On the real algebra of quasianalytic function germs

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Abstract

Given a quasianalytic system $\mathcal{Q} = (\mathcal{Q}_n)_{n\in\mathbb{N}}$ of sheaves, denote by Q_n the local ring of Q-analytic function germs at $0 \in \mathbb{R}^n$. This paper introduces the concepts of Łojasiewicz radical and geometric spectrum $\operatorname{Speg} Q_n \subset \operatorname{Sper} Q_n$. Via the Łojasiewicz inequality, a version of the Nullstellensatz for Q_n is given. We establish a quasianalytic version of the Artin–Lang property for Q_n. Finally, we prove, by means of transformation to normal crossings by blowing up, that the Łojasiewicz radical $\mathcal{L}(I)$ of any ideal $I \subset Q_n$ coincides with the contraction of the real radical $\Re(I\widehat{Q_n})$.

1. Introduction. As in our previous paper [9, 10, 11, 12], we begin by fixing a quasianalytic system $\mathcal{Q} = (\mathcal{Q}_n)_{n\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^n$. For each open subset $U \subset \mathbb{R}^n$, $\mathcal{Q}(U) = \mathcal{Q}_n(U)$

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is thus a subalgebra of the algebra $\mathcal{C}_n^\infty(U)$ of real smooth functions on U. By a Q-analytic function (or a Q-function, for short) we mean any function $f \in \mathcal{Q}(U)$. Similarly,

$$f = (f_1, \ldots, f_k) : U \longrightarrow \mathbb{R}^k$$

is called Q-analytic (or a Q-mapping) if so are its components $f_1, \ldots, f_k$. The following six conditions are imposed on this family of sheaves:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;

2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);

3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \longrightarrow V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^n$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \longrightarrow U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;

4. $\mathcal{Q}$ is closed under differentiation;

5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \ldots, x_{i-1}, a_i, x_{i+1}, \ldots, x_n) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;

6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f}_a$ of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

Q-mappings give rise, in the ordinary manner, to the category Q of Q-manifolds and Q-mappings, which is a subcategory of that of smooth manifolds and smooth mappings. Similarly, Q-analytic, Q-semianalytic and Q-subanalytic sets can be defined. One of the most powerful tools of Q-analytic geometry is transformation to normal crossings by blowing up, which can be used, in particular, the achieve the Łojasioewicz inequality and topological noetherianity (cf. [3]).

Denote by Q_n the local ring of Q-analytic function germs at $0 \in \mathbb{R}^n$; it can be embedded into its completion in the Krull topology which is isomorphic to the formal power series ring $\widehat{Q_n} \equiv \mathbb{R}[[x]]$, $x = (x_1, \ldots, x_n)$.

Denote by $\mathcal{R} = \mathcal{R}_Q$ the expansion of the real field $\mathbb{R}$ by restricted Q-analytic functions, i.e. functions of the form

$$\widetilde{f}(x) = \begin{cases} f(x), & \text{if } x \in [-1,1]^n \\ 0, & \text{otherwise} \end{cases}$$

where $f(x)$ is a Q-function in the vicinity of the compact cube $[-1,1]^n$. $\mathcal{R}_Q$ is a polynomially bounded o-minimal structure which is model complete and admits smooth cell decompositions (cf. [17, 9]). Therefore, the definable subsets in $\mathcal{R}_Q$ coincide with those subsets in $\mathbb{R}^n$ which are Q-subanalytic in a semialgebraic compactification of $\mathbb{R}^n$. From now on, the word "definable" means "definable with parameters" in the structure $\mathcal{R}$.

In Section 2, we introduce the concept of the Łojasiewicz radical of an ideal of Q_n along with a quasianalytic version of the Nullstellensatz. In the next section, we define formal zero locus (which consists of formal arcs through zero) and prove the Artin–Lang property for the ring of formal power series. The latter was first established by Artin [1], in relation to Hilbert's 17-th problem, for the rings of polynomials with coefficients from a real closed field R. It says that there exists an ordering of the field of rational finctions $R(x)$, $x = (x_1, \dots, x_n)$, for which a finite number of polynomials $g_1, \dots, g_l$ are positive iff there exists a point $a \in R^n$ for which $g_1(a) > 0, \dots, g_l(a) > 0$. This result was later generalized by Lang [7] to finitely generated extensions of real closed fields. Our proof of the Artin–Lang property relies on the Weierstrass preparation theorem and the Cherlin–Dickmann theorem on quantifier elimination for real closed, non-trivially and convexly valued fields.

In Section 4, we introduce the notion of the geometric spectrum of the quasianalytic local ring Q_n and formulate a weak version of the Artin–Lang property for Q_n. The last section provides a relation between Łojasiewicz radical and real radical: the Łojasiewicz radical $\pounds(I)$ of an ideal I of Q_n coincides with the contraction of the real radical $\Re(I \cdot \mathbb{R}[[x]])$, $x = (x_1, \dots, x_n)$.

2. Zero locus and Łojasiewicz radical. In this section, we wish to introduce the notion of the zero locus $V(I)$ of any ideal I in the quasianalytic local ring Q_n. This concept is clear for an ideal generated by a finite number of Q-analytic function germs, because one can take their representatives $f_1, \dots, f_k$ which are Q-analytic in a common neighbourhood U of zero, and then the zero locus $V(I)$ is determined by the zero set

$$V(f_1, \dots, f_k) := \{x \in U : f_1(x) = \dots = f_k(x) = 0\}.$$

We shall make no distinction in notation between germs and their representatives; this ambiguity will not lead to confusion.

In order to give the general definition of zero locus in Corollary 3 below, we still define the zero set

$$\widetilde{V}(f_1, \dots, f_k) := \{\gamma(\tau) = (\gamma_1(\tau), \dots, \gamma_n(\tau)) \in Q_1^n :$$

$$\gamma(0) = 0 \quad \text{and} \quad f_1(\gamma) = \dots = f_k(\gamma) = 0\}.$$

Clearly, there is a one-to-one correspondence between the set of Q-analytic arcs $\gamma(\tau) = (\gamma_1(\tau), \dots, \gamma_n(\tau))$ through $0 \in \mathbb{R}^n$ and the set of homomorphisms $\gamma : Q_n \longrightarrow Q_1$; namely, $\gamma_i(\tau) = \gamma(x_i)$, $i = 1, \dots, n$.

Further, we shall regard Q_1 as an ordered ring of function germs in one variable $\tau > 0$. As before, we may identify homomorphisms $\gamma : Q_n \longrightarrow Q_1$ into the ordered ring Q_1 with the half-branches of the corresponding Q-analytic arcs $\gamma(\tau) = (\gamma_1(\tau), \dots, \gamma_n(\tau))$ through $0 \in \mathbb{R}^n$. The theorem below demonstrates that those homomorphisms into the ordered ring Q_1 can be treated on a par with the ordinary points in the vicinity of $0 \in \mathbb{R}^n$.

Theorem 1. *For $f, g_1, \dots, g_l \in Q_n$, put*

$$S := \{x \in \mathbb{R}^n : \; f(x) = 0, \; g_1(x) > 0, \; \dots \; , \; g_l(x) > 0\}$$

and

$$\widetilde{S} := \{\gamma(\tau) \in Q_1^n : \; \gamma(0) = 0, \; f(\gamma) = 0, \quad g_1(\gamma) > 0, \; \dots \; , \; g_l(\gamma) > 0\}.$$

Then $S = \emptyset$ iff $\widetilde{S} = \emptyset$.

Theorem 1 will be proven once we know that any Q-semianalytic set germ at zero has an arbitrarily small representative which is a union of Q-analytic arcs through zero. The latter follows from our theorem on decomposition of a Q-semianalytic set into special cubes (cf. [9]). Indeed, a Q-semianalytic representative of the germ S in an arbitrarily small ball is a finite union of special cubes, i.e. sets of the form $S_j = \varphi_j((0,1)^{d_j})$, $j = 1, \dots, s$, where each φ_j is a mapping Q-analytic in the vicinity of $[0,1]^{d_j}$ such that its restriction to $(0,1)^{d_j}$ is a diffeomorphism onto S_j; actually, the mappings φ_j can be taken as the restrictions of certain compositions of finitely many blowings-up along smooth centers. Then the germ S is represented by the finite union of those special cubes which are adjacent to zero. Clearly, this representative is a union of Q-analytic arcs through zero, as required.

It follows immediately from this theorem that the above two concepts of zero locus $V(f_1, \ldots, f_k)$ and $\widetilde{V}(f_1, \ldots, f_n)$ are equivalent:

Corollary 1. *If $f_1, \ldots, f_k, g_1, \ldots, g_l \in Q_n$, then*

$$V(f_1, \ldots, f_k) = V(g_1, \ldots, g_l) \quad \textit{iff} \quad \widetilde{V}(f_1, \ldots, f_k) = \widetilde{V}(g_1, \ldots, g_l).$$

Corollary 2. *Let $f_1, \ldots, f_k, f \in Q_n$ and suppose that*

$$f \in \sum_{i=1}^{k} f_i \cdot \mathbb{R}[[x]] \cap Q_n = \sum_{i=1}^{k} f_i \cdot Q_n \cap Q_n.$$

Then $V(f_1, \ldots, f_k) = V(f_1, \ldots, f_k, f)$.

Corollary 3. (on zero locus) *Let $I \subset Q_n$ be an ideal, $f_1, \ldots, f_k \in I$ be generators of the ideal $I \cdot \mathbb{R}[[x]]$ and $V = V(f_1, \ldots, f_k)$ be their zero set germ. Then each Q-analytic function germ $f \in I \cdot \mathbb{R}[[x]] \cap Q_n$ vanishes on V.*

Therefore, one can define the zero locus $V(I)$ by putting $V(I) := V$.

Corollary 4. (on topological noetherianity) *Every descending sequence $(V_j)_{j \in \mathbb{N}}$ of Q-analytic set germs stabilizes, i.e. there is an integer N such that $V_j = V_N$ for all $j \geq N$.*

Indeed, let $I_j \subset Q_n$ be the ideal of those Q-analytic function germs which vanish on V_j, $j \in \mathbb{N}$. Then

$$I_j = I_j \cdot \mathbb{R}[[x]] \cap Q_n \quad \text{for} \quad j \in \mathbb{N}.$$

Therefore, since the ascending sequence of ideals $(I_j \cdot \mathbb{R}[[x]])_{j \in \mathbb{N}}$ stabilizes, so do the sequences $(I_j)_{j \in \mathbb{N}}$ and $(V_j)_{j \in \mathbb{N}}$, as required.

We now turn to the concept of Łojasiewicz radical. Let $I \subset Q_n$ be a proper ideal, $V = V(I)$ its zero locus and suppose that a Q-analytic function germ $g \in Q_n$ vanishes on V. By topological noetherianity, there are

$f_1, \ldots, f_k \in I$ such that

$$V = V(f_1, \ldots, f_k) = V(f_1^2 + \ldots + f_k^2) = V(f)$$

with $f = f_1^2 + \ldots + f_k^2 \in I$. It follows from the Łojasiewicz inequality that there are a positive integer s and a positive constant C for which

$$|g(x)|^s \leq C|f(z)| \quad \text{whence} \quad |g(x)|^{s+1} \leq |f(x)|$$

in the vicinity of zero. This reasoning leads to the following definition: the Łojasiewicz radical of an ideal $I \subset Q_n$ is the ideal

$$\text{Ł}(I) := \{g \in Q_n : |g|^r \leq |f| \quad \text{for some} \quad f \in I \quad \text{and} \quad r \in \mathbb{N}.$$

The name is thus justified by a quasianalytic version of the Nullstellensatz presented below, which comes immediately from the Łojasiewicz inequality.

Proposition 1. *The Łojasiewicz radical Ł(I) of any ideal $I \subset Q_n$ coincides with the zero ideal $I(V(I))$ of those Q-analytic function germs which vanish on $V(I)$: Ł$(I) = I(V(I))$.*

3. Formal zero locus and formal Artin-Lang property. In this section, we recall a well-known theorem concerning the real spectrum of the real formal power series ring (Theorem 2) and a formal version of the real Nullstellensatz. The latter result was established by Riesler [15] (convergent version) and Lassalle [8] (formal version). Inspired by these papers, Ruiz proved a certain convergent version of the former (cf. [18], Chap. IV, Prop. 3.4).

In our approach, we shall formulate and prove Theorem 2^* which is a strengthening of the former theorem. Our short proof applies the following two tools: the Weierstrass preparation theorem and the Cherlin–Dickmann theorem (cf. [6]) on quantifier elimination for real closed, non-trivially and convexly valued fields. As an immediate corollary to Theorem 2 we shall obtain the Nullstellensatz for real formal power series.

The real spectrum of a ring A shall be denoted by Sper A. One may regard an element $\sigma \in \operatorname{Sper} A$ as a homomorphism $\sigma : A \longrightarrow R$ into a real closed field R containing the ordered residue field of ker σ. We are now concerned with the ring $\mathbb{R}[[x]]$ of real formal power series in several variables $x = (x_1, \ldots, x_n)$. We establish the following formal Artin–Lang property:

Theorem 2. *Take* $f, g_1, \dots, g_m \in \mathbb{R}[[x]]$. *Then there is a* $\sigma \in \operatorname{Sper} \mathbb{R}[[x]]$ *such that*

$$\sigma(f) = 0,\ \sigma(g_1) > 0,\ \dots,\ \sigma(g_m) > 0,$$

iff there is a homomorphism $\gamma : \mathbb{R}[[x]] \longrightarrow \mathbb{R}[[\tau]]$ *into the ordered ring of formal power series in one variable* $\tau > 0$ *such that*

$$\gamma(f) = 0,\ \gamma(g_1) > 0,\ \dots,\ \gamma(g_m) > 0.$$

Clearly, the *if part* of the equivalence is trivial, and so we must prove the converse implication. We are, in fact, going to establish its strengthening, stated below, which enables us to make use of the Cherlin–Dieckmann theorem. Let R be a real closed field with a non-trivial valuation v whose valuation ring V is a convex subset of R. Consider the language $\mathcal{L}$ of ordered rings with an extra unary relation symbol to denote V. The latter theorem can be formulated as follows. The real closed valued field R admits quantifier elimination in the language $\mathcal{L}$ augmented by a binary relation symbol $\prec$ construed by putting $a \prec b$ if $v(a) < v(b)$ or, equivalently, if $a/b \notin V$. In what follows we shall regard the ordered ring $\mathbb{R}[[\tau]]$ of formal power series in one variable $\tau > 0$ as a subring of the quotient field F of the formal Puiseux series in the variable τ. The latter is a real closed field, and the order function is a non-trivial valuation; the set of elements bounded with respect to the real field $\mathbb{R}$ forms its valuation ring, whose maximal ideal coincides with the set of all infinitesimals.

Theorem 2*. *Take* $f, g_1, \dots, g_m, a_1, b_1 \dots, a_p, b_p \in \mathbb{R}[[x]]$. *If*

$$\sigma(f) = 0,\ \sigma(g_1) > 0,\ \dots\ ,\ \sigma(g_m) > 0,\ \sigma(b_1) \succ \sigma(a_1),\ \dots\ ,\ \sigma(b_p) \succ \sigma(a_p),$$

for some $\sigma \in \operatorname{Sper} \mathbb{R}[[x]]$, *then there is a homomorphism* $\gamma : \mathbb{R}[[x]] \longrightarrow \mathbb{R}[[\tau]]$ *such that*

$$\gamma(f) = 0,\ \gamma(g_1) > 0,\ \dots\ ,\ \gamma(g_m) > 0,\ \gamma(b_1) \succ \gamma(a_1),\ \dots\ , \gamma(b_p) \succ \gamma(a_p).$$

We proceed by induction with respect to the number n of variables. The case $n = 1$ is evident. Suppose the assertion holds for n; we will prove it for $n + 1$. By the Weierstrass preparation theorem, we can assume that all the formal power series under consideration are polynomial with respect to the variable x_{n+1}, and thus there are polynomials

$$F, G_1, \dots, G_m, A_1, B_1 \dots, A_p, B_p \in \mathbb{R}[x_{n+1}, C], \quad C = (C_1, \dots, C_s),$$

and

$$h(x') = (h_1(x'), \dots, h_s(x')) \in \mathbb{R}[[x']]^s, \quad x' = (x_1, \dots, x_n),$$

such that

$$f(x) = F(x_{n+1}, h(x'), \quad g_i(x) = G_i(x_{n+1}, h(x')),$$

$$a_j(x) = A_j(x_{n+1}, h(x')), \quad b_j(x) = B_j(x_{n+1}, h(x')).$$

Then

$$\sigma(f) = F(\sigma(x_{n+1}), \sigma(h)) = 0, \quad \sigma(g_i) = G_i(\sigma(x_{n+1}), \sigma(h)) > 0,$$

$$\sigma(b_j) = B_j(\sigma(x_{n+1}), \sigma(h)) \succ \sigma(a_j) = A_j(\sigma(x_{n+1}), \sigma(h)).$$

Consider the formula

$$\exists\, y\; F(y, C) = 0, \quad \bigwedge_{i=1}^{m} G_i(y, C) > 0, \quad \bigwedge_{j=1}^{p} B_j(y, C) \succ A_j(y, C).$$

By the Cherlin–Dickmann theorem, this formula is equivalent in (R, V) to a quantifier-free formula, which can obviously be taken as a finite disjunction of some finite conjunctions of the form as above. Then one of those conjunctions defines a subset of R^s containing $\sigma(h)$. It follows from the induction hypothesis that there is a homomorphism $\gamma : \mathbb{R}[[x']] \longrightarrow \mathbb{R}[[\tau]]$ for which $\gamma(h) \in F^s$ satisfies that conjunction construed over the valued field F. Via quantifier elimination, there is an element $\gamma_{n+1} \in F$ such that

$$F(\gamma_{n+1}, \gamma(h)) = 0, \quad G_i(\gamma_{n+1}, \gamma(h)) > 0, \quad B_j(\gamma_{n+1}, \gamma(h)) \succ A_j(\gamma_{n+1}, \gamma(h)).$$

But we may always assume, without loss of generality, that the conditions $\sigma(x_i) \succ 1$, $i = 1, \dots, n+1$, occur among the initial ones. We can thus find an element $\gamma_{n+1} \in F$ as above which is a formal Puiseux series with $\gamma_{n+1}(0) = 0$; say $\gamma_{n+1} \in \mathbb{R}[[\tau^{1/r}]]$ for a positive integer r. By putting $\gamma(x_{n+1}) = \gamma_{n+1}$, we extend the homomorphism γ to a homomorphism $\mathbb{R}[[x]] \longrightarrow F$ which fulfills the required conditions. Since the assignment $\tau \mapsto \tau^r$ determines an increasing automorphism φ of the field F, the composition $\varphi \circ \gamma$ is a homomorphism $\mathbb{R}[[x]] \longrightarrow \mathbb{R}[[\tau]]$ we are looking for. This completes the proof.

Clearly, there is a one-to-one correspondence between the set of homomorphisms $\gamma : \mathbb{R}[[x]] \longrightarrow \mathbb{R}[[\tau]]$ into the ordered ring $\mathbb{R}[[\tau]]$, $\tau > 0$, and the set of the half-branches $\tau > 0$ of formal arcs $\gamma(\tau) = (\gamma_1(\tau), \ldots, \gamma_n(\tau))$ through $0 \in \mathbb{R}^n$; namely, $\gamma_i(\tau) = \gamma(x_i)$, $i = 1, \ldots, n$. Hence and by Theorem 2, we can treat those half-branches of formal arcs as "basic points" in the real spectrum.

This legitimizes the following definition of the formal zero locus $\widehat{V}(I)$ of an ideal $I \subset \mathbb{R}[[x]]$: $\widehat{V}(I)$ is the set $\widehat{\mathbb{R}^n_0}$ of all homomorphisms $\gamma : \mathbb{R}[[x]] \longrightarrow \mathbb{R}[[\tau]]$ or, equivalently, of all formal arcs $\gamma(\tau) = (\gamma_1(\tau), \ldots, \gamma_n(\tau))$ through $0 \in \mathbb{R}^n$, such that

$$\gamma(f) = f(\gamma_1(\tau), \ldots, \gamma_n(\tau)) = 0$$

for every $f \in I$; here $\gamma_i(\tau) = \gamma(x_i)$ for $i = 1, \ldots, n$. Such an approach to zero locus may be connected to the model theoretic ones of Robinson [16] and Prestel [14].

For a subset V of $\widehat{\mathbb{R}^n_0}$, define the zero ideal $\widehat{I}(V)$ by putting

$$\widehat{I}(V) := \{f \in \mathbb{R}[[x]] : \gamma(f) = 0 \ \text{ for all } \ \gamma \in V\} = \bigcap \{\ker \gamma : \ \gamma \in V\}.$$

As an immediate consequence of Theorem 2, we obtain the following result due to Lassalle [8]:

Corollary. (formal Nullstellensatz) *The real radical $\Re(I)$ of an ideal $I \subset \mathbb{R}[[x]]$ coincides with $\widehat{I}(\widehat{V}(I))$:*

$$\Re(I) = \widehat{I}(\widehat{V}(I)) = \bigcap \{\ker \gamma : \ \gamma \in \widehat{\mathbb{R}^n_0}, \ \gamma | I = 0\}.$$

Remark. Theorems 2, 2^* and the above corollary remain valid, with the same proof, for the case of convergent power series.

4. Geometric spectrum and quasianalytic Artin–Lang property.

We begin with the following

Theorem 3. *If $\mathfrak{q} \in \operatorname{Sper} \mathbb{R}[[x]]$ and $\mathfrak{p} := \mathfrak{q} \cap Q_n \in \operatorname{Sper} Q_n$, then $\pounds(\mathfrak{p}) = \mathfrak{p}$.*

Our proof makes use of Theorem 2 and the formal positivstellensatz (cf. [4], Prop. 4.4.1) applied to the power series ring. Denote by Σ^2 the cone of finite suns of squares of formal power series from $\mathbb{R}[[x]]$.

Suppose $g \in \pounds(\mathfrak{p})$, i.e. $g^{2k} \le f$ for some $f \in \mathfrak{p}$ and $k \in \mathbb{N}$. Then $h := f - g^{2k} \ge 0$, and thus $\gamma(h) \ge 0$ for every $\gamma \in \widehat{\mathbb{R}_0^n}$. Hence and by Theorem 2, $\sigma(h) \ge 0$ for every $\sigma \in \operatorname{Sper} \mathbb{R}[[x]]$. Therefore, it follows from the formal positivstellensatz that

$$ha = h^{2m} + b \quad \text{for some} \quad a, b \in \Sigma^2,$$

and thus

$$h = \frac{h^2 a}{h^{2m} + b} = \frac{h^2 a \, (h^{2m} + b)^{2k-1}}{(h^{2m} + b)^{2k}}.$$

Putting $c := h^{2m} + b$, we get

$$hc^{2k} \in \Sigma^2 \quad \text{whence} \quad fc^{2k} \in ((gc)^{2k} + \Sigma^2) \cap \mathfrak{p} \cdot \mathbb{R}[[x]].$$

Consequently,

$$gc \in \Re(\mathfrak{p} \cdot \mathbb{R}[[x]]) \subset \mathfrak{q}.$$

We now have the following dichotomy: either $c \not\in \mathfrak{q}$ or $c \in \mathfrak{q}$. In the first case, we get $g \in \mathfrak{q} \cap Q_n = \mathfrak{p}$, as required. In the other one, $h^{2m} + b \in \mathfrak{q}$, and since $b \in \Sigma^2$, we get

$$h = f - g^{2k} \in \Re(\mathfrak{q}) = \mathfrak{q}.$$

Therefore, $g^{2k} \in \mathfrak{q}$ and again we get $g \in \mathfrak{q} \cap Q_n = \mathfrak{p}$, concluding the proof.

Hence and by the quasianalytic Nullstellensatz (Proposition 1), we obtain immediately the

Corollary. *Under the foregoing assumptions, we have* $\mathfrak{p} = I(V(\mathfrak{p}))$.

This legitimizes the following definition: the geometric spectrum $\operatorname{Speg} Q_n$ of the quasianalytic local ring Q_n is the subset of the real spectrum $\operatorname{Sper} Q_n$ of all ideals $\mathfrak{p} = \mathfrak{q} \cap Q_n$ with $\mathfrak{q} \in \operatorname{Sper} \mathbb{R}[[x]]$.

Before establishing a quasianalytic version of the Artin–Lang property, we achieve, via transformation to normal crossings by blowing up, a partial result, stated below.

Proposition 2. *For* $f, g_1, \ldots, g_m \in Q_n$*, put*

$$\widetilde{S} := \{\gamma(\tau) \in Q_1^n : \ \gamma(0) = 0, \ f(\gamma) = 0, \ g_1(\gamma) > 0, \ \ldots \ , \ g_l(\gamma) > 0\}$$

and

$$\widehat{S} := \{\gamma(\tau) \in \mathbb{R}[[\tau]]^n : \ \gamma(0) = 0, \ f(\gamma) = 0, \ g_1(\gamma) > 0, \ \dots \ , \ g_l(\gamma) > 0\}.$$

Then $\widehat{S} = \emptyset$ *iff* $\widetilde{S} = \emptyset$.

Suppose $f, g_1, \dots, g_m$ are functions Q-analytic in a ball around zero, and regard S as a Q-semianalytic subset in this ball. Consider a transformation $\varphi : M \longrightarrow B$ to the normal crossings:

$$f^\varphi := f \circ \varphi, \ g_1^\varphi := g_1 \circ \varphi, \ \dots \ , \ g_l^\varphi := g_l \circ \varphi,$$

and assume, for the sake of simplicity, that φ is a finite composition of definable blowings-up with global smooth Q-analytic centers (cf. [2, 3]). Then $E := \varphi^{-1}(0)$ is a compact Q-analytic subset of M.

Since every Q-analytic and every formal arc can be lifted when blowing up, $\widetilde{S} \neq \emptyset$ or, respectively, $\widehat{S} \neq \emptyset$, iff there exist a point $b \in E$ and a Q-analytic arc $\widetilde{\vartheta}(\tau)$ or, respectively, a formal arc $\widehat{\vartheta}(\tau)$, through b such that

$$f^\varphi(\widetilde{\vartheta}) = 0, \ g_1^\varphi(\widetilde{\vartheta}) > 0, \ \dots \ , \ g_l^\varphi(\widetilde{\vartheta}) > 0$$

or, respectively,

$$f^\varphi(\widehat{\vartheta}) = 0, \ g_1^\varphi(\widehat{\vartheta}) > 0, \ \dots \ , \ g_l^\varphi(\widehat{\vartheta}) > 0.$$

But the existence of such arcs $\widetilde{\vartheta}(\tau)$ and $\widehat{\vartheta}(\tau)$ is equivalent, because the functions $f^\varphi, g_1^\varphi, \dots, g_l^\varphi$ are normal crossings at each point $b \in E$. This completes the proof.

By virtue of Theorems 1 and 2, Proposition 2 yields the following version of the Artin–Lang property:

Theorem 4. *For* $f, g_1, \dots, g_m \in Q_n$, *put*

$$\widehat{S}_f := \{\sigma \in \operatorname{Sper} \mathbb{R}[[x]] : \ \gamma(0) = 0, \ f(\gamma) = 0, \ g_1(\gamma) > 0, \ \dots \ , \ g_l(\gamma) > 0\},$$

and let S *be the set germ at* $0 \in \mathbb{R}^n$ *determined by the conditions*

$$f(x) = 0, \ g_1(x) > 0, \ \dots, \ g_m(x) > 0.$$

Then $\widehat{S}_f = \emptyset$ *iff* $S = \emptyset$.

Remark. In the case of function germs in two variables, $n = 2$, this result was proven by Pieroni (cf. [13], Section 9) under the additional assumption that the quasianalytic local ring Q_2 is closed under division, i.e. if $f, g \in Q_2$ are such that $f = gh$ with a smooth function germ h, then $h \in Q_2$.

On the other hand, for smooth, non-flat function germs in several variables $x = (x_1, \dots, x_n)$, Broglia–Pernazza [5] proved a special case of the above result where the function germ f, which occurs in the equation defining S, is an analytic one.

From the definition of geometric spectrum, we immediately obtain a weak quasianalytic version of the Artin–Lang property:

Corollary. *Take $f, g_1, \dots, g_m \in Q_n$. Then*

$$\sigma(f) = 0, \ \sigma(g_1) > 0, \ \dots \ , \ \sigma(g_m) > 0,$$

for some $\sigma \in \operatorname{Speg} Q_n$ iff the set germ

$$S := \{x \in \mathbb{R}^n : \ f(x) = 0, \ g_1(x) > 0 \ \dots \ , \ g_l(x) > 0\} \neq \emptyset$$

is non-empty.

5. Łojasiewicz radical as the contraction of real radical. We first establish the following

Proposition 3. *Let $I \subset Q_n$ be an ideal, and $V := V(I)$ and $\widehat{V} := \widehat{V}(I)$ be their zero and formal zero loci, respectively. Then the zero ideals $\widehat{I}(V)$ and $\widehat{I}(\widehat{V})$ coincide: $\widehat{I}(V) = \widehat{I}(\widehat{V})$.*

As for Proposition 2, the proof relies again on transformation to normal crossings by blowing up. By Corollary 4 to Theorem 1, we may assume that the ideal I is principal, say, $I = (f)$ with f being a function Q-analytic in a ball B around $0 \in \mathbb{R}^n$. Consider a transformation $\varphi : M \longrightarrow B$ to the normal crossing $f^{\varphi} := f \circ \varphi$, which is a finite composition of definable blowings-up with global smooth Q-analytic centers. As before, $E := \varphi^{-1}(0)$ is a compact Q-analytic subset of M and, for any $g \in \mathbb{R}[[x]]$, we have the equvalences:

$g \in \widehat{I}(V)$ iff for every Q-analytic arc $\gamma(\tau)$ through zero

$$f \circ \gamma = 0 \ \Rightarrow \ g \circ \gamma = 0$$

iff for each point $b \in E$ and every Q-analytic arc $\vartheta(\tau)$ through b

$$f^\varphi \circ \gamma = 0 \;\Rightarrow\; g^\varphi \circ \gamma = 0.$$

Similarly, $g \in \widehat{I}(\widehat{V})$ iff for every formal arc $\gamma(\tau)$ through zero

$$f \circ \gamma = 0 \;\Rightarrow\; g \circ \gamma = 0$$

iff for each point $b \in E$ and every formal arc $\vartheta(\tau)$ through b

$$f^\varphi \circ \gamma = 0 \;\Rightarrow\; g^\varphi \circ \gamma = 0.$$

Since f^φ is a normal crossing at each point $b \in E$, it is of the form $f^\varphi(y) = u(y) \cdot y^\beta$ in local coordinates near b with $y(b) = 0$, where $u(y)$ is a Q-analytic unit at b, $u(b) \neq 0$, and $\beta \in \mathbb{N}^n$. Therefore, both the last conditions in the above two sequences of equivalences mean the same, namely, that g^φ vanishes on the hypersurface y^β near b. Consequently, $g \in \widehat{I}(V)$ iff $g \in \widehat{I}(\widehat{V})$, as asserted.

Corollary. *Put*

$$I(V) := \{g \in Q_n : \; g \circ \gamma = 0 \;\; \textit{for every } \gamma \in V\}$$

and

$$I(\widehat{V}) := \{g \in Q_n : \; g \circ \gamma = 0 \;\; \textit{for every } \gamma \in \widehat{V}\}.$$

Then $I(V) = I(\widehat{V})$.

We conclude this paper with a theorem on Łojasiewicz radical, stated below. It follows directly from Proposition 3 and the quasianalytic and formal versions of the Nullstellensatz.

Theorem 5. *If I is an ideal of the ring Q_n, then*

$$\pounds(I) = \Re(I \cdot \mathbb{R}[[x]]) \cap Q_n.$$

Indeed, with the foregoing notation, we get the following equalities:
$I(V(I)) = \pounds(I)$ — by Proposition 1;
$\widehat{I}(\widehat{V}(I)) = \Re(I \cdot \mathbb{R}[[x]])$ — by the corollary to Theorem 2;
$\widehat{I}(\widehat{V}(I)) = \widehat{I}(V(I))$ — by Proposition 3.

Hence

$$\pounds(I) = \widehat{I}(V(I)) \cap Q_n = \widehat{I}(\widehat{V}(I)) \cap Q_n = \Re(I \cdot \mathbb{R}[[x]]) \cap Q_n,$$

which completes the proof.

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