Modified operations on fuzzy sets

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Abstract

In this paper we introduce and study new fuzzy algebraic operations: $\oplus$-sum and scalar $\odot$-multiplication defined by

$$(A \oplus B)(z) := \sup_{x+y=z} A(x) \cdot B(y) \quad \text{and} \quad (\lambda \odot A)(z) := (A(z/\lambda))^{\lambda},$$

where A, B are fuzzy subsets and $\lambda \in (0, \infty)$. This allows us to investigate a new definition of fuzzy convexity – a fuzzy set A is called *log-convex* if

$$\lambda \odot A \oplus (1-\lambda) \odot A \subset A \quad \text{for } \lambda \in (0,1).$$

It occures that the class of upper semicontinuous fuzzy log-convex sets with nonempty compact supports can be embedded isometrically and isomorphically as a closed convex cone into a Banach space. In particular, fuzzy log-convex sets have the cancellation law

$$A \oplus B = A \oplus C \quad \text{iff} \quad B = C.$$

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1 Introduction

To explain our main idea we first establish some basic notation. The algebraic sum of fuzzy subsets A, B of a Banach space X and the multiplication of fuzzy set by scalar $\lambda \in (0, \infty)$ are defined by the Zadeh extension principle [19]:

$$(A + B)(z) := \sup_{x+y=z} A(x) \wedge B(y), \tag{1}$$

$$(\lambda \cdot A)(z) := A(z/\lambda), \tag{2}$$

where $\wedge$ denotes the minimum operation. One can easily notice that such definitions coincide with classical definitions on crisp (sharp) sets.

The most common metric d_∞ on the upper semicontinuous (usc) normal[1] fuzzy sets is defined by the supremum of the Hausdorff metric[2] on the level sets

$$\mathrm{d}_\infty(A, B) := \sup_{\alpha \in [0,1]} \mathrm{d}_\mathrm{H}([A]^\alpha, [B]^\alpha), \tag{3}$$

where $[A]^\alpha := \{x \in X : A(x) \geq \alpha\}$ for $\alpha \in (0, 1]$ and $[A]^0 := \operatorname{supp} A = \overline{\{x \in X : A(x) > 0\}}$.

Observe that such definition of metric requires normality, in the opposite case the distance would be infinite.

An important subclass of fuzzy sets consists of *fuzzy convex sets* [18], that is of sets satisfying the inequality

$$A(\lambda x + (1 - \lambda)y) \geqslant \min(A(x), A(y)) \quad \text{for } \lambda \in (0, 1). \tag{4}$$

Let us note that usc normal convex fuzzy sets with compact supports can be embedded isometrically as a complete convex cone in a Banach space [11].

Our aim in this article is to investigate the consequences of exchanging the $\wedge$ operation in the formula (1) with standard real multiplication.

Remark 1.1. In general one can exchange the minimum operation with another continuous associative operation on $[0, 1]$ such that 0 acts as zero

[1] Fuzzy set A is called normal if there exists x_0 such that $A(x_0) = 1$.

[2] Hausdorff distance between $K, L \subset X$ is defined by the following: $\mathrm{d}_\mathrm{H}(K, L) := \inf\{\varepsilon \geqslant 0 : K \subset L + \varepsilon \cdot \mathcal{B}$ and $L \subset K + \varepsilon \cdot \mathcal{B}\}$, where $\mathcal{B}$ denotes unit ball centered at zero.

and 1 is a neutral element. A characterization of such operations is given in [9], for a general description of continuous associative functions on $[0,1]$ see also [8]. An example of such an exchange in fuzzy sets theory can be found in [2], where authors consider generalizations of fuzzy lattice operations

$$(A \cap B)(x) = p(A(x), B(x)) \quad \text{and} \quad (A \cup B)(x) = s(A(x), B(x))$$

with $p, s : [0,1]^2 \longrightarrow [0,1]$ associative monotonic operations.

Simultaneously with the change of the algebraic sum we modify the formula (2) for multiplication of fuzzy sets. The new algebraic $\oplus$-sum of fuzzy sets and $\odot$-multiplication of fuzzy set by positive scalar are given by

$$(A \oplus B)(z) := \sup_{x+y=z} A(x) \cdot B(y),$$
$$(\lambda \odot A)(z) := (A(z/\lambda))^{\lambda}.$$

We want to resign from the normality limitation. Thus we put our interest on a subclass of fuzzy sets consisting of usc mappings with nonempty compact supports and we equip this class with a metric that does not depend on the existence of all level sets. Namely we make use of the concept of hypograph. A similar definition of metric based on sendographs has been studied before [5, 6, 7, 16, 17]. Our *logarithmic metric* measures the distance between hypographs of logarithms of fuzzy sets

$$\mathrm{d}_{\log}(A, B) := \mathrm{d}_{\mathrm{H}}(\mathrm{hyp}\,(\ln A), \mathrm{hyp}\,(\ln B)),$$

where d_{H} denotes the Hausdorff distance in the cartesian product $X \times \mathbb{R}$.

The convexity of fuzzy sets is a wide area for studies. For the properties of convex and strongly convex fuzzy sets see [4] and [18]. Modifications of fuzzy convexity can be found in [10], [13] and [14], where authors consider convex, pseudo-convex, preinvex and pseudo-invex fuzzy mappings. Moreover one can find the discussion on preinvex and Φ_1-convex fuzzy mappings in [15].

The modified algebraic operations allow us to reformulate the definition of fuzzy convexity (4). We call a fuzzy set A *log-convex* if

$$\lambda \odot A \oplus (1-\lambda) \odot A \subset A \qquad \text{for } \lambda \in (0,1).$$

It occurs that every fuzzy log-convex set is also fuzzy convex, however the converse is not true. The use of the phrase "logarithmic" in the name of new convexity concept is justified by the fact that the hypographs of logarithms of fuzzy log-convex sets are crisp convex sets.

An important subclass, with respect to modified algebraic operations, consists of usc fuzzy log-convex sets with nonempty compact supports. This class has a Rådström type embedding [12] property:

Main Result (see Theorem 5.6). *The space of usc fuzzy log-convex sets with nonempty compact supports can be embedded isometrically and isomorphically as a closed convex cone into a Banach space.*

Let us recall that one can use the embedding property to study another method of integration of fuzzy log-convex-valued functions or to investigate fuzzy differential equations.

Concluding, we show that within the modified algebraic operations a theory of fuzzy sets similar to the classical one can be build, a theory that has certain advantages over the first one:

- the metric which we define behaves in some cases more naturally then d_∞;
- we do not have to restrict to normal sets;
- fuzzy log-convex sets are more regular then classical fuzzy convex sets.

In our opinion in some problems our approach may be more appropriate then the classical one.

2 Modified algebraic operations

In this section we introduce new algebraic operations: $\oplus$-sum and scalar $\odot$-multiplication and show that they generalize standard algebraic operations on subsets of a Banach space X.

We start with recalling some basic informations regarding the classical theory of fuzzy sets, see [3]. By $\mathcal{F}(X)$ we denote the class of fuzzy subsets of a Banach space X, that is the class of mappings

$$A : X \longrightarrow [0,1].$$

Note that any subset K of X can be embedded in $\mathcal{F}(X)$ by means of its characteristic function

$$\mathbb{1}_K(x) := \begin{cases} 1 & \text{if } x \in K, \\ 0 & \text{if } x \in X \backslash K. \end{cases}$$

The space of fuzzy sets is equipped with two algebraic operations defined in (1) and (2). It is well-known that in case of normal sets the algebraic operations can be described on α-level (cut) sets $[A]^\alpha$. We have

$$[A + B]^\alpha = [A]^\alpha + [B]^\alpha,$$

$$[\lambda \cdot A]^\alpha = \lambda \cdot [A]^\alpha,$$

where $A, B \in \mathcal{F}(X)$, $\lambda \in (0, \infty)$.

Partial ordering in the space $\mathcal{F}(X)$ is given by inclusions. We say that $A \subset B$ if $A(x) \leqslant B(x)$ for all $x \in X$. Note that $A \subset B$ iff $[A]^\alpha \subset [B]^\alpha$ for all $\alpha \in [0, 1]$.

There are several methods of introducing metric in the space of fuzzy sets [3]. In this paper we consider the most common one, that is the d_∞ distance defined by (3). Note that such definition of metric requires normality.

By $\mathcal{F}_{usc}(X)$ we denote the subclass of $\mathcal{F}(X)$ consisting of upper semicontinuous mappings with nonempty compact supports. As elements of $\mathcal{F}_{usc}(X)$ does not have to have nonempty α-level sets for all $\alpha \in [0, 1]$ there is a need to equip this space with metric different to the d_∞ one. We do this in Section 5.

We denote by $\mathcal{E}(X)$ the class of usc normal convex fuzzy subsets of X with compact supports. Space $(\mathcal{E}(X), \mathrm{d}_\infty)$ is an important subset of $\mathcal{F}(X)$ as it can be embedded isometrically as a complete convex cone in a Banach space [11].

Now we are ready to proceed with our modification of the classical definition of fuzzy algebraic operations.

Definition 2.1. Let $A, B \in \mathcal{F}(X)$. We define $\oplus$-*sum* of fuzzy sets A and B by the formula

$$(A \oplus B)(z) := \sup_{x+y=z} A(x) \cdot B(y) \qquad \text{for } z \in X.$$

By $\odot$-*multiplication* of fuzzy set A by scalar $\lambda \in (0, \infty)$ we understand the operation

$$(\lambda \odot A)(z) := (A(z/\lambda))^\lambda \qquad \text{for } z \in X.$$

Remark 2.2. The definition of $\odot$-multiplication comes in a natural way from operation $\oplus$. Let $x \in X, a \in (0,1)$ and $k \in \mathbb{N}$. Then

$$\begin{aligned}
\underbrace{((a\cdot \mathbb{1}_{\{x\}}) \oplus \ldots \oplus (a\cdot \mathbb{1}_{\{x\}}))}_{k}(z) &= \underbrace{(a\cdot \mathbb{1}_{\{x\}})(z/k)\cdot \ldots \cdot (a\cdot \mathbb{1}_{\{x\}})(z/k)}_{k} \\
&= a^k \cdot \mathbb{1}_{\{x\}}(z/k) = ((a\cdot \mathbb{1}_{\{x\}})(z/k))^k \\
&= k \odot (a\cdot \mathbb{1}_{\{x\}})(z),
\end{aligned}$$

The following example shows that fuzzy sets obtained as the results of two methods of addition are usually different. In fact

$$A \oplus B \subset A + B \quad \text{and} \quad \lambda \odot A \subset \lambda \cdot A$$

for $A, B \in \mathcal{F}(X)$ and $\lambda \in (0, \infty)$.

Example 2.3. Consider fuzzy sets $A = \frac{1}{2}\mathbb{1}_{\{1\}}$ and $B = \frac{1}{3}\mathbb{1}_{\{2\}}$. The standard sum of fuzzy sets results from the Zadeh extension principle

$$(A+B)(z) = \sup_{x+y=z} A(x) \wedge B(y) = \sup_{x+y=z} \frac{1}{2}\mathbb{1}_{\{1\}}(x) \wedge \frac{1}{3}\mathbb{1}_{\{2\}}(y)$$

$$= \begin{cases} \frac{1}{3} & \text{for } z = 3, \\ 0 & \text{for } z \neq 3. \end{cases}$$

The $\oplus$-sum can be found by direct computations

$$(A \oplus B)(z) = \sup_{x+y=z} A(x) \cdot B(y) = \sup_{x+y=z} \frac{1}{2}\mathbb{1}_{\{1\}}(x) \cdot \frac{1}{3}\mathbb{1}_{\{2\}}(y)$$

$$= \begin{cases} \frac{1}{6} & \text{for } z = 3, \\ 0 & \text{for } z \neq 3. \end{cases}$$

It follows that

$$A + B = \frac{1}{3}\mathbb{1}_{\{3\}} \qquad \text{and} \qquad A \oplus B = \frac{1}{6}\mathbb{1}_{\{3\}}.$$

When looking on fuzzy sets through theirs supports it occurs that the $\oplus$-sum acts similarly to the standard algebraic addition of subsets of X.

Observation 2.4. *Let $A, B \in \mathcal{F}(X)$. Then one can easily check that*

$$\{x \in X : (A \oplus B)(x) > 0\} = \{x \in X : A(x) > 0\} + \{x \in X : B(x) > 0\}.$$

Consequently

$$\text{supp}\,(A \oplus B) = \overline{\text{supp}\,A + \text{supp}\,B}.$$

If A, B have compact supports, then

$$\text{supp}\,(A \oplus B) = \text{supp}\,A + \text{supp}\,B.$$

If one attempts to define the algebraic operations of fuzzy sets then its definition should generalize the standard algebraic operations of subsets of X. By Observation 2.4 this requirement is satisfied by the $\oplus$-sum, that is

$$\mathbb{1}_K \oplus \mathbb{1}_L = \mathbb{1}_{(K+L)} = \mathbb{1}_K + \mathbb{1}_L \quad \text{for } K, L \subset X.$$

An analogous observation holds for the $\odot$-multiplication

$$\lambda \odot \mathbb{1}_K = \mathbb{1}_{\lambda \cdot K} = \lambda \cdot \mathbb{1}_K \quad \text{for } K \subset X, \lambda \in (0, \infty).$$

The operations $\oplus$ and $\odot$ have the following properties.

Proposition 2.5. *Space $(\mathcal{F}(X), \oplus)$ is a comutative semigroup with neutral element $\mathbb{1}_{\{0\}}$. Moreover for $A, B \in \mathcal{F}(X)$ and $\lambda, \mu \in (0, \infty)$ we have the following equalities:*

$$\begin{aligned} \lambda \odot (A \oplus B) &= (\lambda \odot A) \oplus (\lambda \odot B), \\ (\lambda \cdot \mu) \odot A &= \lambda \odot (\mu \odot A). \end{aligned}$$

Proof. We only present the proof of the distributivity of $\odot$-multiplication over $\oplus$-sum. Other proofs are similar to the presented one.

$$\begin{aligned} (\lambda \odot (A \oplus B))(x) &= ((A \oplus B)(x/\lambda))^\lambda = (\sup_{a+b=x/\lambda} A(a) \cdot B(b)\ )^\lambda \\ &= (\sup_{\lambda a + \lambda b = x} A(a) \cdot B(b)\ )^\lambda = (\sup_{a+b=x} A(a/\lambda) \cdot B(b/\lambda)\ )^\lambda \\ &= \sup_{a+b=x} (A(a/\lambda))^\lambda \cdot (B(b/\lambda))^\lambda = ((\lambda \odot A) \oplus (\lambda \odot B))(x). \end{aligned}$$

□

Note that if $\mathbb{1}_\varnothing$ denotes an empty fuzzy set, that is if $\mathbb{1}_\varnothing(x) \equiv 0$ for all $x \in X$, then

$$\mathbb{1}_\varnothing \oplus B = \mathbb{1}_\varnothing \quad \text{and} \quad \lambda \odot \mathbb{1}_\varnothing = \mathbb{1}_\varnothing$$

for $B \in \mathcal{F}(X)$, $\lambda \in (0, \infty)$.

We show that $\oplus$-sum is an internal operation in the class of usc fuzzy sets.

Theorem 2.6. *Let $A, B \in \mathcal{F}_{usc}(X)$. Then*

$$A \oplus B \in \mathcal{F}_{usc}(X).$$

Proof. We have to show that the result of the $\oplus$-sum of elements A and B is an upper semicontinuous mapping with nonempty compact support.

Set $x \in X$. One can easily see that the upper semicontinuity of A and B ensures that mapping

$$f_x : X \ni y \longrightarrow A(x-y) \cdot B(y) \in [0,1]$$

is upper semicontinuous. Note that $\operatorname{supp} f_x \subset x - \operatorname{supp} A$ is a compact set. Applying the fact that upper semicontinuous mapping reaches its maximum on a compact set we obtain that there exists $y_0 \in x - \operatorname{supp} A$ such that

$$\sup_{y \in x - \operatorname{supp} A} A(x-y) \cdot B(y) = A(x-y_0) \cdot B(y_0). \tag{5}$$

We want to prove that for any $x \in X$ and any sequence $x_n \to x$ we have

$$\limsup_{n\to\infty} (A \oplus B)(x_n) \leqslant (A \oplus B)(x).$$

Suppose that there exist a point $x_0 \in X$ and a sequence $x_n \to x_0$ such that

$$\limsup_{n\to\infty} (A \oplus B)(x_n) > (A \oplus B)(x_0). \tag{6}$$

By (5) for any $n \in \mathbb{N}$ there exists y_n such that

$$\sup_{y \in x_n - \operatorname{supp} A} A(x_n-y) \cdot B(y) = A(x_n-y_n) \cdot B(y_n).$$

Sequence $\{y_n\}_{n\in\mathbb{N}}$ is contained in a compact set $(\bigcup_{n\in\mathbb{N}}\{x_n\}\cup\{x_0\}) - \operatorname{supp} A$. Thus there exist $y_0 \in X$ and a subsequence $\{y_{n_k}\}_{k\in\mathbb{N}}$ such that $y_{n_k} \to y_0$. Then $(x_{n_k} - y_{n_k}) \to x_0 - y_0$ and

$$\begin{aligned}\limsup_{k\to\infty}(A\oplus B)(x_{n_k}) &= \limsup_{k\to\infty} A(x_{n_k}-y_{n_k})\cdot B(y_{n_k})\\ &\leqslant \limsup_{k\to\infty} A(x_{n_k}-y_{n_k})\cdot \limsup_{n\to\infty} B(y_{n_k})\\ &\leqslant A(x_0-y_0)\cdot B(y_0) \leqslant \sup_{y\in X} A(x_0-y)\cdot B(y)\\ &= (A\oplus B)(x_0).\end{aligned}$$

It contradicts (6) and proves upper semicontinuity of $A\oplus B$.

Sets $\operatorname{supp} A$ and $\operatorname{supp} B$ are nonempty and compact. This implies that the set $\operatorname{supp} A + \operatorname{supp} B$ is also nonempty and compact. By Observation 2.4 we know that

$$\operatorname{supp}(A\oplus B) = \operatorname{supp} A + \operatorname{supp} B.$$

Thus $\operatorname{supp}(A\oplus B)$ is nonempty and compact. □

In the following we show that the compactness of supports of $A, B \in \mathcal{F}_{usc}(X)$ is an essential assumption in Theorem 2.6. In general the $\oplus$-sum of two usc mapppings without compact supports does not have to be an usc mapping.

Example 2.7. Put $G = \{(x, \tan(\frac{\pi}{2}x)) \in \mathbb{R}^2 : x \in (-1,1)\}$. Consider $A, B \in \mathcal{F}(\mathbb{R}^2)$ given by

$$A = \mathbb{1}_{\{0\}\times\mathbb{R}} \qquad \text{and} \qquad B = \mathbb{1}_G.$$

Both A and B are upper semicontinuous, however the mapping

$$A\oplus B = \mathbb{1}_{(-1,1)\times\mathbb{R}}$$

is lower semicontinuous and not upper semicontinuous.

3 Fuzzy convex sets

Let us first observe that there is another way to formulate the classical convexity condition (4).

Observation 3.1. *A fuzzy set A is convex* iff $\lambda A + (1-\lambda)A \subset A$ *for* $\lambda \in (0,1)$.

It seems that Observation 3.1 is a known fact, however we did not come across it in the literature. Thus for the convenience of the reader we include its proof.

Proof. First observe that inclusion $A \subset \lambda A + (1-\lambda)A$ always holds. Indeed,

$$\begin{aligned}(\lambda A + (1-\lambda)A)(w) &= \sup_{x+y=w} \lambda A(x) \wedge (1-\lambda)A(y) \\ &= \sup_{x+y=w} A(x/\lambda) \wedge A(y/(1-\lambda)) \\ &= \sup_{\lambda x+(1-\lambda)y=w} A(x) \wedge A(y)\end{aligned} \tag{7}$$

Taking $x = y = w$ we obtain

$$(\lambda A + (1-\lambda)A)(w) \geqslant A(w).$$

Set $\lambda \in (0,1)$. Now it is time to show the equivalence

$$A(\lambda x + (1-\lambda)y) \geqslant A(x) \wedge A(y) \text{ for } x, y \in X \quad \Longleftrightarrow \quad \lambda A + (1-\lambda)A \subset A.$$

Assume that the convexity condition holds. It is equivalent to the fact that

$$A(w) \geqslant \sup_{\lambda x+(1-\lambda)y=w} A(x) \wedge A(y) \quad \text{ for } w \in X.$$

By (7) the last can be rewritten as

$$A(w) \geqslant (\lambda A + (1-\lambda)A)(w) \quad \text{ for } w \in X.$$

This ends the proof of the equivalence. □

One can easily check the following characterization of fuzzy convex sets. The proof of the equivalence between 1 and 3 can be found in [18].

Theorem C. *Let $A \in \mathcal{F}(X)$. The following conditions are equivalent:*

1. *Set A is fuzzy convex.*
2. *Mapping $A : X \to [0,1]$ is quasi-concave.*
3. *Sets $[A]^\alpha$ are convex subsets of X for all $\alpha \in [0,1]$.*
4. *$(\lambda + \mu) \cdot A = (\lambda \cdot A) + (\mu \cdot A)$ for all $\lambda, \mu \in (0,1)$.*

Now we present the analogs of Observation 3.1 and Theorem C for modified algebraic operations.

Definition 3.2. We say that fuzzy set A is *log-convex* if

$$\lambda \odot A \oplus (1-\lambda) \odot A \subset A \qquad \text{for } \lambda \in (0,1).$$

Before we proceed with an analog of Theorem C let us introduce the notion of *hypograph* of a mapping. Let $W : X \longrightarrow [-\infty, \infty)$. By a hypograph of W we understand the set

$$\operatorname{hyp}(W) := \{(x, r) \in X \times \mathbb{R} : r \leqslant W(x)\}. \tag{8}$$

Theorem 3.3. *Let $A \in \mathcal{F}(X)$. The following conditions are equivalent:*

1. *Fuzzy set A is log-convex.*
2. *Mapping $\ln A : X \to [-\infty, 0]$ is concave.*
3. *Set $\operatorname{hyp}(\ln A)$ is a convex subset of $X \times \mathbb{R}$.*
4. *$(\lambda + \mu) \odot A = (\lambda \odot A) \oplus (\mu \odot A)$ for all $\lambda, \mu \in (0,1)$.*

Proof. Note that the equivalence between 2 and 3 follows immediately from the definition of convex set, whereas the equivalence between 1 and 4 is a simple consequence of Proposition 2.5. Thus we only show the equivalence between 1 and 2.

Assume that fuzzy set A is log-convex. Set $\lambda \in (0,1)$. By the definition of $\oplus$-sum the condition for fuzzy log-convexity can be rewritten as

$$\sup_{x+y=z} (\lambda \odot A)(x) \cdot ((1-\lambda) \odot A)(y) \leqslant A(z) \qquad \text{for } z \in X.$$

By the definition of $\odot$-multiplication we have

$$\sup_{x+y=z} (A(x/\lambda))^{\lambda} \cdot (A(y/(1-\lambda)))^{1-\lambda} \leqslant A(z) \qquad \text{for } z \in X.$$

Substituting $x = \lambda u$ and $y = (1-\lambda)w$ we obtain

$$\sup_{\lambda u + (1-\lambda) w = z} (A(u))^{\lambda} \cdot (A(w))^{1-\lambda} \leqslant A(z) \qquad \text{for } z \in X.$$

The last inequality is equivalent to the following

$$(A(u))^\lambda \cdot (A(w))^{1-\lambda} \leqslant A(\lambda u + (1-\lambda)w) \quad \text{for } u, w \in X.$$

Mapping $\ln : (0,\infty) \to \mathbb{R}$ is strictly increasing, thus by taking the logarithm of the both sides of the above inequality we obtain

$$\lambda \cdot \ln(A(u)) + (1-\lambda) \cdot \ln(A(w)) \leqslant \ln(A(\lambda u + (1-\lambda)w)) \quad \text{for } u, w \in X.$$

The last means that mapping $\ln A$ is concave. $\square$

Observe that fuzzy log-convexity implies fuzzy convexity.

Proposition 3.4. *If $A \in \mathcal{F}(X)$ is fuzzy log-convex then it is fuzzy convex.*

Proof. Let $A \in \mathcal{F}(X)$ be fuzzy log-convex and pick $x, y \in X$ and $\lambda \in (0,1)$. By Theorem 3.3 we know that fuzzy log-convexity of A implies concavity of mapping $\ln A$. The definition of a concave mapping gives us the following condition

$$\ln(A(\lambda x + (1-\lambda)y)) \geqslant \lambda \cdot \ln(A(x)) + (1-\lambda) \cdot \ln(A(y)).$$

By the properties of mapping $\ln : (0,\infty) \to \mathbb{R}$ we have

$$\ln(A(\lambda x + (1-\lambda)y)) \geqslant \ln((A(x))^\lambda \cdot (A(y))^{1-\lambda}).$$

Mapping $\ln : (0,\infty) \to \mathbb{R}$ is strictly increasing, thus

$$A(\lambda x + (1-\lambda)y) \geqslant (A(x))^\lambda \cdot (A(y))^{1-\lambda}.$$

It is left to show that

$$(A(x))^\lambda \cdot (A(y))^{1-\lambda} \geqslant \min(A(x), A(y)).$$

Assume first that $A(x) \geqslant A(y)$. Then

$$(A(x))^\lambda \cdot (A(y))^{1-\lambda} = (A(x))^\lambda \cdot A(y) \cdot (A(y))^{-\lambda} = A(y) \cdot \Big(\frac{A(x)}{A(y)}\Big)^\lambda \geqslant A(y).$$

Suppose now that $A(y) \geqslant A(x)$. Then

$$(A(x))^\lambda \cdot (A(y))^{1-\lambda} = A(x) \cdot (A(x))^{\lambda-1} \cdot (A(y))^{1-\lambda} = A(x) \cdot \Big(\frac{A(y)}{A(x)}\Big)^{1-\lambda} \geqslant A(x).$$

$\square$

Note that the converse is not true; there are fuzzy convex sets that are not log-convex.

Example 3.5. Consider fuzzy set $A \in \mathcal{F}(\mathbb{R})$ given by

$$A(x) = \begin{cases} 1 & \text{if } x = 0, \\ \frac{1}{2} & \text{if } x \in (0,1], \\ 0 & \text{otherwise.} \end{cases}$$

Then A satisfies condition (4) and as a consequence is a fuzzy convex set. Notice also that the hypograph of $\ln A$ is not a convex subset of $\mathbb{R}^2$ (e.g. line segment connecting $(0,0)$ with $(1, \ln \frac{1}{2})$ does not lie in $\operatorname{hyp}(\ln A)$), so A is not fuzzy log-convex.

If one attempts to find log-convex envelope of a given fuzzy set then he shoud find the convex envelope of the hypograph of its logarithm. The logarithm of A is given by

$$(\ln A)(x) = \begin{cases} 0 & \text{if } x = 0, \\ -\ln 2 & \text{if } x \in (0,1], \\ -\infty & \text{otherwise.} \end{cases}$$

The convex envelope of its hypograph is described by

$$\operatorname{conv}(\operatorname{hyp}(\ln A)) = (\{0\} \times (-\infty, 0]) \cup \{(x,r) \in \mathbb{R}^2 : x \in (0,1], r \leqslant -x \cdot \ln 2\}.$$

By taking the exponent of the above we obtain

$$(\operatorname{conv} A)(x) = \begin{cases} e^0 & \text{if } x = 0, \\ e^{-x \cdot \ln 2} & \text{if } x \in (0,1], \\ 0 & \text{otherwise,} \end{cases} = \begin{cases} 1 & \text{if } x = 0, \\ 2^{-x} & \text{if } x \in (0,1], \\ 0 & \text{otherwise.} \end{cases}$$

4 Isomorphism

This section contains mostly technical results that will be usefull in section 5.

By a *dominium* of a mapping $W : X \longrightarrow [-\infty, \infty)$ we understand set $\operatorname{dom} W := \{x \in X : W(x) > -\infty\}$.

We denote by $\mathcal{M}(X)$ a class of mappings

$$W : X \longrightarrow [-\infty, 0].$$

By $\mathcal{M}_{usc}(X)$ we understand a subclass of $\mathcal{M}(X)$ consisting of upper semi-continuous mappings W such that $\overline{\operatorname{dom} W}$ is a nonempty compact set.

We introduce two mutually inverse operations between $\mathcal{F}(X)$ and $\mathcal{M}(X)$.

Definition 4.1. Let $A \in \mathcal{F}(X)$. We define operation $\ln : \mathcal{F}(X) \longrightarrow \mathcal{M}(X)$ by

$$(\ln A)(x) := \ln(A(x)).$$

Function exp maps $\mathcal{M}(X) \longrightarrow \mathcal{F}(X)$ and for $W \in \mathcal{M}(X)$ it is defined by

$$(\exp W)(x) := \exp(W(x)).$$

The space $\mathcal{M}(X)$ may now be seen as the image of $\mathcal{F}(X)$ through the logarithm operation

$$\mathcal{M}(X) = \{\ln A : A \in \mathcal{F}(X)\}.$$

In this section we show that mappings exp and log are isomorphisms between $\mathcal{F}(X)$ and $\mathcal{M}(X)$ – see Proposition 4.4 – and that there exists a natural isomorphism between the space $\mathcal{M}_{usc}(X)$ and the class of hypographs of elements of $\mathcal{M}_{usc}(X)$ – see Theorem 4.5 and Theorem 4.7.

In the following we introduce algebraic operations on the space $\mathcal{M}(X)$.

Definition 4.2. Let $U, W \in \mathcal{M}(X)$. A $\boxplus$-*sum* of U and W is defined by

$$(U \boxplus W)(x) := \sup_{u+w=x} (U(u) + W(w)) \qquad \text{for } x \in X.$$

By $\boxdot$-*multiplication* of an element W by scalar $\lambda \in (0, \infty)$ we understand the operation

$$(\lambda \boxdot W)(x) := \lambda \cdot W(x/\lambda) \qquad \text{for } x \in X.$$

After defining the algebraic operations we can investigate the properties of the space $(\mathcal{M}(X), \boxplus, \boxdot)$. It occures that these are similar to that listed in Proposition 2.5.

Observation 4.3. *Space $(\mathcal{M}(X), \boxplus)$ is a comutative semigroup with $V = \{0\} \times (-\infty, 0]$ neutral element. Moreover for $U, W \in \mathcal{M}(X)$ and $\lambda, \mu \in (0, \infty)$ we have the following equalities:*

$$\lambda \boxdot (U \boxplus W) = (\lambda \boxdot U) \boxplus (\lambda \boxdot W),$$
$$(\lambda \cdot \mu) \boxdot W = \lambda \boxdot (\mu \boxdot W).$$

We are now in a position to show that operations exp and ln are isomorphisms between $\mathcal{F}(X)$ and $\mathcal{M}(X)$.

Proposition 4.4. *Let $A, B \in \mathcal{F}(X)$, $\lambda \in (0, \infty)$. Then*

$$A \oplus B = \exp(\ln A \boxplus \ln B)$$

and

$$\lambda \odot A = \exp(\lambda \boxdot \ln A).$$

Proof. To prove the first equation it is sufficient to show that for any $U, W \in \mathcal{M}(X)$

$$\exp(U \boxplus W) = (\exp U) \oplus (\exp W).$$

We have

$$\begin{aligned}
\exp(U \boxplus W)(x) &= \exp((U \boxplus W)(x)) = \exp(\sup_{u+w=x} (U(u) + W(w))) \\
&= \sup_{u+w=x} \exp(U(u) + W(w)) = \sup_{u+w=x} \exp(U(u)) \cdot \exp(W(w)) \\
&= \sup_{u+w=x} (\exp U)(u) \cdot (\exp W)(w) = ((\exp U) \oplus (\exp W))(x).
\end{aligned}$$

The proof of the second equation is straightforward

$$\begin{aligned}
\exp(\lambda \boxdot \ln A)(x) &= \exp(\lambda \cdot (\ln A)(x/\lambda)) = (\exp(\ln(A(x/\lambda))))^\lambda \\
&= (A(x/\lambda))^\lambda = (\lambda \odot A)(x).
\end{aligned}$$

$\square$

Given a set $C \subset X \times \mathbb{R}$, by $p_X(C)$ we denote the projection of C onto X, that is

$$p_X(C) := \{x \in X \mid \exists r \in \mathbb{R} : (x, r) \in C\}.$$

Observe that if $C = \operatorname{hyp} W$ for $W \in \mathcal{M}(X)$ then $p_X(C) = \operatorname{dom} W$.

We have the following representation theorem for the class $\mathcal{M}_{usc}(X)$.

Theorem 4.5. *Let $W \in \mathcal{M}_{usc}(X)$ and denote $C := \operatorname{hyp}(W)$. Then:*

- *C is a closed subset of $X \times (-\infty, 0]$,* (9)
- *$\overline{p_X(C)}$ is a nonempty compact subset of X,* (10)
- *if $(x, r) \in C$ and $s \leqslant r$ then $(x, s) \in C$.* (11)

Conversely, if $C \subset X \times \mathbb{R}$ *satisfies (9)–(11) then* $W : X \to [-\infty, 0]$ *defined by*

$$W(x) := \sup\{r \in \mathbb{R} : (x, r) \in C\} \tag{12}$$

is an element of $\mathcal{M}_{usc}(X)$ *such that* $\operatorname{hyp}(W) = C$.

Proof. The proof of the first part of the theorem follows immediately from the definition of hypograph (8).

Assume that C satisfies (9)–(11) and let W be defined by (12). By (10) we have that $\overline{\operatorname{dom} W} = \overline{p_X(C)}$ is a nonempty compact set and (9) and (11) imply that $\operatorname{hyp} W = C$. It is left to show that mapping W is upper semicontinuous.

The proof goes by a reduction to a contradiction. Suppose that there exist $x_0 \in X$ and a sequence $x_n \to x_0$ such that $\limsup_{n\to\infty} W(x_n) > W(x_0)$. Restricting to a subsequence, denoted also by $\{x_n\}_{n\in\mathbb{N}}$, we have $\lim_{n\to\infty} W(x_n) > W(x_0)$.

If $x_0 \in X \backslash \overline{p_X(C)}$ then we have by (10) that for n sufficiently large also $x_n \in X \backslash \overline{p_X(C)}$. Thus $\lim_{n\to\infty} W(x_n) = -\infty = W(x_0)$, a contradiction.

Assume that $x_0 \in p_X(C)$. Again if for n sufficiently large $x_n \in X \backslash p_X(C)$ then $\lim_{n\to\infty} W(x_n) = -\infty < W(x_0)$, a contradiction. So we have that there exists a subsequence, that we also denote by $\{x_n\}_{n\in\mathbb{N}}$, such that $x_n \in p_X(C)$ for all n. We have by (9) that $x \in p_X(C)$ iff $(x, W(x)) \in C$. Thus $\{(x_n, W(x_n))\}_{n\in\mathbb{N}}$ is a convergent sequence contained in C. By (9) its limit $(x_0, \lim_{n\to\infty} W(x_n))$ is also an element of C. By the definition of mapping W we obtain

$$\lim_{n\to\infty} W(x_n) \leqslant \sup\{r \in \mathbb{R} : (x_0, r) \in C\} = W(x_0),$$

a contradiction.

The last case to consider holds when $x_0 \in (X \backslash p_X(C)) \cap \overline{p_X(C)}$. If again for n sufficiently large $x_n \in X \backslash p_X(C)$ then $\lim_{n\to\infty} W(x_n) = -\infty = W(x_0)$, a contradiction. So assume that $x_n \in p_X(C)$ for all n. It is sufficient to prove that $\lim_{n\to\infty} W(x_n) = -\infty$. Suppose that the last does not hold, that is $\lim_{n\to\infty} W(x_n) > -\infty$. Then $\{(x_n, W(x_n))\}_{n\in\mathbb{N}}$ is a convergent sequence contained in C. By (9) its limit $(x_0, \lim_{n\to\infty} W(x_n))$ is also an element of C. On the other hand, as $x_0 \in X \backslash p_X(C)$, we have that $(x_0, \lim_{n\to\infty} W(x_n))$ does not belong to C, a contradiction. $\square$

There is a method to represent the algebraic operations on $\mathcal{M}_{usc}(X)$ in a simpler form. Before we proceed we need the following observation.

Observation 4.6. *For $U \in \mathcal{M}_{usc}(X)$ and $\alpha \in (-\infty, 0]$ denote*

$$[\operatorname{hyp} U]^\alpha := \{(x, r) \in \operatorname{hyp} U : r \geqslant \alpha\}.$$

Let $U, W \in \mathcal{M}_{usc}(X)$ and $\lambda \in (0, \infty)$. Then for all $\alpha \in (-\infty, 0]$

$$[\operatorname{hyp} U + \operatorname{hyp} W]^\alpha = [[\operatorname{hyp} U]^\alpha + [\operatorname{hyp} W]^\alpha]^\alpha$$

and

$$[\lambda \cdot \operatorname{hyp} U]^\alpha = \lambda \cdot [\operatorname{hyp} U]^{\alpha/\lambda}.$$

The following theorem describes the algebraic operations as operations on hypographs.

Theorem 4.7. *Let $U, W \in \mathcal{M}_{usc}(X)$, $\lambda \in (0, \infty)$. Then*

$$\operatorname{hyp} (U \boxplus W) = \operatorname{hyp} U + \operatorname{hyp} W$$

and

$$\operatorname{hyp} (\lambda \boxdot W) = \lambda \cdot \operatorname{hyp} W,$$

where $+$ stands for algebraic sum of sets, and $\cdot$ denotes usual multiplication of set by scalar.

Proof. We start with the proof that algebraic sum of hypographs is also a hypograph. We want to show that $\operatorname{hyp} U + \operatorname{hyp} W$ satisfies conditions (9)–(11) of Theorem 4.5.

For the proof that $\operatorname{hyp} U + \operatorname{hyp} W$ is a closed subset of $X \times \mathbb{R}$ let us pick $\{(x_n, r_n)\}_{n \in \mathbb{N}}$ a convergent sequence contained in $\operatorname{hyp} U + \operatorname{hyp} W$. We want to show that $\lim_{n \to \infty}(x_n, r_n)$ belongs to $\operatorname{hyp} U + \operatorname{hyp} W$. By Observation 4.6 we have $\{(x_n, r_n)\}_{n \in \mathbb{N}} \subset [\operatorname{hyp} U + \operatorname{hyp} W]^\alpha = [[\operatorname{hyp} U]^\alpha + [\operatorname{hyp} W]^\alpha]^\alpha$ for $\alpha = \min_{n \in \mathbb{N}} r_n$. Sets $[\operatorname{hyp} U]^\alpha$ and $[\operatorname{hyp} W]^\alpha$ are compact, so $[[\operatorname{hyp} U]^\alpha + [\operatorname{hyp} W]^\alpha]^\alpha$ is also compact. This implies in particular that $[\operatorname{hyp} U + \operatorname{hyp} W]^\alpha$ is a closed subset of $X \times \mathbb{R}$. Thus $\lim_{n \to \infty}(x_n, r_n) \in [\operatorname{hyp} U + \operatorname{hyp} W]^\alpha \subset \operatorname{hyp} U + \operatorname{hyp} W$.

Nonemptiness and compactness of $\overline{\operatorname{dom} U + \operatorname{dom} W}$ follows from the algebraic properties of compact sets. We have that $\overline{\operatorname{dom} U + \operatorname{dom} W} \subset \overline{\operatorname{dom} U} + \overline{\operatorname{dom} W}$. An algebraic sum of two compact sets is a compact set. Thus set $\overline{\operatorname{dom} U + \operatorname{dom} W}$ is nonempty and compact, as a nonempty closed subset of compact set $\overline{\operatorname{dom} U} + \overline{\operatorname{dom} W}$.

For the proof of (11) let $(x,r) \in \operatorname{hyp} U + \operatorname{hyp} W$ and let $s \leqslant r$. There exist $(x_1, r_1) \in \operatorname{hyp} U$ and $(x_2, r_2) \in \operatorname{hyp} W$ such that $x = x_1 + x_2$ and $r = r_1 + r_2$. Denote $s_1 = r_1$ and $s_2 = r_2 - r + s$. By $s \leqslant r$ we have $s_2 \leqslant r_2$, thus $(x_1, s_1) \in \operatorname{hyp} U$ and $(x_2, s_2) \in \operatorname{hyp} W$. Then $(x, s) = (x_1 + x_2, s_1 + s_2) = (x_1, s_1) + (x_2, s_2) \in \operatorname{hyp} U + \operatorname{hyp} W$.

By Theorem 4.5 there exists $Z \in \mathcal{M}_{usc}(X)$ such that $\operatorname{hyp} Z = \operatorname{hyp} U + \operatorname{hyp} W$. So it is left to show that $Z = U \boxplus W$. We have by Theorem 4.5 that Z has the following form

$$Z(x) = \sup\{r \in \mathbb{R} : (x,r) \in \operatorname{hyp} U + \operatorname{hyp} W\}.$$

Let $x \in \operatorname{dom} U + \operatorname{dom} W$. Then

$$\begin{aligned} Z(x) &= \sup\{r_1 + r_2 \in \mathbb{R} : (x_1, r_1) \in \operatorname{hyp} U, (x_2, r_2) \in \operatorname{hyp} W, x_1 + x_2 = x\} \\ &= \sup_{x_1 + x_2 = x} \sup\{r_1 \in \mathbb{R} : (x_1, r_1) \in \operatorname{hyp} U\} + \sup\{r_2 \in \mathbb{R} : (x_2, r_2) \in \operatorname{hyp} W\} \\ &= \sup_{x_1 + x_2 = x} U(x_1) + W(x_2) = (U \boxplus W)(x). \end{aligned}$$

For the proof of the second assertion of the theorem notice that for $\lambda \in (0, \infty)$

- $\lambda \cdot (\operatorname{hyp} W)$ is a closed subset of $X \times (-\infty, 0]$,
- $\overline{\lambda \cdot \operatorname{dom} W}$ is a compact subset of X,
- if $(x,r) \in \lambda\cdot(\operatorname{hyp} W)$ and $s \leqslant r$ then $(x,s) = \lambda\cdot(x/\lambda, s/\lambda) \in \lambda\cdot(\operatorname{hyp} W)$.

Thus conditions (9)–(11) of Theorem 4.5 are satisfied and as a consequence there exists $Z \in \mathcal{M}_{usc}(X)$ such that $\operatorname{hyp} Z = \lambda \cdot \operatorname{hyp} W$. Theorem 4.5 states that Z has the following form

$$Z(x) = \sup\{r \in \mathbb{R} : (x,r) \in \lambda \cdot \operatorname{hyp} W\}.$$

Let $x \in \lambda \cdot (\operatorname{dom} W)$. Then

$$\begin{aligned} Z(x) &= \sup\{r \in \mathbb{R} : r = \lambda \cdot s, x = \lambda \cdot y, (y,s) \in \operatorname{hyp} W\} \\ &= \sup\{\lambda s \in \mathbb{R} : x = \lambda \cdot y, (y,s) \in \operatorname{hyp} W\} \\ &= \lambda \cdot \sup\{s \in \mathbb{R} : (x/\lambda, s) \in \operatorname{hyp} W\} \\ &= \lambda \cdot W(x/\lambda) = (\lambda \boxdot W)(x). \end{aligned}$$

$\square$

5 Logarithmic metric and embedding theorem

The definition of metric in the space $\mathcal{F}_{usc}(X)$ is closely related to the concept of the distance in $\mathcal{M}_{usc}(X)$.

We equip $\mathcal{M}_{usc}(X)$ with a metric given as a Hausdorff distance between hypographs of elements of $\mathcal{M}_{usc}(X)$. We apply the maximum metric in the cartesian product $X \times \mathbb{R}$. From the properties of the Hausdorff distance on the space of nonempty closed subsets of the Banach space $X \times \mathbb{R}$ it follows that metric $\mathrm{d_H}$ is positively homogeneous and that $(\mathcal{M}_{usc}(X), \mathrm{d_H})$ is a complete metric space.

The distance between elements of $\mathcal{F}_{usc}(X)$ is given as a distance between their images in the space $\mathcal{M}_{usc}(X)$.

Definition 5.1. Let $A, B \in \mathcal{F}_{usc}(X)$. *Logaritmic distance* between fuzzy sets A and B is defined by

$$\mathrm{d_{log}}(A, B) := \mathrm{d_H}(\mathrm{hyp}\,(\ln A), \mathrm{hyp}\,(\ln B)).$$

Equipping spaces $\mathcal{F}_{usc}(X)$ and $\mathcal{M}_{usc}(X)$ with mertics $\mathrm{d_{log}}$ and $\mathrm{d_H}$ respectively one sees that operations exp and ln are isometries. Indeed

$$\begin{aligned}\mathrm{d_{log}}(\exp U, \exp W) &= \mathrm{d_H}(\mathrm{hyp}\,(\ln(\exp U)), \mathrm{hyp}\,(\ln(\exp W))) \\ &= \mathrm{d_H}(\mathrm{hyp}\, U, \mathrm{hyp}\, W).\end{aligned}$$

Because cancelation law does not hold without convexity assumption put on elements of $\mathcal{F}_{usc}(X)$, logarithmic metric is not invariant under translation.

Example 5.2. Let

$$A = B = [0, 1] \times (-\infty, 0], \qquad C = \{0, 1\} \times (-\infty, 0].$$

Then sets A, B and C are hypographs of elements of $\mathcal{M}_{usc}(\mathbb{R})$. One sees that $A + B = [0, 2] \times (-\infty, 0] = A + C$, but $B \neq C$.

Although logarithmic metric $\mathrm{d_{log}}$ on $\mathcal{F}_{usc}(X)$ is not invariant under translation it is possible to deduce its weaker property.

Observation 5.3. *Let $A, B, C \in \mathcal{F}_{usc}(X)$. Then*

$$\mathrm{d_{log}}(A \oplus B, A \oplus C) \leqslant \mathrm{d_{log}}(B, C).$$

Observation 5.3 allows to show that $\oplus$-sum is Lipschitz continuous.

Corollary 5.4. *The $\oplus$-sum satisfies Lipschitz condition, that is for $A, B, C, D \in \mathcal{F}_{usc}(X)$*

$$\mathrm{d}_{\log}(A \oplus C, B \oplus D) \leqslant 2 \cdot \max(\mathrm{d}_{\log}(A, B), \mathrm{d}_{\log}(C, D)).$$

Proof. The proof follows simply from Observation 5.3 and the triangle inequality

$$\begin{aligned} \mathrm{d}_{\log}(A \oplus C, B \oplus D) &\leqslant \mathrm{d}_{\log}(A \oplus C, A \oplus D) + \mathrm{d}_{\log}(A \oplus D, B \oplus D) \\ &\leqslant \mathrm{d}_{\log}(C, D) + \mathrm{d}_{\log}(A, B) \\ &\leqslant 2 \cdot \max(\mathrm{d}_{\log}(A, B), \mathrm{d}_{\log}(C, D)). \end{aligned}$$

□

An important subclass of $\mathcal{F}_{usc}(X)$ consists of fuzzy log-convex sets.

Definition 5.5. We denote by $\mathcal{E}_{log}(X)$ a class of usc fuzzy log-convex sets with nonempty compact supports.

A significant result for our considerations on the class $\mathcal{E}_{log}(X)$ comes from [1].

Theorem E. *Let Y be a Banach space and V be a closed convex cone*[3] *in Y. Let us denote by C_V the class of nonempty closed convex subsets of Y such that their Hausdorff distance from V is finite. We equip C_V with the following algebraic operations:*

$$\begin{aligned} U \dotplus W &:= \overline{\{u + w \in Y : u \in U, w \in W\}}, \\ \lambda \cdot W &:= \{\lambda \cdot w : w \in W\}, \end{aligned}$$

for $U, W \in C_V$, $\lambda \in (0, \infty)$. Then

- *Hausdorff metric d_{H} on C_V is positively homogeneous and invariant under translations.*
- *The class $(C_V, \mathrm{d}_{\mathrm{H}})$ is a complete metric space.*

[3]By a cone in a Banach space Y we understand a convex subset V of Y such that $\lambda x \in V$ for $x \in V$, $\lambda \in (0, \infty)$.

- *The class* $(C_V, \dot{+})$ *is a commutative semigroup with cancellation law and with neutral element* V*.*

As a consequence, C_V *can be embedded isometrically and isomorphically as a closed convex cone into a Banach space.*

As a direct conclusion of Theorem E we obtain

Theorem 5.6. *The following statements holds true:*

- *Logarithmic metric* $\mathrm{d}_{\log}$ *on the space* $\mathcal{E}_{log}$ *is positively homogeneous and invariant under translations.*
- *The class* $(\mathcal{E}_{log}(X), \mathrm{d}_{\log})$ *is a complete metric space.*
- *The class* $(\mathcal{E}_{log}(X), \oplus)$ *is a commutative semigroup with cancellation law and with neutral element* $\mathbb{1}_{\{0\}}$*.*

As a consequence, space $\mathcal{E}_{log}(X)$ *can be embedded isometrically and isomorphically as a closed convex cone into a Banach space.*

Proof. Let us denote $V = \{0\} \times (-\infty, 0] \subset X \times \mathbb{R}$. Then V is a closed convex cone in a Banach space $X \times \mathbb{R}$.

Let C_V be defined as in the Theorem E. Then the class of hypographs of logarithms of elements of $\mathcal{E}_{log}(X)$ is a closed subclass of C_V and as a consequence the asserts of Theorem E apply to it.

Let h denote a mapping from $\mathcal{M}_{usc}(X)$ to the class of hypographs of elements of $\mathcal{M}_{usc}(X)$ defined by

$$h(W) := \mathrm{hyp}\,(W) \qquad \text{for } W \in \mathcal{M}_{usc}(X).$$

By Theorem 4.5 mapping h is one-to-one and by Theorem 4.7 we have

$$h(U \boxplus W) = h(U) + h(W) \quad \text{and} \quad h(\lambda \boxdot W) = \lambda \cdot h(W),$$

for $U, W \in \mathcal{M}_{usc}(X)$, $\lambda \in (0, \infty)$. Thus h is an isomorphism. As the metric in the space $\mathcal{M}_{usc}(X)$ measures the Hausdorff distance between corresponding hypographs, we obtain that mapping h is an isometry.

Note that by Proposition 4.4 and by the definition of logarithmic metric mapping $\ln_{|\mathcal{F}_{usc}(X)} : \mathcal{F}_{usc}(X) \to \mathcal{M}_{usc}(X)$ is also an isometric isomorphism. Thus mapping $(h \circ \ln_{|\mathcal{F}_{usc}(X)})$ is an isometric isomorphism between the space $(\mathcal{F}_{usc}(X), \mathrm{d}_{\log})$ and the class of the hypographs of elements of $\mathcal{M}_{usc}(X)$

equipped with the Hausdorff metric. Restriction of $(h \circ \ln_{|\mathcal{F}_{usc}(X)})$ to $\mathcal{E}_{log}(X)$ gives an isometric isomorphism between the space $(\mathcal{E}_{log}(X), \mathrm{d}_{\log})$ and the class of hypographs of logarithms of elements of $\mathcal{E}_{log}(X)$. As a consequence, the asserts of Theorem E apply to $\mathcal{E}_{log}(X)$. □

The importance of the last theorem follows from the fact that this allows another method of defining integration of fuzzy log-convex-valued mappings.

Remark 5.7. Let i denote the embedding isomorphism from Theorem 5.6. Then one can study the integral of fuzzy log-convex-valued mapping f given by

$$\int f(t)\, dt := i^{-1}\Big(\int i(f(t))\, dt \Big).$$

Note that the integral is defined correctly. Structure of the cone and its completeness ensure that integration in the Banach space does not move $i(f(t))$ out of the cone.

One can investigate the fuzzy differential equations in an analogous way.

Problem. It is well-known that a closed set is convex iff $A + A = 2A$. This allows to define convexity for arbitrary associative continuous operation. Two most important examples of such operations are given by the $\min(x, y)$ operation and by $g^{-1}(g(x)g(y))$ for a continuous strictly increasing $g : [0, 1] \to [0, 1]$, see [8, Figure 2]. Clearly, the semigroup $[0, 1]$ with the operation $x * y \to g^{-1}(g(x)g(y))$ is isomorphic with the semigroup $[0, 1]$ with the standard multiplication operation. That is why in our paper we restrict our attention to multiplication. However, it remains a problem of the study in the general case.

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