# Hyperbolic dynamics in graph-directed IFS

T. Kułaga J. Tabor

IM UJ Preprint 2010/09

**Abstract**

A *cone space* is a complete metric space $(X, d)$ with a pair of functions $c_s, c_u : X \times X \to \mathbb{R}$, such that there exists $K > 0$ satisfying

$$\frac{1}{K} d(x, x') \le \max(c_s(x, x'), c_u(x, x')) \le K d(x, x') \quad \text{for } x, x' \in X.$$

For a partial map $f$ between cone spaces $X$ and $Y$ we introduce $|f|_s$ which measures the stable contraction rate and $\langle f \rangle_u$ which measures the unstable expansion rate. We say that $f$ is *cone-hyperbolic* if

$$|f|_s < 1 < \langle f \rangle_u.$$

Using cone field and graph-directed IFS we build an abstract metric model which decribes the dynamics of the hyperbolic-like systems. This allows to obtain estimations from below and above of the fractal dimension of the hyperbolic invariant set.

## 1 Introduction

Graph description of dynamical behaviour, and in particular graph-directed iterated function theory, is one of the most important and fruitful ideas in modern theory of dynamical systems [3, 5, 6, 8, 10, 13, 15, 18].

We generalize the notion of the graph-directed IFS from contracting to hyperbolic case. As a consequence we obtain a simple and applicable tool, based on the graph-directed IFS and cone condition, to characterize the dynamics of the hyperbolic-like systems in general metric spaces. It allows to estimate the dimension of the invariant set from above and below or show the Lipschitz semiconjugacy between the invariant set and the model graph-directed system. Before proceeding further let us show a direct consequence of our results on the classical Smale's horseshoe (see Theorem 6.1).

**Smale's horseshoe.** *Consider the horizontal* $H_1 = [-1,1] \times \left[-\frac{3}{4}, -\frac{1}{4}\right]$, $H_2 = [-1,1] \times \left[\frac{1}{4}, \frac{3}{4}\right]$ *and vertical* $V_1 = \left[-\frac{3}{4}, -\frac{1}{4}\right] \times [-1,1]$, $V_2 = \left[\frac{1}{4}, \frac{3}{4}\right] \times [-1,1]$ *strips. Let* $S = H_1 \cup H_2$

*Let* $f : S \to \mathbb{R}^2$ *be such that* $f_i := f|H_i$ *is an affine mapping,* $f_i(H_i) = V_i$ *and* $df_1 = \begin{bmatrix} 1/4 & 0 \\ 0 & 4 \end{bmatrix}$ *and* $df_2 = \begin{bmatrix} 1/4 & 0 \\ 0 & -4 \end{bmatrix}$.

*Let us take a Lipschitz function* $p : S \to \mathbb{R}^2$ *and consider* $g := f + p$. *If*

$$\|p\|_{\sup} < 1/4 \text{ and } \operatorname{lip}(p) < 1/4,$$

*then the dynamics of* $g$ *on* $\operatorname{inv}(g, S)$ *is conjugated to* $f$ *on* $\operatorname{inv}(f, S)$ *by a homeomorphism* $\Phi$, *such that the Hölder constant of* $\Phi$ *is less then* $\log_{1/4}(1/4 + \operatorname{lip}(p))$ *and Hölder constant of* $\Phi^{-1}$ *is less then* $\log_{1/4-\operatorname{lip}(p)}(1/4)$. *Moreover* [1],

$$\overline{\dim}_B(\operatorname{inv}(f_p, S)) \leq (\log_2(4 - \operatorname{lip}(p))^{-1} - (\log_2(1/4 + \operatorname{lip}(p))^{-1},$$

$$\dim_H(\operatorname{inv}(f_p, S)) \geq (\log_2(4 + \operatorname{lip}(p))^{-1} - (\log_2(1/4 - \operatorname{lip}(p))^{-1}.$$

*Observe that when* $\operatorname{lip}(p) \to 0$ *then the Hölder constant and the dimension converge to* 1.

To describe our ideas more precisely let us quote the Mauldin and Williams graph-directed generalization [4, Theorem 3.5] of the classical Moran Theorem [12] (for the original paper see [11]). Let $G = (V, E)$ be a directed graph (where $V$ denotes the set of vertices and $E$ the set of edges). Given an edge $e$ by $i(e)$ we denote its initial and by $t(e)$ its terminal vertex. By a path $\alpha = (\alpha_j)_{j \in J}$ in $G$ we denote a sequence of edges such that $t(\alpha_j) = i(\alpha_{j+1})$ for $j \in J$: $j + 1 \in J$.

We call $G$ a *contracting graph* if we are given a labeling function $S : E \to (0,1)$. If $G$ is a strongly connected contracting graph then we call $G$ a *Mauldin-Williams graph.* With every Mauldin-Williams graph we associate the *Mauldin-Williams dimension*, which is the unique $r := r_G(S) \in [0, \infty)$ such that there exist $(x_v)_{v \in V} \subset (0,1)$ satisfying

$$x_v = \sum_{e \in E:\, t(e) = v} (S_e)^r x_{i(v)} \quad \text{for } v \in V. \tag{1}$$

[1]We are particulary interested in these estimates since we have $\dim_H \leq \dim \leq \overline{\dim}_B$ for most reasonable fractal dimension dim.

By $\mathbb{Z}_-$ we understand $\{k \in \mathbb{Z} : k < 0\}$.

**Moran Theorem [4, Theorem 3.5].** *Let $G = (V, E)$ be a strongly connected graph, let $X_v$ be a bounded complete metric space for every $v \in V$, and let $F_e : X_{i(e)} \to X_{t(e)}$ be such that*

- $\mathrm{lip}(F_e) < 1$ *for every $e \in E$.*

*By $\mathrm{inv}^-(X_v)$ we denote the backward invariant set of $X_v$, that is the set of all points $x \in X_v$ for which there exists a path $\alpha = (\alpha_k)_{k\in\mathbb{Z}_-}$ in $G$, $t(\alpha_{-1}) = v$ and an $\alpha$-orbit $\mathrm{z} = (z_k)_{k\in\mathbb{Z}_-\cup\{0\}}$ ($F_{\alpha_k}(z_k) = z_{k+1}$) such that $\mathrm{z}_0 = x$.*

*Then for every $v \in V$*

- $\mathrm{inv}^-(X_v)$ *is compact;*
- $\overline{\dim}_B\left(\mathrm{inv}^-(X_v)\right) \leq r_G(\mathrm{lip}(F_e))$.

*Moreover, in general the above estimation cannot be improved.*

One of the disadvantages of the classical IFS theory is that it does not cope well with typical hyperbolic behaviour, while there exists a large and advanced theory dealing with the properties (in particular global and local dimension) of $C^1$ hyperbolic systems [1, 16].

We show that by a relatively simple adaptation of the cone condition [2, 8, 14, 18] to a metric case we can generalize the notion of classical graph-directed IFS to the "hyperbolic-like" case. This allows us to obtain estimations from above and below of the dimension of the invariant set of graph-directed IFS with hyperbolic structure.

Let us briefly describe the contents of the paper. In the next section we adapt the notion of a cone field to metric spaces by modifying the approach of S. Newhouse from [14]. By a *cone space* we understand a complete metric space $(X, d)$ with a pair of functions $c_u, c_s : X \times X \to \mathbb{R}$, such that there exists $K > 0$ satisfying

$$\frac{1}{K}d(x, x') \leq \max(c_s(x, x'), c_u(x, x')) \leq Kd(x, x') \quad \text{for } x, x' \in X.$$

Given a cone space $X$ we define the *stable and unstable cones* by the formula

$$\begin{aligned} C_s(X) &:= \{(x, x') : c_s(x, x') \geq c_u(x, x')\}, \\ C_u(X) &:= \{(x, x') : c_u(x, x') \geq c_s(x, x')\}. \end{aligned}$$

For a partial map $f$ between cone spaces $X$ and $Y$ we introduce

$$\begin{aligned} |f|_s &:= \sup_{(f(x),f(x'))\in C_s(Y)} \frac{c(f(x),f(x'))}{c(x,x')}, \\ \langle f\rangle_u &:= \inf_{(x,x')\in C_u(X)} \frac{c(f(x),f(x'))}{c(x,x')}. \end{aligned}$$

We call $|f|_s$ the *s-contraction rate* and $\langle f\rangle_u$ the *u-expansion rate*[2]. Roughly speaking $|f|_s$ measures the contraction rate on the stable cone while $\langle f\rangle_u$ the expansion rate on the unstable one. We say that $f$ is *cone-hyperbolic* if

$$|f|_s < 1 < \langle f\rangle_u.$$

In 3 section we establish some notation related to the graph-directed IFS. Sections 4 and 5 contain main theorems of the paper. Let us present one of the results of section 5 (see Corollary 5.1). Given a directed graph $G=(V,E)$ and $e\in E$, by $e^{-1}$ we denote the inversed edge (that is $i(e^{-1})=t(e)$, $t(e^{-1})=i(e)$). By $G^{-1}=(V,E^{-1})$ we denote the graph with the same vertices and inversed edges.

**Hyperbolic Moran Theorem.** *Let $G=(V,E)$ be a strongly connected graph, let $X_v$ be a bounded cone space for every $v\in V$, and let $F_e$ be a partial map with a closed graph between $X_{i(e)}$ and $X_{t(e)}$ such that*

- *$F_e$ is cone-hyperbolic for every $e\in E$.*

*By* $\mathrm{inv}(X_v)$ *we denote the invariant set of $X_v$, that is the set of all points $x\in X_v$ for which there exists a doubly infinite path $\alpha=(\alpha_k)_{k\in\mathbb{Z}}$ in $G$ and an $\alpha$-orbit* $\mathrm{z}=(z_k)_{k\in\mathbb{Z}}$ *such that* $\mathrm{z}_0=x$.

*Then for every $v\in V$*

- *$inv_v(X_v)$ is compact;*
- $\overline{\dim}_B\,(\mathrm{inv}(X_v)) \le r_G(|F_e|_s) + r_{G^{-1}}(\langle F_e\rangle_u^{-1})$.

*Moreover, in general the above estimation cannot be improved.*

In the last section we present an application of our tools on an example of the Smale's horseshoe with a Lipschitz perturbation.

[2] These constants correspond to minimal expansion and the minimal co-expansion used by S. Newhouse [14].

## 2 Cone fields in metric spaces

Let $(X, d)$ be a metric space. It is often convenient to modify the original metric $d$ to some other function $c : X \times X \to \mathbb{R}_+$. In our case we just need the single assumption on $c$ that there exists $K > 0$ such that

$$\frac{1}{K}d(x, x') \le c(x, x') \le Kd(x, x') \quad \text{for } x, x' \in X.$$

From now on we assume that on every metric space we have an additional function $c$ which satisfies the above condition (if we are not given $c$ directly we simply take $c = d$).

For an interval $J \subset \mathbb{R}$ we define $J_{\mathbb{Z}} := J \cap \mathbb{Z}$. We say that $I \subset \mathbb{Z}$ is a $\mathbb{Z}$-interval if there exists an interval $J \subset \mathbb{R}$ such that $I = J_{\mathbb{Z}}$. For a $\mathbb{Z}$-interval we put $I^+ := I \cup (I + 1)$.

Given metric spaces $X, Y$ and a partial map $f : X \rightharpoonup Y$ we define the Lipschitz and the co-Lipschitz constants of $f$ (with respect to the function $c$) by the formula

$$\begin{aligned} |f| &:= \inf \{R \in [0, \infty] : c(f(x), f(x')) \le R \cdot c(x, x') \text{ for } x, x' \in \mathrm{dom} f\}, \\ \langle f \rangle &:= \sup \{R \in [0, \infty] : c(f(x), f(x')) \ge R \cdot c(x, x') \text{ for } x, x' \in \mathrm{dom} f\}. \end{aligned}$$

**Observation 2.1.** *Let $E$ and $F$ be Banach spaces and let $A : E \to F$ be an invertible linear operator. Let $U \subset E$, $p : U \to F$ and let $g : U \to F$ be defined by*

$$g(x) := Ax + p(x) \quad \text{for } x \in U.$$

*We put $c(x, x') = \|x - x'\|$. Then one can easily notice that*

$$\langle g \rangle \ge \|A^{-1}\|^{-1} - \mathrm{lip}(g), \ |g| \le \|A\| + \mathrm{lip}(g).$$

Let $f : Y \rightharpoonup Z$ and $g : X \rightharpoonup Y$. Then

$$|f \circ g| \le |f| \cdot |g|, \ \langle f \circ g \rangle \ge \langle f \rangle \cdot \langle g \rangle. \tag{2}$$

**Remark 2.1.** Consider mappings $f_i : X_i \rightharpoonup X_{i+1}$ for $i \in I = [k, n)_{\mathbb{Z}}$ where $k, n \in Z, k < n$. Let $(x_i)_{i \in I^+}$, $(x'_i)_{i \in I^+}$ be such that:

$$x_i, x'_i \in \mathrm{dom} f_i, x_{i+1} = f_i(x_i), \ x'_{i+1} = f_i(x'_i) \quad \text{for } i \in I.$$

Then obviously

$$d(x_n, x'_n) \le K^2 \cdot |f_{n-1}| \cdot \ldots \cdot |f_k| \cdot d(x_k, x'_k).$$

In this section we generalize the notion of a cone-field to metric spaces (our aim is to obtain the analogue of Remark 2.1). To estimate the distance between orbits from above and below in the case when the given map has hyperbolic-like structure we need an additional structure of a cone field. We adapt some of the notation and ideas from [14].

**Definition 2.1.** Let $(X, d)$ be a complete metric space. By a *cone field* on $X$ we understand a pair of functions $c_s, c_u : X \times X \to \mathbb{R}_+$, such that there exists $K > 0$ satisfying

$$\frac{1}{K} d(x, y) \leq c(x, y) \leq K d(x, y) \quad \text{for } x, y \in X,$$

where $c(x, y) := \max(c_s(x, y), c_u(x, y))$. In this case we call $X$ a *cone metric space* (cone space shortly).

Given a cone space $X$ we introduce the cones $C_s(X)$ and $C_u(X)$ by the formula:

$$\begin{aligned} C_s(X) &:= \{(x, x') \in X \times X : c_s(x, y) \geq c_u(x, x')\}, \\ C_u(X) &:= \{(x, x') \in X \times X : c_u(x, x') \geq c_s(x, x')\}. \end{aligned}$$

**Definition 2.2.** For $f : X \rightharpoonup Y$ we define

$$\begin{aligned} |f|_s &:= \inf \{R \in [0, \infty] \ : \ c(f(x), f(x')) \leq R \cdot c(x, y) \\ &\qquad\qquad \text{for } x, x' \in \mathrm{dom} f : (f(x), f(x')) \in C_s(Y)\}, \\ \langle f \rangle_u &:= \sup \{R \in [0, \infty] \ : \ c(f(x), f(x')) \geq R \cdot c(x, x') \\ &\qquad\qquad \text{for } x, x' \in \mathrm{dom} f : (x, x') \in C_u(X)\}. \end{aligned}$$

We call $|f|_s$ the $s$-contraction rate and $\langle f \rangle_u$ the $u$-expansion rate.

In the following we give an estimate of $|f|_s$ and $\langle f \rangle_u$.

**Observation 2.2.** *We assume that we have two Banach spaces $E = E_s \oplus E_u$ and $F = F_s \oplus F_u$. For each $x \in E$ we have $x = x_s + x_u$ where $x_s \in E_s$, $x_u \in E_u$ and as functions $c_s$ and $c_u$ we take*

$$c_s(x, x') := \|x_s - x'_s\|, \quad c_u(x, x') := \|x_u - x'_u\|, \quad x, x' \in E.$$

*The same holds for $F$. Additionally we assume that both norms satisfy $\|x\| = \max(\|x_s\|, \|x_u\|)$.*

*Let $A : E \to F$ be a linear operator given in the matrix form by*

$$A = \left[ \begin{array}{cc} A_s & A_{su} \\ A_{us} & A_u \end{array} \right].$$

*and let $p = (p_s, p_u) : U \to F_s \oplus F_u$, where $U \subset E$, be a given Lipschitz mapping. Let $g : U \to F$, $g(x) := Ax + p(x)$. Then*

$$\begin{aligned} |g|_s \ &\leq \ \|A_s\| + \|A_{su}\| + \mathrm{lip}(p_s), && (3) \\ \langle g \rangle_u \ &\geq \ \|A_u^{-1}\|^{-1} - \|A_{us}\| - \mathrm{lip}(p_u). && (4) \end{aligned}$$

*Proof.* To prove (3) let us choose $x, x' \in U$ such that $(g(x), g(x')) \in C_s(F)$. Then

$$\begin{aligned} \|g(x) - g(x')\| \ &\leq \ \|A_s(x_s - x'_s)\| + \|A_{su}(x_u - x'_u)\| + \mathrm{lip}(p_s)\|x - x'\| \\ &\leq \ (\|A_s\| + \|A_{us}\| + \mathrm{lip}(p_s)) \cdot \|x - x'\|. \end{aligned}$$

We show (4). Fix $x, x' \in U$ such that $(x, x') \in C_u(E)$. Then $\|x - x'\| = \|x_u - x'_u\| \geq \|x_s - x'_s\|$, and consequently

$$\begin{aligned} \|g(x) - g(x')\| \ &\geq \ \|A_u(x_u - x'_u) + A_{us}(x_s - x'_s) + (p_u(x) - p_u(x'))\| \\ &\geq \ (\|A_u^{-1}\|^{-1} - \|A_{us}\| - \mathrm{lip}(p_u)) \cdot \|x - x'\|. \end{aligned}$$

$\square$

**Definition 2.3.** The function $f$ is *dominating* if

$$|f|_s < \langle f \rangle_u,$$

and *cone-hyperbolic* if

$$|f|_s < 1 < \langle f \rangle_u.$$

Trivially, a cone-hyperbolic mapping is dominating.

**Proposition 2.1.** *Let $f : X \rightharpoonup Y$ be dominating. Then $f$ is cone invariant, that is for $x, x' \in \mathrm{dom} f$ we have*

$$\begin{aligned} (x, x') \in C_u(X) \ &\Rightarrow \ (f(x), f(x')) \in C_u(Y), && (5) \\ (f(x), f(x')) \in C_s(Y) \ &\Rightarrow \ (x, x') \in C_s(X). && (6) \end{aligned}$$

*Proof.* Let $x, x' \in \mathrm{dom} f$ be such that $(x, x') \in C_u(X)$. Then

$$c(f(x), f(x')) \geq \langle f \rangle_u c(x, x').$$

We want to show that $(f(x), f(x')) \in C_u(Y)$. If this was not the case, then

$$(f(x), f(x')) \in C_s(Y),\ c(f(x), f(x')) > 0.$$

This gives us

$$0 < c(f(x), f(x')) \leq |f|_s c(x, x').$$

Consequently, $|f|_s > 0$ and $c(x, y) > 0$. Summarizing, we obtain that

$$|f|_s c(x, x') \geq c(f(x), f(x')) \geq \langle f \rangle_u c(x, x').$$

Since $c(x, x') > 0$ we get $|f|_s \geq \langle f \rangle_u$, a contradiction.

The proof for $f^{-1}$ is analogous. □

As an important consequence we get an analogue of (2).

**Theorem 2.1.** *Let $f : Y \rightharpoonup Z$ and $g : X \rightharpoonup Y$ be dominating. Then*

$$|f \circ g|_s \leq |f|_s \cdot |g|_s,\ \langle f \circ g \rangle_u \geq \langle f \rangle_u \cdot \langle g \rangle_u.$$

*Proof.* It is a direct consequence of the cone invariance (see (5) and (6)) and Definition 2.2. □

In the following we show that in the case of dominating functions we can estimate the rate of increase of distance between two orbits.

**Corollary 2.1.** *Consider dominating mappings $f_i : X_i \rightharpoonup X_{i+1}$ for $i \in I$, where $I = [k, n)_{\mathbb{Z}}$. Let $(x_i)_{i \in I^+}$, $(x'_i)_{i \in I^+}$ be such that:*

$$x_i, x'_i \in \mathrm{dom}(f_i), x_{i+1} = f_i(x_i),\ x'_{i+1} = f_i(x'_i) \quad \text{for } i \in I.$$

*Then for every $l \in I^+ = [k, n]_{\mathbb{Z}}$ we have:*

$$c(x_l, x'_l) \leq \max(\langle f_{n-1} \rangle_u^{-1} \cdot \ldots \cdot \langle f_l \rangle_u^{-1} \cdot c(x_n, x'_n), |f_{l-1}|_s \cdot \ldots \cdot |f_k|_s \cdot c(x_k, x'_k)).$$

*Proof.* Consider first the case when $(x_l, x'_l) \in C_u(X_l)$. Applying Proposition 2.1 we obtain that $(x_i, x'_i) \in C_u(X_i)$ for $i = l, \ldots, n$. By Theorem 2.1 we get

$$c(x_n, x'_n) \geq \langle f_{n-1} \rangle_u \cdot \ldots \cdot \langle f_l \rangle_u \cdot c(x_l, x'_l),$$

and consequently

$$c(x_l, x'_l) \leq \langle f_{n-1} \rangle_u^{-1} \cdot \ldots \cdot \langle f_l \rangle_u^{-1} \cdot c(x_n, x'_n).$$

Now let us discuss the case when $(x_l, x'_l) \in C_s(X_l)$. By Proposition 2.1 we have $(x_i, x'_i) \in C_u(X_i)$ for $i = k, \ldots, l$, and therefore

$$c(x_l, x'_l) \leq |f_{l-1}|_s \cdot \ldots \cdot |f_k|_s \cdot c(x_k, x'_k).$$

□

# 3 Graph notation

Let $G = (V, E)$ be a directed graph. Given a (possibly empty) $\mathbb{Z}$-interval $J$ we consider the set of paths in $G$

$$E(J, G) := \{\alpha : J \to E : t(\alpha_j) = i(\alpha_{j+1}) \text{ for } j \in J : j + 1 \in J\}.$$

If $G$ is fixed, we usually omit it and write $E(J)$. By $E(G)$ (or simply $E()$) we denote the set of all paths in the graph $G$. For a $\mathbb{Z}$-interval $I$ such that $0 \in I^+$ and $v \in V$ we define

$$E_v(I, G) := \{\alpha \in E(I, G) \;:\; t(\alpha_{-1}) = v \text{ or } i(\alpha_0) = v\}.$$

We also put $E_v(G) := \bigcup_{I :\, 0 \in I^+} E_v(I, G)$.

**Definition 3.1.** Let $G = (V, E)$ be a given directed graph. By a *graph $G$-directed iterated function system* (or $G$-graph system shortly) we understand labeling functions $X$ and $F$ over vertices and edges of $G$: every vertex $v$ is labeled by a complete metric space $X_v$, and every edge $e$ is labeled with a partial function $F_e : X_{i(e)} \rightharpoonup X_{t(e)}$ with a closed graph[3]. To denote the whole $G$-graph system we usually write $(V, v \to X_v; E, e \to F_e)$ (in that case we speak simply of a graph system).

[3]In fact to shorten the notation we informally allow $F_e$ to have a larger domain then $X_{i(e)}$ or image not contained in $X_{t(e)}$ and then restrict automatically $F_e$ to $F_e \cap (X_{i(e)} \times X_{t(e)})$

Let us explain why we assume in the definition that every $F_e$ has a closed graph.

**Observation 3.1.** *Let $\alpha \in E(G, I)$ be a given path in $G$ and let $\mathrm{z}^n = (\mathrm{z}_i^n)_{i \in I^+}$ be a pointwise convergent (to some $\mathrm{z} = (\mathrm{z}_i)_{i \in I^+}$) sequence of $\alpha$-orbits. Then by the fact that $F_e$ has a closed graph we obtain that $\mathrm{z}$ is also an $\alpha$-orbit.*

We say that a graph system $\Gamma$ is *contracting* if $|F_e| < 1$ for every $e \in E$. Since we modify the standard approach let us now briefly show how one usually proves the Moran Theorem [4, Theorem 3.5]. The idea is based on building an abstract graph system and using the semiconjugacy.

We define the left shift operator $P$ on $E()$, where $P(E(I)) = E(I-1)$, by the formula

$$(P\alpha)_k := \alpha_{k+1} \quad \text{for } \alpha \in E(I), k \in I - 1.$$

Given $\alpha, \alpha' \in E(G)$, we define a two-sided analogue of the longest common prefix

$$\alpha \wedge \alpha' := (\alpha \cap \alpha')|_I \in E(G),$$

where

$$I := \bigcup \{J \mid J \text{ is } \mathbb{Z}\text{-interval}, \, 0 \in J^+, \alpha|_J = \alpha'|_J\}.$$

Given a labeling function $T : E \to [0, 1]$ we naturally extend it to the space of all paths by

$$T(\alpha) := \begin{cases} \prod\limits_{i \in I} T(\alpha_i) \text{ if } I \neq \emptyset, \\ 1 \text{ otherwise,} \end{cases}$$

for $\alpha \in E(I)$.

**Definition 3.2.** Let $\alpha \in E(I)$. We say that $\mathrm{z} : I^+ \to \bigcup_{v \in V} X_v$ is an *$\alpha$-orbit* if $\mathrm{z}_i \in \mathrm{dom} F_{\alpha_i}$ for $i \in I$ and

$$F_{\alpha_i}(\mathrm{z}_i) = \mathrm{z}_{i+1} \quad \text{for } i \in I.$$

The set of all $\alpha$-orbits we denote by $\mathrm{orb}(\alpha)$.

Let $v \in V$ and $I$ be such that $0 \in I^+$. Then for $\alpha \in E_v(I)$ we define its coding multimap

$$\mathcal{C}_v^I(\alpha) := \{x \in X_v \mid \exists \mathrm{z} \in \mathrm{orb}(\alpha) : \, \mathrm{z}_0 = x\}.$$

Dually, given $x \in X_v$, we denote its $I$-address multimap by

$$\mathcal{A}_v^I(x) := \{\alpha \in E_v(I) \,|\, \exists \mathrm{z} \in \mathrm{orb}(\alpha) : \mathrm{z}_0 = x\}.$$

If $I = \mathbb{Z}$ then we simply write $\mathcal{A}_v(x)$, $\mathcal{C}_v(\alpha)$. One can easily observe that $\mathcal{C}_v^I$ and $\mathcal{A}_v^I$ are inverse multimaps. Now we are ready to define the *invariant set* for $X_v$ (we assume that $0 \in I^+$):

$$\mathrm{inv}^I(X_v) := \{x \in X_v \,|\, \mathcal{A}_v^I(x) \neq \emptyset\}.$$

If $I = \mathbb{Z}$ then we simply write $\mathrm{inv}(X_v)$.

**Definition 3.3.** Let $I$ be a $\mathbb{Z}$-interval such that $0 \in I^+$ and let $\Gamma = (V, v \to X_v; E \to F_e)$ be a graph system. For $e \in E$ we put

$$F_e^I := F_e \cap (\mathrm{inv}^I(X_{i(e)}) \times \mathrm{inv}^I(X_{t(e)})), \tag{7}$$

and define the graph system

$$\mathrm{inv}^I(\Gamma) := (V, v \to \mathrm{inv}^I(X_v); E, e \to F_e^I).$$

In the case when $I = \mathbb{Z}$ we simply write $\mathrm{inv}(\Gamma)$.

# 4 Metric hyperbolic case

With the use of the metric one can estimate the Hausdorff dimension of the invariant set from below. To do this, we will need a hyperbolic equivalent of a Mauldin-Williams graph:

**Definition 4.1.** Let $G = (V, E)$ be a directed graph. We say that $G$ is *hyperbolic* if we are given two labeling functions $S, U : E \to (0, \infty)$ such that

$$S_e \in (0, 1),\, U_e \in (1, \infty) \quad \text{for } e \in E.$$

We say that $G$ is a *hyperbolic Mauldin-Williams graph* if $G$ is a strongly connected hyperbolic graph.

For $\alpha \in E(\mathbb{Z})$ we put $\alpha_- = \alpha|_{\mathbb{Z}_-}$, $\alpha_+ = \alpha|_{\mathbb{N}}$. Let us observe that we have natural isomorphisms:

$$E_v(\mathbb{Z}, G) \ni \alpha \to (\alpha_-, \alpha_+) \in E_v(\mathbb{Z}_-, G) \times E_v(\mathbb{N}, G),$$

$$E_v(\mathbb{N},G) \ni \alpha \to \alpha^{-1} \in E_v(\mathbb{Z}_-,G^{-1}),$$
$$E_v(\mathbb{Z},G) \ni \alpha \to (\alpha_-,\alpha_+^{-1}) \in E_v(\mathbb{Z}_-,G) \times E_v(\mathbb{Z}_-,G), \tag{8}$$

where $(\alpha^{-1})_k := (\alpha_{-1-k})^{-1}$. In the following we generalize the abstract contracting graph construction from the previous section to the hyperbolic case. One can easily verify that the following construction is correct:

**Model hyperbolic graph construction.** *Let $G$ be a hyperbolic graph. We define a graph system $\Gamma_G[S,U]$ by:*

- *we label every $v \in V$ with the space $\mathbf{X}_v := E_v(\mathbb{Z})$;*
- *for every $e \in E$ we consider the partial map $P_e : \mathbf{X}_{i(e)} \rightharpoonup \mathbf{X}_{t(e)}$ which is the restriction of the left shift $P$ to $\{\alpha \in E(\mathbb{Z}) : \alpha_0 = e\}$;*
- *we define the cone structure and complete metric $\rho_S^U$ on $\mathbf{X}_v$:*

$$c_S(\alpha,\alpha') := S(\alpha_- \wedge \alpha'_-),\; c_U(\alpha,\alpha') := (1/U)(\alpha_+ \wedge \alpha'_+) \quad \text{for } \alpha,\alpha' \in E_v(\mathbb{Z}),$$

$$\rho_S^U(\alpha,\alpha') := \max(c_S(\alpha,\alpha'), c_U(\alpha,\alpha')).$$

- *we have*

$$|P_e|_s = S_e,\; \langle P_e \rangle_u = U_e \quad \text{for } e \in E. \tag{9}$$

In the contracting case we have the following.

**Dimension Theorem [3, Theorem 6.4.2].** *Let $G$ be a Mauldin-Williams graph and let $r := r_G(S)$ denote the Mauldin-Williams dimension of $G$ (see* (1)*). Then*

$$\dim_H(E_v(\mathbb{Z}_-)) = \underline{\dim}_B(E_v(\mathbb{Z}_-)) = \overline{\dim}_B(E_v(\mathbb{Z}_-)) = r,$$

*and $\mathcal{H}^r(E_v(\mathbb{Z}_-)) \in (0,\infty)$, where in $E_v(\mathbb{Z}_-)$ we take the metric $\rho_S$ defined as $\rho_S(\alpha,\alpha') := S(\alpha \wedge \alpha')$. The space $(E_v(\mathbb{Z}_-),\rho_S)$ is a compact and complete metric space.*

**Observation 4.1.** *Let us observe that* (8) *induces a natural isometry*

$$(E_v(\mathbb{Z},G),\rho_S^U) \approx (E_v(\mathbb{Z}_-,G),\rho_S) \times (E_v(\mathbb{Z}_-,G^{-1}),\rho_{1/U}).$$

*Consequently, $(E_v(\mathbb{Z}),\rho_S^U)$ is a compact and complete metric space.*

Modifying of the standard argument (see [4]) from one-sided to two-sided case one can get the following.

**Hyperbolic Dimension Theorem.** *Let $G$ be a hyperbolic Mauldin-Williams graph and let $r := r_G(S) + r_{G^{-1}}(1/U)$. Then*

$$\dim_H(E_v(\mathbb{Z})) = \underline{\dim}_B(E_v(\mathbb{Z})) = \overline{\dim}_B(E_v(\mathbb{Z}) = r,$$

*and $\mathcal{H}^r(E_v(\mathbb{Z})) \in (0, \infty)$, where in $E_v(\mathbb{Z})$ we take the metric $\rho_S^U$.*

To proceed further we need notions of semiconjugacy between two graph systems.

**Definition 4.2.** Let $\Gamma$ and $\Gamma'$ be two $G$-graph systems. We say that a sequence of surjections $\Phi_v : X_v \to X'_v$ is a *semiconjugacy* between $\Gamma$ and $\Gamma'$ if

$$F'_e \circ \Phi_{i(e)} = \Phi_{t(e)} \circ F_e \quad \text{for } e \in E.$$

If all the functions are homeomorphisms then the sequence $(\Phi_v)_{v \in V}$ is called a *conjugacy.*

Now we are ready to formulate the main result of this section. We recall that $F_e^{\mathbb{Z}} = F_e \cap (\mathrm{inv}(X_{i(e)}) \times \mathrm{inv}(X_{t(e)}))$ (see Definition 3.3).

**Theorem 4.1.** *Let $\Gamma$ be a graph system. We assume that*

- *there exists $\varepsilon > 0$ such that for $e, e' \in E, e \neq e'$*

$$t(e) = t(e') \;\Rightarrow\; \mathrm{dist}_{t(e)}(\mathrm{im}F_e^{\mathbb{Z}}, \mathrm{im}F_{e'}^{\mathbb{Z}}) \geq \varepsilon, \tag{10}$$

$$i(e) = i(e') \;\Rightarrow\; \mathrm{dist}_{i(e)}(\mathrm{dom}F_e^{\mathbb{Z}}, \mathrm{dom}F_{e'}^{\mathbb{Z}}) \geq \varepsilon; \tag{11}$$

- *$S_e := \langle F_e \rangle \in (0, 1)$, $U_e := |F_e| \in (1, \infty)$ for every $e \in E$;*
- *$\mathrm{orb}(\alpha) \neq \emptyset$ for every $\alpha \in E(\mathbb{Z})$.*

*Then the maps $\mathcal{A}_v : \mathrm{inv}(X_v) \to E_v(\mathbb{Z})$ give a Lipschitz semiconjugacy between $\mathrm{inv}(\Gamma)$ and $\Gamma_G[S, U]$.*

*Proof.* By the definition $\mathcal{A}_v(x) \neq \emptyset$ for $x \in \mathrm{inv}(X_v)$. Moreover, by the assumptions we know that $\mathrm{orb}(\alpha) \neq \emptyset$ for every $\alpha \in E(\mathbb{Z})$, which implies that $\mathcal{A}_v(\mathrm{inv}(X_v)) = E_v(\mathbb{Z})$.

Let us now show that $\mathcal{A}_v : \mathrm{inv}(X_v) \to E_v(\mathbb{Z})$ is a well-defined single-valued map. Let $x, x' \in \mathrm{inv}(X_v)$ and $\alpha \in \mathcal{A}_v(x)$, $\alpha' \in \mathcal{A}_v(x')$ be arbitrarily chosen. We show that

$$\rho_S(\alpha_-, \alpha'_-) \leq \frac{K^2}{\varepsilon} d(x, x'),\ \rho_{1/U}(\alpha_+, \alpha'_+) \leq \frac{K^2}{\varepsilon} d(x, x')$$

We prove the first inequality (the second is analogous). It is enough to consider the case when $\alpha_- \neq \alpha'_-$. Let $k \in \mathbb{Z}_-$ be such that

$$\alpha_k \neq \alpha'_k,\ \alpha_i = \alpha'_i \quad \text{for } i \in \mathbb{Z}_-, i > k.$$

Let $\mathrm{z}$ be an $\alpha$-orbit such that $\mathrm{z}_0 = x$, and $\mathrm{z}'$ be an $\alpha'$-orbit such that $\mathrm{z}'_0 = x'$. By the assumption (10) we conclude that

$$d(\mathrm{z}_k, \mathrm{z}'_k) \geq \varepsilon.$$

Consequently

$$d(x, x') = d(\mathrm{z}_0, \mathrm{z}'_0) \geq \frac{1}{K} c(\mathrm{z}_0, \mathrm{z}'_0) \geq \prod_{i=k}^{-1} \langle F_{\alpha_i} \rangle \cdot \frac{1}{K} c(\mathrm{z}_k, \mathrm{z}'_k) \geq \frac{\varepsilon}{K^2} \rho_S(\alpha_-, \alpha'_-).$$

Thus $\rho_S^U(\alpha, \alpha') \leq \frac{K^2}{\varepsilon} d(x, x')$ which implies that $\mathcal{A}_v$ is a single valued and Lipschitz map. $\square$

As a direct consequence of the above theorem and the Hyperbolic Dimension Theorem we get:

**Corollary 4.1.** *Assume additionally (to assumptions of Theorem 4.1) that $G$ is a strongly connected graph. Let $r = r_G(S) + r_{G^{-1}}(1/U)$. Then*

$$\mathcal{H}^r(\mathrm{inv}(X_v)) > 0 \quad \textit{for } v \in V.$$

# 5 Cone-hyperbolic graph

We show that cone graph system (graph system in which every space $X_v$ is a cone space) under some additional assumptions is conjugated to the model hyperbolic graph.

Let us begin with a direct consequence of Corollary 2.1.

**Proposition 5.1.** *Let $\Gamma$ be a cone graph system such that $F_e$ is dominating for every $e \in E$. We assume that*

$$C = \max_{X \in V} \operatorname{diam}(X_v) < \infty.$$

*Let $v \in V$ be fixed and let $\alpha, \alpha' \in E_v()$, and $\mathrm{z} \in \operatorname{orb}(\alpha)$, $\mathrm{z}' \in \operatorname{orb}(\alpha')$. Then*

$$d_v(\mathrm{z}_0, \mathrm{z}'_0) \leq K^2 C \max(|F(\alpha_- \wedge \alpha'_-)|_s, \langle F(\alpha_+ \wedge \alpha'_+)\rangle_u^{-1}).$$

We say that a cone graph system $\Gamma = (V, v \to X_v; E, e \to F_e)$ is *hyperbolic* if

$$|F_e|_s < 1 < \langle F_e \rangle_u \quad \text{for } e \in E.$$

**Theorem 5.1.** *Let $\Gamma = (V, v \to X_v; E, e \to F_e)$ be a hyperbolic cone graph system such that*

$$C := \max_{v \in V} \operatorname{diam}(X_v) < \infty.$$

*Let $S_e := |F_e|_s$, $U_e := \langle F_e \rangle_u$ and let*

$$\mathbf{X}_v := \{\alpha \in E_v(\mathbb{Z}) : \operatorname{orb}(\alpha) \neq \emptyset\}.$$

*Then*

- *the space $\mathbf{X}_v$ is a cone-space with cone field $c_S$, $c_U$ and the metric $\rho_S^U$;*
- *for every $e \in E$ the partial map $P_e : \mathbf{X}_{i(e)} \rightharpoonup \mathbf{X}_{t(e)}$ defined as the restriction of the left shift $P$ to $\{\alpha \in \mathbf{X}_{i(e)} : \alpha_0 = e\}$ satisfies*

$$|P_e|_s \leq S_e, \ \langle P_e \rangle_u \geq U_e \quad \textit{for } e \in E; \tag{12}$$

- *the maps $\mathcal{C}_v$ give a Lipschitz semiconjugacy between the hyperbolic graph system $(V, v \to (\mathbf{X}_v, \rho_S); E, e \to P_e)$ and $\operatorname{inv}(\Gamma)$;*
- $\operatorname{inv}(X_v)$ *is a compact subset of $X_v$.*

*Proof.* Let $\alpha, \alpha' \in \mathbf{X}_v$ and $\mathrm{z} \in \operatorname{orb}(\alpha)$, $\mathrm{z}' \in \operatorname{orb}(\alpha')$ be arbitrarily chosen. Directly from Proposition 5.1 we conclude that

$$d_v(\mathrm{z}_0, \mathrm{z}'_0) \leq C K^2 \rho_S^U(\alpha, \alpha'). \tag{13}$$

This implies that the map $\mathcal{C}_v : \mathbf{X}_v \to \operatorname{inv}(X_v)$ is a single-valued Lipschitz map (directly from the definition it is a surjection).

Let us now show that $\mathbf{X}_v$ is a cone space. Since $(E_v(\mathbb{Z};G), \rho_S^U)$ is a compact (and consequently complete) metric space, to show that $\mathbf{X}_v$ is complete it is enough to prove that it is a closed subset of $E_v(\mathbb{Z})$. So let $(\alpha^n)_{n\in\mathbb{N}} \subset \mathbf{X}_v$ be a sequence convergent to $\alpha \in E_v(\mathbb{Z})$. Our aim is to prove that $\alpha \in \mathbf{X}_v$, or in other words that $\mathrm{orb}(\alpha) \neq \emptyset$. For $n \in \mathbb{N}$ let $z^n \in \mathrm{orb}(\alpha^n)$ be arbitrarily chosen. Let us fix $j \in \mathbb{Z}$. Then by (13)

$$d_v(\mathrm{z}_j^k, \mathrm{z}_j^l) \leq CK^2 \rho_S^U(P^j\alpha^k, P^j\alpha^l) \to 0, \text{ as } k, l \to \infty.$$

Because spaces $\{X_u\}_{u\in V}$ are complete, we obtain that $\mathrm{z}_j^n \to \mathrm{z}_j$ for some $z_j \in X_{i(\alpha_j)}$. We are going to show that such defined z is an $\alpha$-orbit. Take $k \in \mathbb{N}$ such that $j \in [-k,k]_{\mathbb{Z}}$. The set $U_k := \{\beta \in E_v(\mathbb{Z}), \alpha_l = \beta_l, l \in [-k,k]_{\mathbb{Z}}\}$ is an open neighbourhood of $\alpha$ in $E_v(\mathbb{Z})$. This yields that there exists $n_k \in \mathbb{N}$ such that $\alpha^n \in U_k$ for $n \geq n_k$. Consequently

$$F_{\alpha_j}(z_j^n) = F_{\alpha_j^n}(z_j^n) = z_j^{n+1}, n \geq n_k.$$

Since each $F_e$, $e \in E$ has a closed graph we conclude that $F_{\alpha_j}(\mathrm{z}_j) = z_{j+1}$ for every $j \in \mathbb{Z}$, and therefore $\mathrm{z} \in \mathrm{orb}(\alpha)$, which implies that $\alpha \in \mathbf{X}_v$.

One can easily notice that (12) is a direct consequence of (9). Also $\mathrm{inv}(X_v)$ is compact as an image of a compact set $\mathbf{X}_v$ through the continuous map $\mathcal{C}_v$. □

As a direct corollary of the above theorem and Hyperbolic Dimension Theorem from Section 4 we obtain:

**Corollary 5.1.** *Let $G = (V,E)$ be a strongly connected graph, let $X_v$ be a bounded cone space for every $v \in V$, and let $F_e$ be a partial map with a closed graph between $X_{i(e)}$ and $X_{t(e)}$ such that*

- *$F_e$ is cone-hyperbolic for every $e \in E$.*

*Then for every $v \in V$*

- *$\mathrm{inv}_v(X_v)$ is compact;*
- $\overline{\dim}_B\,(\mathrm{inv}(X_v)) \leq r_G(|F_e|_S) + r_{G^{-1}}(\langle F_e\rangle_{1/U})$.

*Moreover, in general the above estimation cannot be improved.*

Theorem 5.1 has a disadvantage since it does not give a semiconjugacy with the model hyperbolic system we know well, but only with its subset. To obtain semiconjugacy we need an additional assumption.

**Corollary 5.2.** *Let all the assumptions of Theorem 5.1 hold. Then* $\mathrm{orb}(\alpha) \neq \emptyset$ *for every path* $\alpha \in E(\mathbb{Z})$ *if and only if*

$$\mathrm{orb}(\alpha) \neq \emptyset \textit{ for every finite path } \alpha \in E(). \tag{14}$$

*Consequently, if* (14) *holds then* $\mathbf{X}_v = E_v(\mathbb{Z})$ *for every* $v \in \mathbb{Z}$.

*Proof.* Let $\alpha \in E(\mathbb{Z})$ be fixed and let $\alpha^n := \alpha|_{[-n,n)_{\mathbb{Z}}}$. We choose $\mathrm{z}^n \in \mathrm{orb}(\alpha^n)$. By proceeding as in the proof of Theorem 5.1 one can easily prove that $\mathrm{z}_j^n \to z_j$ for every $j \in \mathbb{Z}$, and that $\{\mathrm{z}_j\}_{j\in\mathbb{Z}}$ is in fact an $\alpha$-orbit. □

**Remark 5.1.** Note that in the classical case of contracting graph-directed IFS condition (14) is automatically satisfied, while in the general hyperbolic case this condition is usually non-trivial.

In general, to verify (14) one needs some additional topological tools like covering relations [7, 19] which "work" for subsets of $\mathbb{R}^n$ or related analogues in general metric spaces [9, 17].

Now we are going to "summarize" the results of this and the previous section in one theorem.

**Theorem 5.2.** *Let* $\Gamma = (V, v \to X_v; E, e \to F_e)$ *be a cone graph system such that*

*i)* $\mathrm{diam}(X_v) < \infty$ *for every* $v \in V$*;*

*ii) for every* $e, e' \in E$, $e \neq e'$*:*

$$\mathrm{dom}F_e \cap \mathrm{dom}F_{e'} = \emptyset,\ \mathrm{im}F_e \cap \mathrm{im}F_{e'} = \emptyset; \tag{15}$$

*iii)* $F_e$ *is bi-Lipschitz and cone-hyperbolic for every* $e \in E$*;*

*iv)* $\mathrm{orb}(\alpha) \neq \emptyset$ *for every finite path* $\alpha \in E_v()$.

*Let* $\mathbf{X}_v := E_v(\mathbb{Z})$ *and let*

$$S'_e := \langle F_e \rangle, S_e := |F_e|_s,\ U_e := \langle F_e \rangle_u, U'_e := |F_e|.$$

*Then*

- $\mathcal{C}_v : (\mathbf{X}_v, \rho_S^U) \to (\mathrm{inv}(X_v), d_v)$ *is a Lipschitz surjection;*

- $\mathcal{A}_v : (\mathrm{inv}(X_v), d_v) \to (\mathbf{X}_v, \rho_{S'}^{U'})$ *is a Lipschitz surjection;*
- $\mathcal{C}_v$ *defines the conjugacy between graph systems* $\Gamma_G[S_e, U_e]$ *and* $\mathrm{inv}(\Gamma)$.

*Proof.* All we need to show is that the assumptions of Theorem 4.1 are satisfied.

By Theorem 5.1 we conclude that $\mathrm{inv}(X_v)$ is compact for every $v \in V$. This together with the fact that $F_e$ has a closed graph yields that the domain and the image of $F_e^{\mathbb{Z}}$ (see Definition 3.3) are compact sets. Finally (15) implies that the $\varepsilon$-disjointness assumption of Theorem 4.1 is satisfied.

By Corollary 5.2 we conclude that $\mathrm{orb}(\alpha) \neq \emptyset$ for every $\alpha \in E(\mathbb{Z})$. Thus all the assumptions of Theorem 4.1 are satisfied. □

As an easy consequence we obtain Hölder conjugacy.

**Corollary 5.3.** *Let all the assumptions of Theorem 5.2 hold. Suppose that we are given constants* $\bar{S}_e, \bar{U}_e$ *and* $\gamma \in (0, 1]$ *such that*

$$\bar{S}_e \in [\langle F_e \rangle^{1/\gamma}, |F_e|_s^{\gamma}],\ \bar{U}_e \in [\langle F_e \rangle_u^{\gamma}, |F|^{1/\gamma}] \quad \text{for } e \in E.$$

*Then the graph systems* $\Gamma_G[\bar{S}, \bar{U}]$ *and* $\mathrm{inv}(\Gamma)$ *are Hölder conjugate, where the conjugacy* $\mathcal{C}_v$ *and its inverse* $\mathcal{A}_v$ *are Hölder continuous with Hölder constant* $\gamma$.

# 6 Smale's horseshoe

Our aim is to show an application of Theorem 5.2 on a relatively simple example. Before that let us comment on the assumptions of Theorem 5.2. Conditions i) and ii) are quite easy to verify using direct computations or interval arithmetics approach. To show iii) one can use estimations obtained in Observations 2.1 and 2.2. Assumption iv) can be checked by the covering relations argument [7, 19]. Let us explain the main idea behind this notion in a simplified $\mathbb{R}^2$ case.

By an $h$-set we denote the set $A \subset \mathbb{R}^2$ and the homeomorphism $h : [0, 1]^2 \to A$. Having two $h$-sets $A_1$, $A_2$ and a continuous map $f : A_1 \to \mathbb{R}^2$ we say that $A_1$ $f$-covers $A_2$ ($A_1 \overset{f}{\Rightarrow} A_2$) if the following conditions are satisfied

- $(h_2^{-1} \circ f \circ h_1)([0, 1]^2) \subset [0, 1] \times \mathbb{R}$,
- $(h_2^{-1} \circ f \circ h_1)([0, 1] \times \{0\})$ is below $[0, 1] \times \{0\}$,

- $(h_2^{-1} \circ f \circ h_1)([0,1] \times \{1\})$ is above $[0,1] \times \{1\}$.

**Covering Relations Theorem [19, Theorem 4].** For a sequence of $h$-sets $(A_i)_{i=1}^{n+1} \subset \mathbb{R}^2$ and continuous functions $f_i : A_i \to \mathbb{R}^2$ such that $A_i \overset{f_i}{\Rightarrow} A_{i+1}$ there exists a sequence of points $(x_i)_{i=1}^{n+1}$ such that

$$x_i \in A_i,\ x_{i+1} = f_i(x_i), \quad \text{for } i = 1, \dots, n.$$

**Example 6.1.** We consider a modified linear horsheshoe based on [1, Section 6.1.3]. Take two horizontal strips

$$H_1 = [-1,1] \times \left[-\frac{3}{4}, -\frac{1}{4}\right] \qquad \text{and} \qquad H_2 = [-1,1] \times \left[\frac{1}{4}, \frac{3}{4}\right]$$

Put $S = H_1 \cup H_2$ and take a function $f : S \to \mathbb{R}^2$ such that $f_i = f|H_i$ are affine mappings and

$$df_1 = \begin{bmatrix} s & 0 \\ 0 & u \end{bmatrix} \text{ and } df_2 = \begin{bmatrix} s & 0 \\ 0 & -u \end{bmatrix},$$

where $0 < s \leq 1/4$ and $u \geq 4$. To keep things simple assume that $f_1(0,1/2) = (-1/2,0)$ and $f_2(0,-1/2) = (1/2,0)$ (see Fig. 1).

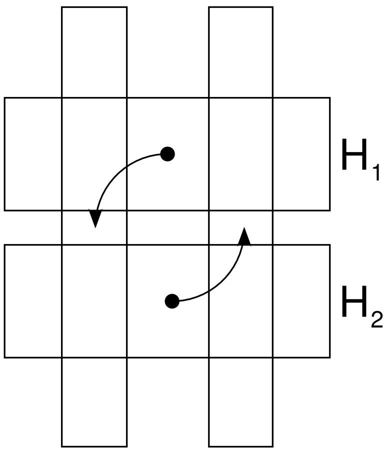


Figure 1: Construction of a modified linear horsheshoe.

By Observations 2.1 and 2.2 we know that

$$\langle f_i \rangle = |f_i|_s = s \qquad \text{and} \qquad \langle f_i \rangle_u = |f_i| = u.$$

Therefore we have a hyperbolic graph system

$$\Gamma = (\{v\}, v \to \mathrm{inv}(f,S); \{e_1, e_2\}, e_i \to f_i)$$

which represents the dynamics of $f$ on the invariant set $\mathrm{inv}(f,S)$.

In our example we have $H_i \overset{f_i}{\Rightarrow} H_j$ for $i,j = 1,2$. By Covering Relation Theorem we know that (14) is satisfied. Therefore by Theorem 5.2 we obtain that $\Gamma$ is Lipschitz conjugated to an abstract cone graph system which is in fact a simple shift on two symbols $\Sigma_2 = \{1,2\}^{\mathbb{Z}}$. The essential difference from the classical approach is that we define the metric on $\Sigma_2$ by

$$\rho(\alpha, \alpha') = \max\left\{s^{-k_-}, u^{-1-k_+}\right\}, \tag{16}$$

where $k_- := \inf\{i \le 0 : \alpha_{-1} = \alpha'_{-1}, \ldots, \alpha_i = \alpha'_i\}$, $k_+ := \sup\{i \ge -1 : \alpha_0 = \alpha'_0, \ldots, \alpha_i = \alpha'_i\}$.

Corollary 4.1 and 5.1 imply that most reasonable fractal dimension of $\mathrm{inv}(f,S)$ is equal to $\log_u 2 - \log_s 2$.[4]

We further modify the above example by introducing a Lipschitz perturbation. Let $g = f + p$ where $p$ is Lipschitz. We are interested in the dynamics of $g$ on the set $\mathrm{inv}(g,S)$ (see Fig. 2).

In the following we present the major consequence of our results. Recall that $|p|$ stands for the Lipschitz constant of $p$.

**Theorem 6.1.** *Let $p : S \to \mathbb{R}^2$ be such that*

$$|p| < s, \tag{17}$$

$$\sup\{\|p(x)\| : x \in S\} < \frac{1}{2} - s. \tag{18}$$

*Let $g = f + p$. Then the dynamics of $g$ on $\mathrm{inv}(g,S)$ is Hölder conjugated to the shift on two symbols $\Sigma_2$ with metric defined as in (16) by a homeomorphism $\Phi$. $\Phi$ is Hölder continuous with constant $\log_s(s + |p|)$ and $\Phi^{-1}$ is Hölder continuous with constant $\log_{s-|p|} s$. Moreover*

$$\overline{\dim}_B(\mathrm{inv}(g,S)) \le (\log_2(u - |p|)^{-1} - (\log_2(s + |p|)^{-1}, \tag{19}$$

$$\dim_H(\mathrm{inv}(g,S)) \ge (\log_2(u + |p|)^{-1} - (\log_2(s - |p|)^{-1}. \tag{20}$$

[4]In the case when $u = 4$ and $s = 1/4$ we obtain that $\dim(\mathrm{inv}(f,S)) = 1$.

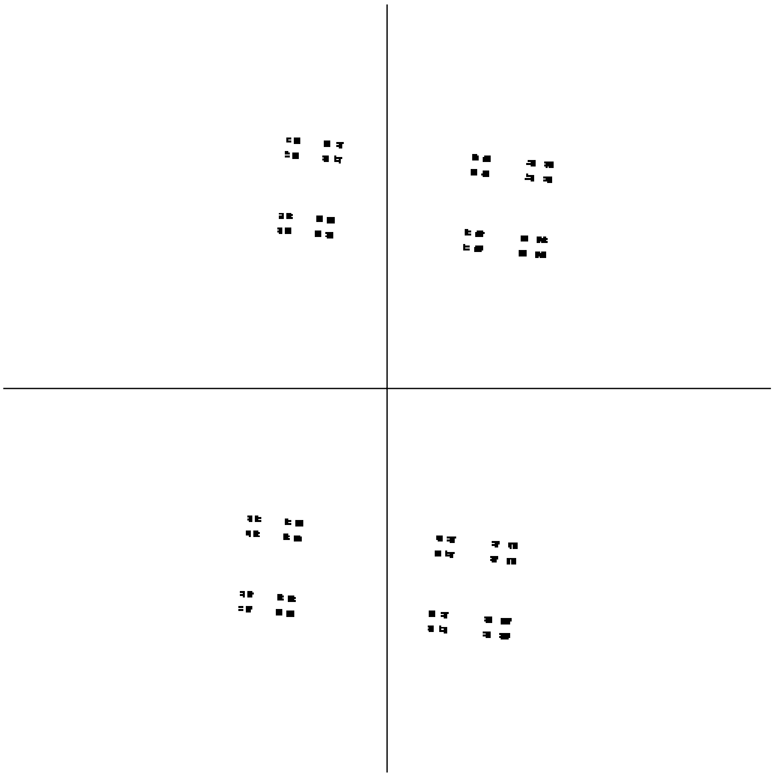

Figure 2: Invariant set for a perturbated linear horsheshoe.

*Proof.* Obviously $\mathrm{diam}(S) < \infty$. From (18) it follows that

$$\mathrm{im} g_1 \cap \mathrm{im} g_2 = \emptyset,$$

where $g_i = f_p | H_i$. By Proposition 2.1 we have

$$\begin{array}{lcl} |g_i| & \geq & s_i - |p|, \\ |g_i|_s & \leq & s_i + |p|, \\ \langle g_i \rangle_u & \geq & u_i - |p|, \\ \langle g_i \rangle & \leq & u_i + |p|. \end{array}$$

Therefore by (17) we know that $g_i$ are bi-Lipschitz and cone-hyperbolic. Assumption (18) yields that $H_i \overset{g}{\Rightarrow} H_j$ for $i, j = 1, 2$. Consequently $\mathrm{orb}(\alpha) \neq \emptyset$ for every finite path $\alpha \in E()$. Theorem 5.2 yields that $\mathcal{C} : (\Sigma_2, \rho_1) \to$

$(\mathrm{inv}(g,S),d)$ and $\mathcal{A} : (\mathrm{inv}(g,S),d) \to (\Sigma_2,\rho_2)$ are Lipschitz, where

$$\rho_1(\alpha,\alpha') = \max\left\{(s+|p|)^{-k_-},(u-|p|)^{-1-k_+}\right\},$$
$$\rho_2(\alpha,\alpha') = \max\left\{(s-|p|)^{-k_-},(u+|p|)^{-1-k_+}\right\},$$

and $d$ is a standard Euclidean metric in $\mathbb{R}^2$. This gives us the fractal dimension estimates (19) and (20). The functions $id_1 : (\Sigma_2,d) \to (\Sigma_2,d_1)$ and $id_2 : (\Sigma_2,d_2) \to (\Sigma_2,d)$ are both Hölder continuous and as the homeomorphism $\Phi$ we take $\mathcal{C} \circ id_1 = (id_2 \circ \mathcal{A})^{-1}$. Hölder constants follow from simple calculations. $\square$

# References


[1] L. Barreira, *Dimension and Recurrence in Hyperbolic Dynamics,* Birkhäuser 2008.

[2] M. Brin, G. Stuck, *Introduction to dynamical systems,* Cambridge Univ. Press 2002.

[3] G. A. Edgar, *Measure, Topology and Fractal Geometry,* Springer-Verlag New York 1990.

[4] G. A. Edgar, J. A. Golds, *A fractal dimension estimate for a graph-directed iterated function system of non-similarities.* Indiana Univ. Math. J. **48/2** (1999), 429–447.

[5] K. Falconer, *Fractal Geometry, Mathematical Foundations and Applications,* John Wiley and Sons 1990.

[6] K. Falconer, *Techniques of Fractal Geometry,* John Wiley and Sons 1997.

[7] M. Gidea, P. Zgliczynski, *Covering relations for multidimensional dynamical systems I, II,* J. Differ. Eq. **202**, 32–58, 59–80 (2004).

[8] A. Katok, B. Hasselblatt : *Introduction to the modern theory of dynamical systems,* Cambridge University Press 1997.

[9] J. Kennedy, J. A. Yorke: *Topological horseshoes,* Transactions of the American Mathematical Society, (2001), 2513–2530.

[10] B. Kitchens, *Symbolic dynamics. One-sided, two-sided and countable state Markov shifts,* Universitext, Springer-Verlag, Berlin, 1998.

[11] R. Mauldin, S. Williams, *Hausdorff dimension in graph directed constructions,* Trans. AMS **309/2** (1988), 811–829.

[12] P. Moran, *Additive functions of intervals and Hausdorff measure,* Proc. Cambridge Philos. Soc. **42** (1946), 15–23.

[13] R. Mauldin, M. Urbański, *Graph-directed Markov Systems: Geometry and Dynamics of Limit Sets,* Cambridge University Press 2003.

[14] S. Newhouse: *Cone-fields, Domination, and Hyperbolicity,* in *Modern Dynamical Systems and Applications,* B. Hasselblatt, M.Brin, Y. Pesin, eds., Cambridge University Press, New York 2004, 419–433.

[15] G. Osipenko, *Dynamical Systems, Graphs and Algorithms,* Lecture Notes in Mathematics, 1889. Springer-Verlag, Berlin, 2007.

[16] Y. Pesin, *Dimension Theory in Dynamical Systems,* Chicago 1997.

[17] M. Pireddu, F. Zanolin, *Cutting surfaces and applications to periodic points and chaotic-like dynamics,* Topol. Methods Nonlinear Anal. **30/2** (2007), 279–319.

[18] R. Sturman, J. M. Ottino, S. Wiggins, *The mathematical foundations of mixing,* Cambridge University Press 2006.

[19] P. Zgliczyński, *Covering relations, cone conditions and stable manifold theorem,* J. of Diff. Eq. **246** (2009) 1774–1819.