C}$ is a finite definable

Q-analytic cell decomposition of $U \times (0,1)$. Consider a cell $D \in \mathcal{D}$ and the definable Q-analytic functions $\xi_i : D \longrightarrow \mathbb{R}$, $i = 0, 1, \dots, s$, with

$$\xi_0 = 0 < \xi_1 < \dots \ \xi_{s-1} < \xi_s = 1,$$

which are involved in the cell decomposition $\mathcal{C}$. In other words, $\mathcal{C}$ consists of precisely s cells of the layer type which lie over D, namely, the cells

$$C_i := \{(x,u) : x \in U, \ \xi_{i-1}(x) < u < \xi_i(x)\}, \quad i = 1, \dots, s.$$

We recursively define s pairs $A_{i,0}, A_{i,1}$, $i = 1, \dots, s$, of subsets of the cell D by putting

$$A_{1,0} := \{x \in D : \xi_1(x) < 1/s\}, \quad A_{1,1} := \{x \in D : \xi_1(x) \geq 1/s\};$$

$$A_{2,0} := \{x \in A_{1,0} : \xi_2(x) < 2/s\}, \quad A_{2,1} := \{x \in A_{1,0} : \xi_2(x) \geq 2/s\};$$

$$A_{3,0} := \{x \in A_{2,0} : \xi_3(x) < 3/s\}, \quad A_{3,1} := \{x \in A_{2,0} : \xi_3(x) \geq 3/s\};$$

and so on ...

It is not difficult to check that $A_{s,0} = \emptyset$ and D is the disjoint union

$$D = A_{1,1} \cup A_{2,1} \cup A_{3,1} \cup \dots \cup A_{s,1}.$$

Since

$$A_{i,1} \times ((i-1)/s, i/s) \subset C_i \quad \text{for} \quad i = 1, \dots, s,$$

the conclusion of the lemma holds over each set $A_{i,1}$. Therefore, as a required refinement $\mathcal{D}'$, we can take any refinement of the cell decomposition $\mathcal{D}$ which is compatible with the sets $A_{i,1}$, $i = 1, \dots, s$, constructed for each cell $D \in \mathcal{D}$.

Due to the above lemma, we may assume that, for each cell $D \in \mathcal{D}$, there is a cell $C_D \in \mathcal{C}$, $C_D \subset D \times \mathbb{R}^n$ such that, for every positive integer k, the restriction to C_D of $\delta_k(x,u)$ is Q-analytic, and we can choose points $p_{k,1}, \dots, p_{k,\mu(k)}$ as above for which

$$D \times \{p_{k,1}, \dots, p_{k,\mu(k)}\} \subset C_D.$$

Consequently, the restriction to each cell $C \in \mathcal{C}$ of every function $v_k(x,u)$, and thus of $w_k(x,u)$ too, is Q-analytic.

Hence and by topological noetherianity along with the foregoing characterization of Gateaux differentiability, the descending sequence of subsets $(\mathcal{G}_k(f))_{k\in\mathbb{N}}$ stabilizes, which is the required result. Finally, we should take as $\mathcal{B}$ any refinement of the Q-analytic cell decomposition $\mathcal{D}$ that is compatible with $\mathcal{G}_\infty(f)$, concluding the proof.

3. Formally composite functions. We begin with the concept of a formally composite function (cf. [3, 19]). For a smooth submanifold M and any $k \in \mathbb{N} \cup \{\infty\}$, denote by $\mathcal{C}^k(M)$ the Fréchet algebra of real functions of class $\mathcal{C}^k$ on M. Let $\varphi : M \longrightarrow N$ be a smooth mapping between two real smooth manifolds with closed image $T := \varphi(M) \subset N$. Denote by $(\varphi^*\mathcal{C}^k(N))^\wedge$ the subalgebra of $\mathcal{C}^k(M)$ of all those functions $g \in \mathcal{C}^k(M)$ that are formally $\mathcal{C}^k$-composite with φ, i.e., for each $a \in T$, there is $h \in \mathcal{C}^k(N)$ such that the function $g - \varphi^*(h)$ is k-flat on the fibre $\varphi^{-1}(a)$. Since this definition is local with respect to the target space, we may assume that $N = \mathbb{R}^n$. Then, by virtue of Borel's lemma, a function g is formally composite with φ iff, for each point $a \in T$, there is a formal power series

$$H_a \in \mathbb{R}[[x-a]], \quad x = (x_1, \dots, x_n),$$

such that

$$\widehat{\varphi}_b^*(H_a) = \mathrm{T}_b\, g \quad \text{ for all } \; b \in \varphi^{-1}(a),$$

or

$$\widehat{\varphi}_b^*(H_a) - \mathrm{T}_b\, g \quad \text{ is } k\text{-flat for all } \; b \in \varphi^{-1}(a),$$

according as $k = \infty$ or $k \in \mathbb{N}$. In the latter case, of course, one can merely require that H_a be a polynomial of degree not greater than k.

In the proof of Theorem 2, given in Section 4, we shall still need a quasi-analytic version of Glaeser's composite function theorem from our paper [19], recalled below. This theorem reduces the problem whether a function g is composite with φ to the problem whether g is formally composite with φ.

Composite Function Theorem. *Consider a polynomially bounded, o-minimal structure $\mathcal{R}$ which admits smooth cell decomposition. Let $M \subset \mathbb{R}^p$ and $N \subset \mathbb{R}^q$ be smooth definable submanifolds, and $\varphi : M \longrightarrow N$ be a smooth definable mapping with closed image T, which is generically a submersion. Then*

$$(\varphi^*\mathcal{C}^\infty(T))^\wedge = \varphi^*\mathcal{C}^\infty(T).$$

We shall return to the structure $\mathcal{R} = \mathcal{R}_Q$ investigated in this paper. Our purpose now is to proceed with some stabilization properties linked to formally composite functions. Let $f : U \longrightarrow \mathbb{R}$ be a definable function of class $\mathcal{G}^\infty$ on an open subset U of $\mathbb{R}^n$, i.e. with all (weak) Gateaux differentials $\delta_x^k f$, $k \in \mathbb{N}$, at every point $x \in U$. Consider two Q-analytic mappings

$$\varphi : M \longrightarrow U \quad \text{and} \quad g : M \longrightarrow \mathbb{R}$$

on a definable Q-analytic manifold M, and suppose that φ is surjective and $f \circ \varphi = g$. Let

$$\Psi_a \in \mathbb{R}[[x-a]], \quad \Psi_a(x-a) = \Psi(a; x-a) = \sum_{\alpha \in \mathbb{N}^n} \psi_\alpha(a)(x-a)^\alpha$$

be a unique formal power series determined by the (weak) Gateaux differentials of f at each point $a \in U$.

By Theorem 3 (on Gateaux differentiability), there exists a finite definable Q-analytic cell decomposition $\mathcal{B}$ of U such that the restriction to each cell $B \in \mathcal{B}$ of every (weak) Gateaux differential $\delta^k f$ is Q-analytic. This means exactly that every function ψ_α is Q-analytic on each cell $B \in \mathcal{B}$.

Denote by $\mathcal{Q}_{b,M}$ the local ring of Q-analytic germs at a point b of a Q-analytic manifold M; for simplicity, we shall drop the index M if this is not misleading. Further, let $\widehat{\mathcal{Q}}_{b,M}$ denote the completion of $\mathcal{Q}_{b,M}$ in the Krull topology, and $\widehat{\mathfrak{m}}_{b,M}$ its maximal ideal. For $a \in U$, we may, of course, identify $\widehat{Q}_a$ with $\mathbb{R}[[x-a]]$.

For $b \in M$ and $a = \varphi(b) \in U$, let

$$\widehat{\varphi}_b^* : \widehat{\mathcal{Q}}_a \longrightarrow \widehat{\mathcal{Q}}_b$$

be the local ring homomorphism induced by φ.

Denote by $\mathfrak{G}_k$, $k \in \mathbb{N} \cup \{\infty\}$, the set of those points $a \in U$ at which the series Ψ_a realizes g as formally $\mathcal{C}^k$-composite with φ, i.e.

$$\mathfrak{G}_\infty = \{a \in U : \ \widehat{\varphi}_b^*(\Psi_a) = \mathrm{T}_b\, g \quad \text{for all} \quad b \in \varphi^{-1}(a)\}$$

and

$$\mathfrak{G}_k = \{a \in U : \ \widehat{\varphi}_b^*(\Psi_a) \equiv \mathrm{T}_b\, g \pmod{\widehat{\mathfrak{m}}_b^{k+1}} \quad \text{for all} \quad b \in \varphi^{-1}(a)\}, \quad k \in \mathbb{N}.$$

Theorem 4. (on a Formally Composite Function) *Under the above assumptions, the following two properties hold:*

• *the sets $\mathfrak{G}_k$, $k \in \mathbb{N}$, are definable;*

• *the descending sequence $(\mathfrak{G}_k)_{k\in\mathbb{N}}$ stabilizes, whence there is a positive integer $l \in \mathbb{N}$ such that*

$$\mathfrak{G}_k = \mathfrak{G}_l \quad \text{for all} \;\; k \in \mathbb{N} \cup \{\infty\}, \; k \geq l.$$

For a proof, fix a cell $B \in \mathcal{B}$ and a finite, definable, Q-analytic stratification $\mathcal{S}$ of $\varphi^{-1}(B)$. Take a stratum $V \in \mathcal{S}$ and points $b \in V$, $a = \varphi(b) \in B$. Put

$$\mathfrak{G}_\infty(V) := \{b \in V : \; \widehat{\varphi}_b^*(\Psi_a) = \mathrm{T}_b\, g\}$$

and

$$\mathfrak{G}_k(V) := \{b \in V : \; \widehat{\varphi}_b^*(\Psi_a) \equiv \mathrm{T}_b\, g \pmod{\widehat{\mathfrak{m}}_b^{k+1}}\}, \quad k \in \mathbb{N}.$$

It is easy to check that $(\mathfrak{G}_k(V))_{k\in\mathbb{N}}$ is a descending sequence of sets which are the zero sets of some families of definable Q-analytic functions on V. Hence and again by topological noetherianity, $\mathfrak{G}_\infty(V) = \mathfrak{G}_k(V)$ for k large enough, say for $k > k(V)$. Putting

$$l(B) := \max\{k(V) : \; V \in \mathcal{S}\},$$

we get $\mathfrak{G}_\infty \cap B = \mathfrak{G}_k \cap B$ for all $k > l(B)$. Consequently,

$$\mathfrak{G}_\infty = \mathfrak{G}_k \quad \text{for all} \;\; k > l := \max\{l(B) : \; B \in \mathcal{B}\},$$

which is the required result.

4. Proof of Theorem 2. We shall make use of the stabilization effects linked to Gateaux differentiability and formally composite functions, developed in Sections 2 and 3, as well as a quasianalytic version of the composite function theorem from our paper [19]. It is sufficient to prove that there is a positive integer N such that, for every point $x \in U$, if the function f is of class $\mathcal{C}^N$ near x, it is of class $\mathcal{C}^\infty$ near x.

We may assume, without loss of generality, that the set U is bounded and the function f is bounded. Let $\Gamma \subset \mathbb{R}^{n+1}$ be the closure of the graph of f. It follows from the uniformization theorem that there exist a compact definable Q-manifold M of dimension n, and a definable Q-analytic mapping

$$(\varphi, g) : M \longrightarrow \mathbb{R}^n \times \mathbb{R}$$

such that $(\varphi, g)(M) = \Gamma$. Actually, we can take M to be a finite number of n-dimensional spheres, which follows immediately from the theorem on decomposition into immersion cubes from [16]. Clearly, we can assume that $\varphi : M \longrightarrow \mathbb{R}^n$ is generically a submersion on each connected component of M. It is easy to check that

$$\Omega := (\varphi, g)^{-1}(\mathrm{graph}\, f) = \varphi^{-1}(U)$$

is an open subset of M, and that $f \circ \varphi|\Omega = g|\Omega$. Further, we shall regard φ and g as definable Q-analytic mappings on Ω; obviously, $\varphi : \Omega \longrightarrow U$ is a proper mapping.

It follows from Theorem 3 (on Gateaux differentiability) that the descending sequence of definable subsets $(\mathcal{G}_k(f))_{k\in\mathbb{N}}$ of those points $x \in U$ at which the function $f : U \longrightarrow \mathbb{R}$ has (weak) j-th Gateaux differentials $\delta_x^j f$ for all $j \leq k$, stabilizes, i.e. there is a positive integer N_1 such that

$$\mathcal{G}_{N_1}(f) = \mathcal{G}_{N_1+1}(f) = \ldots = \mathcal{G}_\infty(f).$$

Therefore, at each point $a \in \mathcal{G}_{N_1}(f) = \mathcal{G}_\infty(f)$ there is a unique formal power series

$$\Psi_a \in \mathbb{R}[[x-a]], \quad \Psi_a(x-a) = \Psi(a; x-a) = \sum_{\alpha\in\mathbb{N}^n} \psi_\alpha(a)(x-a)^\alpha$$

determined by the (weak) Gateaux differentials of f. Furthermore, there exists a finite definable Q-analytic cell decomposition $\mathcal{B}$ of $\mathcal{G}_{N_1}(f)$ such that the restriction to each Q-analytic cell $B \in \mathcal{B}$ of every (weak) Gateaux differential $\delta^k f$, $k \in \mathbb{N}$, is Q-analytic. This means that every function $\psi_\alpha(x)$ is Q-analytic on each cell $B \in \mathcal{B}$.

Now, for a positive integer k, let $\mathcal{C}_k(f)$ denote the set of those points $a \in U$ in the vicinity of which f is of class $\mathcal{C}^k$; obviously, the sets $\mathcal{C}_k(f)$, $k \in \mathbb{N}$, are open definable subsets of U. Obviously, $\mathcal{C}_k(f) \subset \mathcal{G}_k(f)$ for any $k \in \mathbb{N}$; in particular,

$$\mathcal{C}_{N_1}(f) \subset \mathcal{G}_{N_1}(f) = \mathcal{G}_\infty(f),$$

and thus the formal power series $\Psi_a(x-a)$ are defined for all $a \in \mathcal{C}_{N_1}(f)$.

We are now going to apply the quasianalytic version of the composite function theorem. For any $k \in \mathbb{N} \cup \{\infty\}$, let $\mathfrak{G}_k$ be the set of those points

$a \in \mathcal{C}_{N_1}(f)$ at which the series Ψ_a realizes g as formally $\mathcal{C}^k$-composite with φ. By Theorem 4 (on a formally composite function), the descending sequence of definable subsets $(\mathfrak{G}_k)_{k\in\mathbb{N}}$ stabilizes, i.e. there is a positive integer $N_2 \geq N_1$ such that

$$\mathfrak{G}_{N_2} = \mathfrak{G}_{N_2+1} = \ldots = \mathfrak{G}_\infty.$$

It is clear that $\mathcal{C}_{N_2}(f) \subset \mathfrak{G}_{N_2}$. Therefore, the restriction of g to the open subset $\varphi^{-1}(\mathcal{C}_{N_2}(f))$ is formally $\mathcal{C}^\infty$-composite with the restriction of φ to $\varphi^{-1}(\mathcal{C}_{N_2}(f))$. Hence and by the composite function theorem, there is a smooth function

$$h : \mathcal{C}_{N_2}(f) \longrightarrow \mathbb{R}$$

such that $h \circ \varphi = g$ on $\mathcal{C}_{N_2}(f)$. Since $g = f \circ \varphi$ and the mapping φ is surjective, we get $f = h$ on $\mathcal{C}_{N_2}(f)$, and thus f is a smooth function on $\mathcal{C}_{N_2}(f)$. This completes the proof of Theorem 2.

5. Final remarks. We conclude this paper with the following comment. Given any polynomially bounded, o-minimal structure $\mathcal{R}$, the smooth functions definable in $\mathcal{R}$ form a quasianalytic system of sheaves Q, and induce a quasianalytic structure $\mathcal{R}_Q$. It may, obviously, contain fewer definable sets than the initial structure $\mathcal{R}$. In particular, while the exponent field of $\mathcal{R}$ may be any subfield of $\mathbb{R}$, that of $\mathcal{R}_Q$ is just $\mathbb{Q}$.

The singular locus of a definable set or function may not be definable if the polynomially bounded structure $\mathcal{R}$ does not admit smooth cell decomposition. An example of such a structure is the one constructed by Le Gal–Rolin [11]. This structure is polynomially bounded with exponent field $\mathbb{Q}$, and does not admit smooth cell decomposition. It is generated by a function $H : \mathbb{R} \longrightarrow \mathbb{R}$ with the following two properties:

• *the restriction of H to the complement of any neighbourhood of $0 \in \mathbb{R}$ is piecewise given by finitely many polynomials;*

• *the germ of H at $0 \in \mathbb{R}$ is not smooth, but is weakly smooth, i.e. is of class $\mathcal{C}^k$ for any positive integer k.*

It is easy to check that the singular locus $\mathrm{Sing}\,(H)$ is a countable set with a unique accumulation point $0 \in \mathbb{R}$; obviously, we have

$$\mathrm{Reg}\,(\mathrm{graph}\,(H)) = \mathrm{graph}\,(H) \cap (\mathrm{Reg}\,(H) \times \mathbb{R}).$$

Clearly, these sets are not definable.

Open Problem. Do the results of our paper about singular locus extend to arbitrary, polynomially bounded, o-minimal structures which admit smooth cell decomposition?

The only structures for which the answer is known to be in the affirmative are $\mathbb{R}_{an}^{K}$ where K is a subfield of $\mathbb{R}$; here, $\mathbb{R}_{an}^{K}$ is the expansion of the real field $\mathbb{R}$ by restricted analytic functions and power functions with exponents from K. This result was established by van den Dries–Miller [4].

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References

[1] E. Bierstone, P.D Milman, *Semianalytic and subanalytic sets*, Inst. Hautes Études Sci. Publ. Math. **67** (1988), 5–42.

[2] —, —, *Resolution of singularities in Denjoy–Caleman classes*, Selecta Math., New Ser. **10** (2004), 1–28.

[3] —, —, W. Pawłucki, *Composite differentiable functions*, Duke Math. J. **83** (1996), 607–620.

[4] L. van den Dries, C. Miller, *Extending Tamm's theorem*, Ann. Inst. Fourier (Grenoble) **44**(5) (1994), 1367–1395.

[5] —, —, *Geometric categories and o-minimal structures*, Duke Math. J. **84** (1996), 497–540.

[6] P.M. Eakin, G.A. Harris, *When $\Phi(f)$ convergent implies f convergent*, Math. Ann. **229** (1977), 201–210.

[7] A. Elkhadiri, *Homomorphism of quasianalytic local rings*, Proc. Amer. Math. Soc. **138** (4) (2010), 1433–1438.

[8] A. Gabrielov, *Formal relations between analytic functions*, Math. USSR Izvestija **7** (1973), 1056–1088; Izvestija Akad. Nauk SSSR **37** (1973), 1056–1090.

[9] H. Komatsu, *The implicit function theorem for ultradifferentiable mappings*, Proc. Japan Acad. **55** (1979), 69–72.

[10] K. Kurdyka, *Points réguliers d'un sous-analytique*, Ann. Inst. Fourier (Grenoble) **38** (1) (1988), 133–156.

[11] O. Le Gal, J.-P. Rolin, *An o-minimal structure which does not admit $\mathcal{C}^\infty$ cellular decomposition*, Ann. Inst. Fourier **59** (2) (2009), 543–562.

[12] B. Malgrange, *Frobenius avec singularités*, Invent. Math. **39** (1) (1977), 67–89.

[13] S. Mandelbrojt, *Séries adhérentes, régularisation des suites, applicationes*, Collection Borel, Gauthiers-Villars, Paris, 1952.

[14] C. Miller, *Infinite differentiability in in polynomially bounded o-minimal structures*, Proc. Amer. Math. Soc. **123** (1995), 2551–2555.

[15] K.J. Nowak, *On the Euler characteristic of the links of a set determined by smooth definable functions*, Ann. Polon. Math. **93** (3) (2008), 231–246.

[16] —, *Decomposition into special cubes and its application to quasi-subanalytic geometry*, Ann. Polon. Math. **96** (1) (2009), 65–74.

[17] —, *Quantifier elimination, valuation property and preparation theorem in quasianalytic geometry via transformation to normal crossings*, Ann. Polon. Math. **96** (3) (2009), 247–282.

[18] —, *Quasianalytic perturbation of multi-parameter hyperbolic polynomials and symmetric matrices*, Ann. Polon. Math. **101** (3) (2011), 275–291.

[19] —, *A note on Bierstone–Milman–Pawłucki's paper "Composite differentiable functions"*, Ann. Polon. Math. **102** (3) (2011), 293–299.

[20] W. Pawłucki, *La théorème de Puiseux pour une application sous analytique*, Bull. Polish Acad. Sci. Math. **32** (1984), 556–560.

[21] J.-B. Poly, G. Raby, *Fonction distance et singularitiés*, Bull. Sci. Math. (2) **108** (1984), 187–195.

[22] J.-P. Rolin, P. Speissegger, A.J. Wilkie, *Quasianalytic Denjoy–Carleman classes and o-minimality*, J. Amer. Math. Soc. **16** (2003), 751–777.

[23] C. Roumieu, *Ultradistributions définies sur $\mathbb{R}^n$ et sur certaines classes de variétés differentiables*, J. Analyse Math. **10** (1962-62), 153–192.

[24] M. Tamm, *Subanalytic sets in the calculus of variations*, Acta Math. **146** (1981), 167–199.

[25] V. Thilliez, *On quasianalytic local rings*, Expo. Math. **26** (2008), 1–23.

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