Strict verification of approximate midconvexity on non-convex sets

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Abstract

Let V be a given (not necessarily convex) subset of a normed space and let $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. We say that $f : V \to \mathbb{R}$ is ω-approximately midconvex if

$$f(\frac{x+y}{2}) \le \frac{f(x)+f(y)}{2} + \omega(\|x-y\|) \quad \text{for } x, y \in V : \frac{x+y}{2} \in V.$$

Our aim is to find/estimate the function

$$\sup\{f\colon \{0, \frac{1}{N}, \ldots, \frac{N-1}{N}, 1\} \to \mathbb{R} |\ f - \omega\text{-midconvex},\ f(0) = f(1) = 0\},$$

for $N \in \mathbb{N}$. We present a computer assisted approach which given $\varepsilon > 0$ and $N \in \mathbb{N}$ enables us, under reasonable assumptions, to find the above supremum with accuracy ε.

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1 Introduction

The main idea of our investigation lies in joining together the notions of approximate convexity and convexity on non-convex sets.

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Let us first recall some basic information concerning approximate convexity. The term "approximate convexity" was introduced by D. H. Hyers and S. M. Ulam [4] in 1952. Its variation adapted to Jensen convexity can be stated as follows:

Definition 1.1 ([8])**.** Let X be a normed space, V be a convex subset of X, and ε be a nonnegative constant. A function $f\colon V \to \mathbb{R}$ is said to be ε-*midconvex* (or ε-*Jensen convex*) if

$$Jf(x,y) := f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \varepsilon \quad \text{for } x,y \in V\colon \frac{x+y}{2} \in V.$$

A natural generalization of this definition for normed spaces lies in replacing the constant ε by a function ω which depends on the norm of the difference $\|x-y\|$:

Definition 1.2. Let V be a convex subset of a normed space X and let $\omega\colon \mathbb{R}_+ \to \mathbb{R}_+$ be a given function. We say that $f\colon V \to \mathbb{R}$ is $\omega(\cdot)$-*midconvex* (or $\omega(\cdot)$-*Jensen convex*) if

$$f\left(\frac{x+y}{2}\right) \le \frac{f(x)+f(y)}{2} + \omega(\|x-y\|) \quad \text{for } x,y \in V : \frac{x+y}{2} \in V.$$

For some recent results we refer the reader to [9, 11]. The general research question lies in verifying how far from convex functions are $\omega(\cdot)$-approximately convex functions. To measure this we will the convexity difference operator defined by

$$Cf(x,y;t) := f(tx+(1-t)y) - tf(x) - (1-t)f(y) \quad \text{for } x,y \in V, t \in [0,1]$$

will be useful. The method of attack of this problem in many cases is based on the reduction to one dimensional case, which is stated in the following trivial observation:

Observation 1.3. Let V be a convex subset of a Banach space and let $f\colon V \to \mathbb{R}$ be given. Then f is $\omega(\cdot)$-midconvex iff for every $x,y \in V$, the function $\varphi_{x,y} : [0,1] \to \mathbb{R}$ defined by

$$\varphi_{x,y}(t)\colon [0,1] \ni t \to Cf(x,y;t) \in \mathbb{R},$$

is $\omega_{x,y}(\cdot)$-midconvex, where $\omega_{x,y}(r) := \omega(\|x-y\|r)$.

Observe that the above mentioned function $\varphi_{x,y}$ satisfies $\varphi_{x,y}(0) = \varphi_{x,y}(1) = 0$. As in general case to obtain convexity from Jensen convexity we need (local) boundedness, we see that the study of $\omega(\cdot)$-approximately convex functions can be reduced to investigate of the set

$$J_\omega([0,1],\{0,1\}) := \{f \in \mathcal{B}([0,1];\{0,1\})\colon f \text{ is } \omega(\cdot)\text{-midconvex}\},$$

where $\omega\colon [0,1] \to \mathbb{R}_+$ is given and by $\mathcal{B}(V;W)$ we denote the set of all real-valued bounded from above functions on set V which are zero on W. It occurs that the optimal bound of this set defined by

$$f_\omega([0,1],\{0,1\}) := \sup\{f \in J_\omega([0,1],\{0,1\})\}$$

are usually interesting fractal-like functions connected to the classical Takagi function, see [1, 8, 12].

Our second motivation lies in the recent generalization of (Jensen) convexity to non-convex sets (or in general arbitrary subsets of groups) proposed and studied by W. Jarczyk and M. Laczkovich [5, 6]:

Definition 1.4 ([6])**.** Let G be an Abelian group and let V be a subset of G. We say that $f\colon V \to \mathbb{R}$ is *convex* if following inequality holds

$$f(x) \le \frac{f(x+\delta)+f(x-\delta)}{2} \quad \text{for } x \in V, \delta \in G \text{ such that } x+\delta, x-\delta \in V.$$

In our paper we generalize the definition of approximate convexity in the spirit of the previous definition:

Definition 1.5. Let V be a subset of an Abelian group G and let $\omega : V \times V \to [0,\infty]$ such that $\omega(x,x) = 0$ for $x \in V$ be given.

We say that a function $f\colon V \to \mathbb{R}$ is $\omega(\cdot,\cdot)$-*midconvex* (or $\omega(\cdot,\cdot)$-*Jensen convex*) if

$$f(x) \le \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta, x+\delta) \quad \text{for } x \in V, \delta \in G\colon x-\delta, x+\delta \in V.$$

Similarly to the standard case, the study of such functions and their understanding can be often deduced from the properties of the set

$$J_\omega(V;W) := \{f \in \mathcal{B}(V;W)\colon f \text{ is } \omega(\cdot,\cdot)\text{-midconvex}\}.$$

Our aim in this paper is to present a computer assisted approach which given a finite set V can find within a specified error bound the optimal estimation from above of $J_\omega(V,W)$, that is

$$f_\omega(V,W) := \sup\{f \in J_\omega(V,W)\}.$$

We illustrate our approach in the simplest case when $V = \{0, 1/N, \ldots, (N-1)/N, 1\}$ and $W = \{0,1\}$.

2 Estimate of optimal $\omega(\cdot,\cdot)$-midconvex functions.

In this section we discuss the construction of optimal ω-Jensen convex functions.

Let V be a given subset of an Abelian group G. By Δ_V we understand the diagonal in $V \times V$, that is $\Delta_V := \{(v,v) : v \in V\}$. From now on we assume that $\omega : V \times V \to [0,\infty]$, $\omega(\Delta_V) = 0$ is fixed.

First of all, we introduce the operation $P_\omega : [-\infty,\infty)^V \to [-\infty,\infty)^V$ as follows

$$P_\omega f(x) := \inf\{\frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta, x+\delta) | \delta \in G \colon x-\delta, x+\delta \in V\},$$

for $f \in [-\infty,\infty)^V$.

Proposition 2.1. Let $f, g \in [-\infty,\infty)^V$ be arbitrary functions and $\omega : V \times V \to [0,\infty]$, $\omega(\Delta_V) = 0$ be fixed function. Then operation P_ω has following properties:

1. $P_\omega g \le g$,
2. if $g \ge f$, then $P_\omega g \ge P_\omega f$,
3. $P_\omega(0) \equiv 0$,
4. $P_\omega g \ge 0$ for $g \ge 0$.

Proof. Ad 1. Suppose the assertion of this properties is false, so there exists $x \in V$ such that $P_\omega g(x) > g(x)$. Thus according to the definition of P_ω for all $\delta \in G \colon x - \delta, x+\delta \in V$ we have $\frac{g(x-\delta)+g(x+\delta)}{2} + \omega(x-\delta, x+\delta) > g(x)$. Which lead us to contradiction because by setting $\delta = 0$ we get $g(x) > g(x)$.

Other properties are obvious and can be proved similarly to the first one. □

Furthermore, the operation $P_\omega^\infty : [-\infty,\infty)^V \to [-\infty,\infty)^V$

$$P_\omega^\infty f := \lim_{n\to\infty} P_\omega^n f$$

is well-defined, because operation P_ω is decreasing. Thus according to Proposition 2.1 we get that $P_\omega^\infty g \ge 0$ for $g \ge 0$.

Using this we can make observation:

Lemma 2.2. Let $f, g \in [-\infty,\infty)^V$ be arbitrary functions and $\omega : V \times V \to [0,\infty]$, $\omega(\Delta_V) = 0$ be fixed. If f is $\omega(\cdot,\cdot)$-midconvex, then $P_\omega f = f$. Thus, if $g \ge f$, then $f = P_\omega f \le P_\omega g$, and consequently

$$f \le P_\omega^\infty g.$$

Proof. Let f and ω fulfills lemma assumptions. If f is $\omega(\cdot,\cdot)$-midconvex, then

$$\begin{aligned} P_\omega f(v) =& \inf\{\frac{f(v-\delta)+f(v+\delta)}{2} + \omega(v-\delta, v+\delta) | \delta \in G\colon v-\delta, v+\delta \in V\} \\ =& \frac{f(v)+f(v)}{2} + \omega(v,v) = f(v) \quad \text{for } v \in V. \end{aligned}$$

Second assertion is obvious. □

We are interested in the class of approximately convex functions which are zero on W, $(W \subset V)$. We want to find the optimal estimation (from above) of elements of this class. We put

$$f_\omega(V;W) := \sup\{f \in J_\omega(V;W)\}.$$

There appears a question how to compute the function $f_\omega(V;W)$.

As in many cases the estimation of the $\omega(\cdot,\cdot)$-Jensen convex function we are interested in, can be deduced from the knowledge of $f_\omega(V;W)$ – for example if we want to find an estimate of f (which we assume to be bounded and ω-Jensen convex) on the interval $[a,b]$, by subtracting the respective affine function (namely $x \to f(a) + \frac{x-a}{b-a}[f(b)-f(a)]$) we can reduce to the case when $f(a) = f(b) = 0$. So we can restrict to investigation of bounded approximately Jensen convex functions on the interval $[0,1]$, which are zero at 0 and 1 (so $V = [0,1]$ and $W = \{0,1\}$).

Next theorem give us the way to estimate upper bound of $f_\omega(V;W)$. We use the notation

$$\mathbb{1}_{V;W}\colon V \in v \to \begin{cases} 1 & \text{for } v \in V \setminus W, \\ 0 & \text{for } v \in W. \end{cases}$$

Theorem 2.3. Let V and $W \subset V$ be given subsets of an Abelian group G. We assume that

$$\exists\, A \geq 0\ \forall\, f \in J_\omega(V;W)\colon f \leq A. \tag{1}$$

Then

$$f_\omega(V;W) = P_\omega^\infty(A\mathbb{1}_{V;W})$$

and f_ω is $\omega(\cdot,\cdot)$-midconvex.

Proof. By the assumptions

$$f_\omega(V;W) \leq A\mathbb{1}_{V;W}$$

and consequently the inequality

$$f_\omega(V;W) \leq P_\omega^\infty(A\mathbb{1}_{V;W})$$

holds.

We prove the opposite inequality. For $n \in \mathbb{N} \cup \{\infty\}$ we put

$$g_n := P_\omega^n(A\mathbb{1}_{V;W}).$$

Clearly, g_n converges pointwise, as $n \to \infty$, to $g_\infty := \lim_{n\to\infty} g_n$. On the other hand directly from the definition we know that

$$g_{n+1}(v) \le \frac{g_n(v-\delta) + g_n(v+\delta)}{2} + \omega(v-\delta, v+\delta)|\delta \in G\colon v-\delta, v+\delta \in V.$$

By taking the limit we get

$$g_\infty(v) \le \frac{g_\infty(v-\delta) + g_\infty(v+\delta)}{2} + \omega(v-\delta, v+\delta)|\delta \in G\colon v-\delta, v+\delta \in V,$$

which implies that g_∞ is $\omega(\cdot,\cdot)$-Jensen convex, and consequently $g_\infty \in J_\omega(V;W)$. □

Example 2.4. The assumption (1) is not redundant. Consider $V = \{0\} \cup [\frac{1}{2}, 1]_N$ and $W = \{0, 1\}$ subsets of $\mathbb{R}$. This situation allows us to calculate P_ω on set V. However, for set $V = \{0, \frac{1}{3}, 1\}$ and $W = \{0, 1\}$ (subsets of $\mathbb{R}$) we cannot established operator P_ω, because we cannot calculate the value $P_\omega(\frac{1}{3})$, so it could be arbitrary large.

Now we can easily obtain lower bound of optimal $\omega(\cdot,\cdot)$-midconvex function.

Theorem 2.5. Let V and $W \subset V$ be given subsets of an Abelian group G. We assume that

$$\exists\, A \ge 0\ \forall\, f \in J_\omega(V;W)\colon f \le A.$$

Let $h\colon V \to \mathbb{R}$ be such that

$$h \ge P_\omega^\infty(A\mathbb{1}_{V;W}).$$

If $(1-\varepsilon)h$ is $\omega(\cdot,\cdot)$-midconvex for some $\varepsilon \in (0,1)$, then

$$(1-\varepsilon)h \le f_\omega(V;W) \le h.$$

Proof. According to Theorem 2.3 we have that $f_\omega \le h$, because function $P_\omega^\infty(A\mathbb{1}_{V;W})$ is $\omega(\cdot,\cdot)$-midconvex. Lower bound is consequence of definition f_w as a supremum of set $J(V;W)$ while directly from the assumptions $(1-\varepsilon)h \in J(V;W)$. □

3 Strict numerical verification

In this section we give two algorithms which help us to encode the results obtained in the previous section and create application which founds bounds of $f_\omega(V;W)$ for V and $W \subset V$ finite subsets of an Abelian group G.

We introduce algorithm that summarizes results obtained in Theorem 2.3 and Theorem 2.5 which give us that outcome function from our construction is $\omega(\cdot,\cdot)$-midconvex:

choose
$A \quad \geq 0$ such that $\forall\, f \in J_\omega(V;W)\colon f \leq A$
$\varepsilon \quad \in (0,1)$ (precision)
$n \quad \leftarrow 1$
repeat
$h_n \quad \leftarrow$ upper bound for $P_\omega^n(A\mathbb{1}_{V;W})$
$n \quad \leftarrow n+1$
until $(1-\varepsilon)h_n$ is not $\omega(\cdot,\cdot)$-midconvex
return we get estimation $(1-\varepsilon)h_n \leq f_\omega(V;W) \leq h_n$

As it occurs the above algorithm is inconvenient for implementation because states *calculate* h_n and *check that* $(1-\varepsilon)h_n$ *is* $\omega(\cdot,\cdot)$*-midconvex* slow it down. Hence we try to modify those calculations to make it faster.

But first we have to answer the question: how we can find upper bound for $P_\omega^n(A\mathbb{1}_{V;W})$ for fixed $n \in \mathbb{N}$? To solve this problem we prepared all calculations using interval aritmetics which allows us to deal with finite precision of computer calculations and control error value [2, 10] (for implementation see [14]). When we work with interval arithmetic, instead of considering real number (ex. $\sqrt{3}$) we work with the interval (ex. $[1.7320; 1.7321]$) which contains our number lies between lower and upper bound of this interval.

Main algorithm

Let us start with useful notations:

$$K(V) = \{(v,\delta) | v \in V, \delta \in G : v-\delta, v+\delta \in V\},$$

where V is given finite subset of Abelian group G ($\operatorname{card} K(V) \leq (\operatorname{card} V)^2$, because for pair $v, v+\delta \in V$ we can recover $\delta \in G$).

Definition 3.1. Let V be given finite subset of an Abelian group G and let $(v,\delta) \in K(V)$. We define operator $\mathcal{P}_{(v,\delta)} : [-\infty,\infty)^V \to [-\infty,\infty)^V$ as follows:

$$\mathcal{P}_{(v,\delta)}f\colon V \ni x \to \begin{cases} \min\left\{f(x), \frac{f(x-\delta)+f(x+\delta)}{2} + \omega(x-\delta, x+\delta)\right\} & \text{for } x = v, \\ f(x) & \text{for } x \neq v, \end{cases}$$

for $f \in [-\infty,\infty)^V$.

As we see for every $f \in [-\infty,\infty)^V$ the operator $\mathcal{P}_{(v,\delta)}$ modifies the function f only at the point v. Also we get that $\mathcal{P}_{(v,\delta)}f \leq f$.

Given a sequence $S = (s_1,\ldots,s_n)$ of elements of $K(V)$ we denote

$$\mathcal{P}_S = \mathcal{P}_{s_n} \circ \ldots \circ \mathcal{P}_{s_1}.$$

From now on $S = \{s_1, \dots, s_n\}$ denotes a fixed sequence such that

$$K(V) = \bigcup_{i=1}^{n} \{s_i\} \text{ and } n = \operatorname{card} K(V).$$

To simplify notation from now on we use the letter $\mathcal{P}$ instead of $\mathcal{P}_S$.

As we show, we can apply it for function $h_A \colon V \ni v \to A\mathbb{1}_{V;W} \in \mathbb{R}_+$ and obtain upper bound for $P_\omega(A\mathbb{1}_{V;W})$.

Lemma 3.2. Let V be a finite subset of Abelian group G. We have $P_\omega^{\operatorname{card} K(V)} f \le \mathcal{P}f \le P_\omega f$ for $f \in [-\infty, +\infty)^V$.

Proof. Let $f \in [-\infty, +\infty)^V$. According to Definition 3.1 we have that $P_\omega f \le \mathcal{P}_{(v,\delta)} f$ for all $(v, \delta) \in K(V)$, which implies $P_\omega^{\operatorname{card} K(V)} f \le \mathcal{P}f$.

We check now second inequality, so we want to show that for every $v \in V$: $\mathcal{P}f(v) \le P_\omega f(v)$. Let us choose arbitrary $v \in V$. We have that

$$P_\omega f(v) = \inf\{\frac{f(v-\delta) + f(v+\delta)}{2} + \omega(v-\delta, v+\delta) | \delta \in G \colon v-\delta, v+\delta \in V\}.$$

Because V is finite there exists such $\delta \in G$ fulfilling those infimum. Thus we obtain $\mathcal{P}_{(v,h)}$ shuch that $\mathcal{P}_{(v,h)} f(v) \le P_\omega f(v)$. This finishes the proof, because v was arbitraty choosen.

□

We see that the operator $\mathcal{P}$ converges faster then P_ω.

What is left is to show that there exists $A \ge 0$ such that for all $f \in J_\omega(V;W) \colon f \le A$? In general case it is hard to verify if there exists such A that condition (1) holds (or even estimate it). However in the case where $V = [0,1]_N = \{0, 1/N, \dots, (N-1)/N, 1\}$ and $W = \{0, 1\}$ we can put (see. [11, Corollary 2.1])

$$A = 2 \sup_{x,y \in [0,1]_N} \omega(x, y).$$

Thus we obtain the following observation (special case of Theorem 2.5).

Theorem 3.3. Let $\omega : [0,1]_N \times [0,1]_N \to \mathbb{R}_+$ and $C \ge 2 \sup \omega$ be given. Let $h \colon [0,1]_N \to \mathbb{R}$ be such that

$$h \ge \mathcal{P}^k(C\mathbb{1}_{[0,1]_N;\{0,1\}})$$

for some $k \in \mathbb{N}$. If $(1-\varepsilon)h$ is $\omega(\cdot,\cdot)$-midconvex for some $\varepsilon \in (0,1)$, then

$$(1-\varepsilon)h \le f_\omega([0,1]_N; \{0,1\}) \le h.$$

So we can conclude by presenting full algorithm for finding estimation of $f_\omega([0,1]_N; \{0,1\})$:

choose
 $C \geq 0$ such that for fixed $\omega\colon V \times V \to \mathbb{R}_+, C \geq 2 \sup \omega$
 $h_C\colon V \ni v \to C\mathbb{1}_{V;W} \in \mathbb{R}_+$
for $n \in \{1, 2, \ldots, N_{MAX}\}$ **do**
   $h_C \leftarrow \mathcal{P} h_C$
end for
return h_C – upper bound of $P_\omega^\infty(C\mathbb{1}_{V;W})$

Estimating the error

Using the operator $\mathcal{P}$ we can get function h_C – upper bound of $f_\omega([0,1]_N; \{0,1\})$. To obtain lower bound we calculate the error considered in Observation 3.3 by choosing $\varepsilon \in (0,1)$ such that

$$\frac{1}{1-\varepsilon} \geq \sup\left\{\frac{h_C(x) - \frac{h_C(x-\delta)+h_C(x+\delta)}{2}}{\omega(x-\delta, x+\delta)} : x-\delta, x, x+\delta \in [0,1]_N, \delta \in \mathbb{R}, \delta \neq 0\right\}. \quad (2)$$

Application example

We created application (using Java programming language and following libraries [13], [14]) which applied operator $\mathcal{P}$ to specified function ω and present obtained function plot.

This application is available to download from:
`http://www.ii.uj.edu.pl/~misztalk/index.php?page=convex`
Plots prepared in this program are presented on Figures 1 and 3. All this pictures presents not one but two functions – lower and upper bound of $J_\omega([0,1]_N; \{0,1\})$, however the distance between them is so small that we cannot separate them from each other.

Numerical experiments

Let us fix $\omega(x,y) = |x-y|$ for $x, y \in [0,1]_{1024}$.

We investigate how many iteration of the operator $\mathcal{P}$ we need to obtain small ε. So we apply operator $\mathcal{P}$ and then calculate ε according to equation (2). The results are presented on Figure 2. Surprising is that we need such few iterations to get high precision level – in this case it is sufficient to take 10 iterations to obtain $\varepsilon = 5.684 \cdot 10^{-14}$.

4 Estimation of optimal midconvexity on $[0,1]_N$

In this section we will recall estimation of optimal midconvex function applied for $[0,1]_N$ for fixed $N \in \mathbb{N}$.

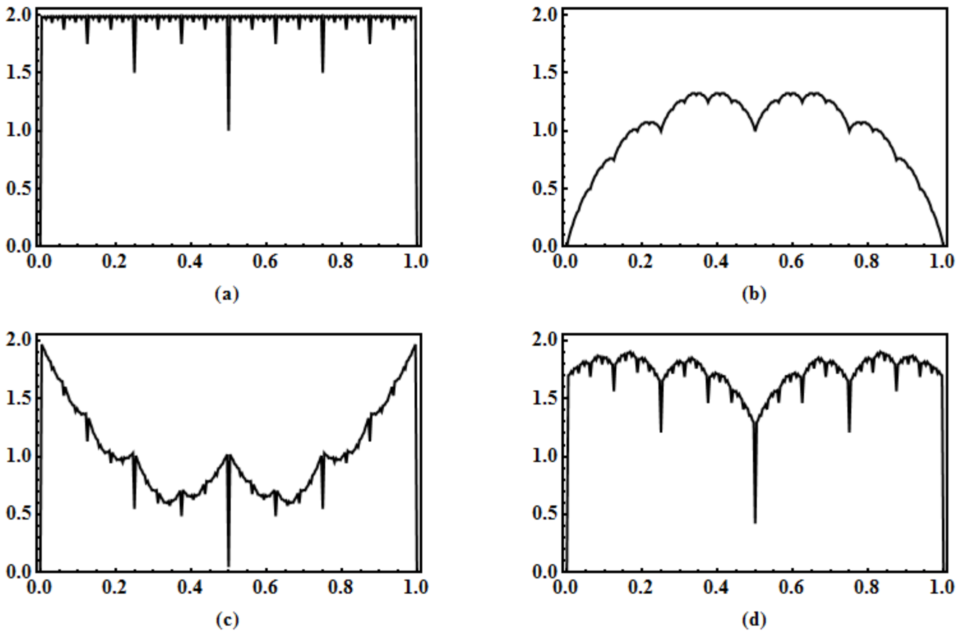

Figure 1: Iteration of operator $\mathcal{P}$ for different functions ω: (a) $\omega(x,y) = |x-y|^{0.001}$, $x, y \in [0,1]_{1024}$. We obtain $\varepsilon = 2.22 \cdot 10^{-16}$. (Compare with [8]). (b) $\omega(x,y) = |x-y|$, $x, y \in [0,1]_{1024}$, $\varepsilon = 4.663 \cdot 10^{-15}$. For this ω we have Takagi-like function [1]. (c) $\omega(x,y) = (\cos|x-y|)^5$, $x \in [0,1]_{1024}$, $\varepsilon = 8.882 \cdot 10^{-16}$. (d) $\omega(x,y) = \sin(\exp|x-y|)$, $x \in [0,1]_{1024}$, $\varepsilon = 2.22 \cdot 10^{-16}$.

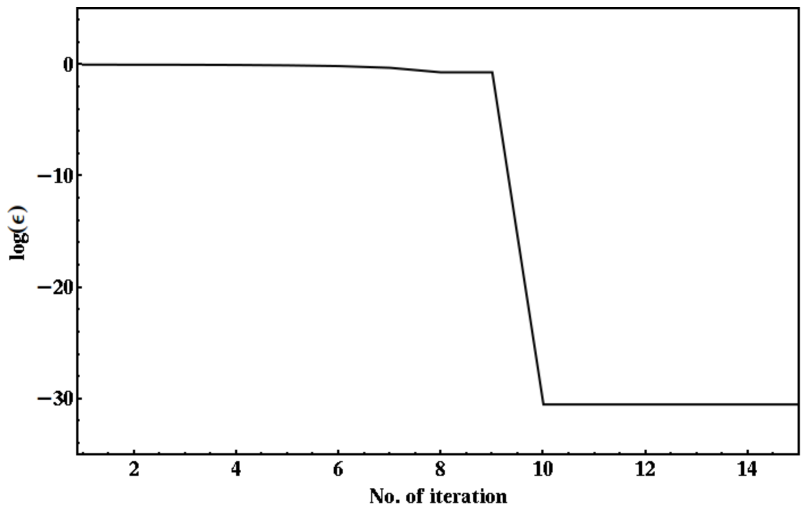

Figure 2: Error ε as a function of iteration the operator $\mathcal{P}$ for $\omega(x,y) = |x-y|$ under interval $[0,1]_{1024}$.

We recall two estimations for locally bounded $\alpha(\cdot)$-midconvex functions on $[0,1]_N$. But firstly let us denote $d(x) := 2\text{dist}(x,\mathbb{Z})$ for $x \in \mathbb{R}$. Then estimation can be stated as follows:

Theorem 4.1 ([11, Corollary 2.1, Proposition 3.1])**.** Let $N = 2^k$ for certain $k \in N$. Let $h : [0,1]_N \to \mathbb{R}$, $h(0) = h(1) = 0$ be an $\alpha(\cdot)$-midconvex function. Then

$$h(q) \le \min\{ \underbrace{\sum_{k=0}^{\infty} \frac{1}{2^k}\alpha(d(2^k q)),}_{\mathbf{E1}} \underbrace{\sum_{k=0}^{\infty} \alpha(1/2^k)d(2^k q)}_{\mathbf{E2}} \} \quad \text{for } q \in [0,1]_N. \tag{3}$$

Observation 4.2. Let V and $W \subset V$ be given subsets of an Abelian group G. If $V \subset \widehat{V}$, then $f_\omega(\widehat{V};W)|_V \le f_\omega(V;W)$.

Theorem 4.3. Let $V = [0,1]_N$ for $N = 2^k$, $k \in \mathbb{N}$, $k \ge 3$ and $W = \{0,1\}$. For $\omega(x,y) = \sin(\cos(|x-y|))$ approximations of $f_\omega([0,1]_N, \{0,1\})$ obtained by (3) are not optimal (see Figure 3).

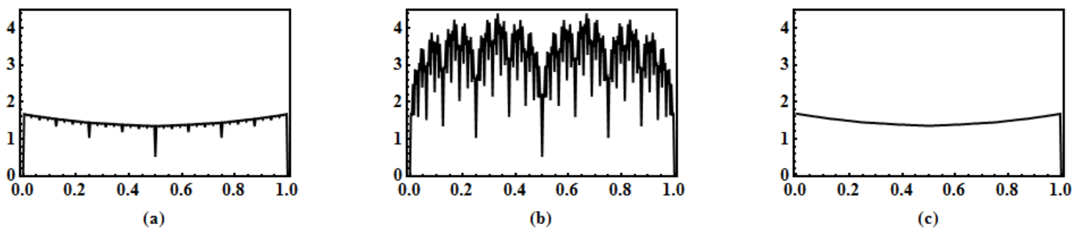

Figure 3: Graph of comparison of three estimators: (a) P_ω^∞, (b) **E1**, (c) **E2** for $\omega(x,y) = \sin(\cos(|x-y|))$ on the set $[0,1]_{256}$.

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