Supplement to the paper "Quasianalytic perturbation of multi-parameter hyperbolic polynomials and symmetric matrices

Krzysztof Jan Nowak

IMUJ PREPRINT 2011/04

Abstract

In our paper [IMUJ Preprint 5 (2009)], we investigated the quasianalytic perturbation of hyperbolic polynomials and symmetric matrices by applying our quasianalytic version of the Abhyankar–Jung theorem from [IMUJ Preprint 2 (2009)], whose proof relied on a theorem by Luengo on ν-quasiordinary polynomials. But those papers of ours were suspended after we had become aware that Luengo's paper contained an essential gap. This gave rise to our subsequent article on quasianalytic perturbation theory, which developed, however, different methods and techniques. A recent paper by Parusiński–Rond validates Luengo's result, which allows us to resume our previous approach.

1. Introduction. Our papers [7, 8] were devoted to carrying the results by Kurdyka–Paunescu [4] concerning the perturbation of hyperbolic polynomials and symmetric matrices with analytic coefficients over to the quasianalytic settings. Our proofs of those results relied on the following

2010 MSC: 14P15, 32B20, 15A18, 26E10.

Key words: quasianalytic functions, hyperbolic quasiordinary polynomials.

theorem on splitting of hyperbolic quasiordinary polynomials (Theorem 1* from [8]):

Main Theorem. *Let Ω be an open, simply connected subset of $\mathbb{R}^m$. Then every hyperbolic quasiordinary polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathcal{Q}(\Omega)[t]$$

with quasianalytic coefficients splits into linear factors of the form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_i(x)), \quad x \in \Omega,$$

where $\psi_i(x)$ are smooth (i.e. of class C^∞) functions quasi-subanalytic on Ω.

The proof of this theorem given in [8] was quite long and technically complicated, making use, inter alia, of the technique of global (canonical) desingularization. The proof presented in [7] was much more elementary, but it applied a quasianalytic version of the Abhjankar–Jung theorem from our paper [6]. The latter was based on a theorem by Luengo [5] about ν-quasiordinary polynomials, which contained, however, an essential gap (cf. [3]). The recent paper [9] by Parusiński–Rond validates Luengo's result, allowing us to resume our previous approach. Actually, they give a short, elementary proof that Luengo's result is equivalent to the formal version of the Abhyankar–Jung theorem.

This paper is organized as follows. The first section recalls the version of the Abhyankar–Jung theorem for certain henselian $k[x]$-algebras, established in [6]. In Section 2, we demonstrate how the main theorem follows from it.

2. The Abhyankar–Jung theorem for henselian subrings of formal power series. We call a polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in k[[x]][t], \quad x = (x_1, \ldots, x_m),$$

quasiordinary if its discriminant $D(x)$ is a normal crossing:

$$D(x) = x^\gamma \cdot u(x) \quad \text{with} \quad \gamma \in \mathbb{N}^m,\ u(x) \in k[[x]],\ u(0) \neq 0.$$

We say that $f(x;t)$ is a Weierstrass polynomial if its coefficients $a_i(x)$ belong to the maximal ideal of $k[[x]]$, i.e. $a_k(0) = 0$. Let us write

$$f(x;t) = \sum_{\alpha \in \mathbb{N}^m} \sum_{k=0}^{n} a_{\alpha,k} \cdot x^\alpha t^k$$

and put

$$E(f) := \{(\alpha_1, \ldots, \alpha_m, k) \in \mathbb{N}^{m+1} : a_{\alpha,k} \neq 0\}.$$

By the Newton polyhedron $N(f)$ of the polynomial $f(x,t)$ we mean the convex hull of $E(f) + \mathbb{N}^{m+1}$. We say, after Hironaka [2], that the polynomial $f(x;v)$ is ν-quasiordinary with an exponent $\delta = (\delta_1 \ldots, \delta_m) \in \mathbb{Q}^m$ if

1) $N(f) \subset S + [0,\infty)^{m+1}$ and $S \cap E(f) \neq \emptyset$, where S is the segment joining the points $(0, \ldots, 0, n)$ and $(\delta_1, \ldots, \delta_m, 0)$;

2) the polynomial

$$P(x,t) := \sum_{(\alpha,k)\in S} a_{\alpha,k} x^\alpha t^k$$

is not a power of a linear form.

The first condition means that the projection of the set $N(f) \cap \{t < n\}$ from the point $(0, \ldots, 0, n)$ onto the hyperplane $t = 0$ is exactly $\delta + [0,\infty)^m$.

Now let us recall the following result due to Luengo [5], Theorem 1 (see also [9], Theorem 1.2):

Proposition. *Every quasiordinary Weierstrass polynomial*

$$f(x;t) = t^n + a_{n-2}(x)t^{n-2} + \cdots + a_0(x) \in k[[x]][t], \quad x = (x_1, \ldots, x_m),$$

with vanishing coefficient of t^{n-1}*, is* ν*-quasiordinary.*

Since $a_{n-1}(x) \equiv 0$, only condition 1) from the above definition needs a verification in the proof of the propositon. By means of the Tschirnhausen transformation

$$t' = t + 1/n \cdot a_{n-1}(x),$$

one can always come to the case of a polynomial with vanishing coefficient of t^{n-1} without changing the discriminant. The converse is not true as shown in the following example from [5].

Example. The polynomial

$$g(x_1, x_2; t) := t^4 - 2x_1x_2^2 \cdot t^2 + x_1^4x_2^4 + x_1^2x_2^7$$

is ν-quasiordinary but not quasiordinary since its discriminant $D(x_1, x_2)$ is divisible by $x_1x_2(x_1^2 + x_2^3)$.

Remark 1. Since the discriminant of a monic polynomial is a weighted polynomial in its coefficients, the discriminant $D(x)$ of the foregoing ν-quasiordinary polynomial $f(x;t)$ with exponent δ is divisible by $x^{(n-1)\delta}$. Therefore, if the discriminant $D(x)$ is a normal crossing $D(x) = x^{\gamma} \cdot u(x)$, then $\gamma \geq (n-1)\delta$, i.e. $\gamma_i \geq (n-1)\delta_i$ for all $i = 1, \ldots, m$. In particular, $\delta_i = 0$ whenever $\gamma_i = 0$.

Let k be an algebraically closed field of characteristic zero. Consider a henselian $k[x]$-subalgebra $k\langle x\rangle$ of the formal power series ring $k[[x]]$, $x = (x_1, \ldots, x_m)$, which is closed under reciprocal (whence it is a local ring), power substitution and division by a coordinate. For positive integers $r_1, \ldots, r_m$ put

$$k\langle x_1^{1/r_1}, \ldots, x_m^{1/r_m}\rangle := \{a(x_1^{1/r_1}, \ldots, x_m^{1/r_m}) : a(x) \in k\langle x\rangle\};$$

when $r_1 = \ldots = r_m = r$, we shall denote the above algebra by $k\langle x^{1/r}\rangle$.

Abhyankar–Jung Theorem. *Under the above assumptions, every quasiordinary polynomial*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in k\langle x\rangle[t]$$

has all its roots in $k\langle x^{1/r}\rangle$*, for some* $r \in \mathbb{N}$*; actually, one can take* $r = n!$.

The proof is by induction with respect to the degree of the polynomial $f(x;t)$. Performing the Tschirnhausen transformation, we may assume that $a_{n-1}(x) \equiv 0$. If $f(x;t)$ is not a Weierstrass polynomial, then $f(0;t)$ is not a power of a linear form. Since the ring $k\langle x\rangle$ of coefficients is henselian, the polynomial $f(x;t)$ is reducible: $f(x;t) = f_1(x;t)f_2(x;t)$. The theorem thus follows from the induction hypothesis.

Otherwise, $f(x;t)$ is a Weierstrass polynomial, and then, by the proposition, $f(x;t)$ is a ν-quasiordinary polynomial with an exponent $\delta \in \mathbb{Q}^m$. Take any multi-index $(\beta_1, \ldots, \beta_m, l) \in E(f)$ that lies on the segment S from the definition of ν-quasiordinarity. This property of the polynomial $f(x;t)$ implies immediately the inequalities:

$$(n-l)\alpha \geq (n-k)\delta \quad \text{for all} \quad (\alpha, k) \in E(f).$$

Moreover, for at least one multi-index from $E(f) \cap S$, we have equality.

Therefore, in the new coordinates

$$x_1 = y_1^{n-l}, \ \ldots, \ x_m = y_m^{n-l}, \ t = w \cdot y_1^{\delta_1} \cdots y_m^{\delta_m},$$

each $a_k(x) = a_k(y_1^{n-l}, \ \ldots, y_m^{n-l})$, $k = 0, 1, \ldots, n-2$, is divisible by

$$y_1^{(n-k)\delta_1} \cdots y_m^{(n-k)\delta_m}.$$

Hence

$$f(x;t) = f(y^{n-l}, \ \ldots, y_m^{n-l}; t) =$$
$$= t^n + a_{n-2}(y_1^{n-l}, \ldots, y_m^{n-l}) \cdot t^{n-2} + \ldots + a_0(y_1^{n-l}, \ldots, y_m^{n-l}) =$$
$$= t^n + y_1^{2\delta_1} \cdots y_m^{2\delta_m} \cdot b_{n-2}(y) \cdot t^{n-2} + \ldots + y_1^{n\delta_1} \cdots y_m^{n\delta_m} \cdot b_0(y),$$

with $b_k(y) \in \ k\langle y \rangle$. Moreover, at least one coefficient from among $b_k(y)$, $k = 0, \ldots, n-2$, is a unit: $b_k(0) \neq 0$. We thus get

$$f(x;t) = y_1^{n\delta_1} \cdots y_m^{n\delta_m} \cdot g(y;w),$$

where

$$g(y;w) = w^n + b_{n-2}(y) w^{n-2} + \ldots + b_0(y) \in k\langle y \rangle[w].$$

Consequently, the polynomial $g(0,w)$ is not a power of a linear form. Since the ring $k\langle y \rangle$ of coefficients is henselian, the polynomial $g(y;w)$ is reducible: $g(y;w) = g_1(y;w) \cdot g_2(y;w)$. Therefore the proof is complete again by the induction hypothesis.

Remark 2. Suppose that the discriminant $D(x)$ of the polynomial $f(x;t)$ is a normal crossing of the form

$$D(x) = x_1^{\gamma_1} \cdots x_p^{\gamma_p} \cdot u(x) \quad \text{with} \quad u(0) \neq 0, \quad 0 \leq p \leq m.$$

Then $\delta_{p+1} = \ldots = \delta_m = 0$ (cf. Remark 1), and thus the inequalities $(n-l)\alpha \geq (n-k)\delta$ from the above proof are trivially satisfied for all α and k. It is therefore sufficient to change only the first p from among the variables x. Consequently, all the roots of the polynomial $f(x;v)$ belong to $k\langle x_1^{1/r}, \ldots, x_p^{1/r}, x_{p+1}, \ldots, x_m \rangle$.

3. Proof of the Main Theorem. Denote by $\mathcal{Q}_m = \mathbb{R}\langle x \rangle$ the ring of germs at $0 \in \mathbb{R}^m$ of smooth quasi-subanalytic functions, and put

$$\mathbb{C}\langle x \rangle := \mathcal{Q}_m \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{R}\langle x \rangle \otimes_{\mathbb{R}} \mathbb{C};$$

here $m \in \mathbb{N}$ and $x = (x_1, \ldots, x_m)$. $\mathbb{C}\langle x \rangle$ may be regarded, of course, as a henselian $\mathbb{C}[x]$-subalgebra of the formal power series ring $\mathbb{C}[[x]]$, which is closed under reciprocal, power substitution and division by a coordinate.

The Abhyankar–Jung theorem from Section 2 yields immediately the following two corollaries.

Corollary 1. *Consider a quasiordinary polynomial*

$$h(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathbb{C}\langle x\rangle[t].$$

Then there exists an $r \in \mathbb{N}$ such that for each closed orthant Q_k in $\mathbb{R}^m$, $k = 1, \ldots, 2^m$, we have in the vicinity of $0 \in \mathbb{R}^m$ a factorization of the form

$$h(x;t) = \prod_{i=1}^{n} (t - \varphi_{ik}(|x_1|^{1/r}, \ldots, |x_m|^{1/r})) \quad \textit{for} \quad x \in Q_k,$$

where $\varphi_{ik} \in \mathbb{C}\langle x\rangle$; actually, one can take $r = n!$.

Corollary 2. (A real version of the Abhyankar–Jung theorem) *Let*

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathbb{R}\langle x\rangle[t]$$

be a quasiordinary polynomial. Then there exists an $r \in \mathbb{N}$ such that for each closed orthant Q_k in $\mathbb{R}^m$, $k = 1, \ldots, 2^m$, we have in the vicinity of $0 \in \mathbb{R}^m$ a factorization of the form

$$f(x;t) = \prod_{i=1}^{p} (t - \varphi_{ik}(|x|^{1/r})) \prod_{j=1}^{q} (t^2 - \alpha_{jk}(|x|^{1/r})t + \beta_{jk}^2(|x|^{1/r})) \quad \textit{for} \quad x \in Q_k,$$

where $p + 2q = n$, $\varphi_{ik}, \alpha_{jk}, \beta_{jk} \in \mathbb{R}\langle x\rangle$ and $|x|^{1/r} = (|x_1|^{1/r}, \ldots, |x_m|^{1/r})$; actually, one can take $r = n!$.

Before turning to hyperbolic polynomials, we still need to look more carefully at Corollary 1. For any closed subset $A \subset \mathbb{R}^m$, let $\mathcal{C}(A)$ and $\mathcal{D}(A)$ be the $\mathbb{R}$-algebras of those quasi-subanalytic functions on A which are continuous and smooth in a neighbourhood of A, respectively; put

$$\mathcal{C}(A, \mathbb{C}) := \mathcal{C}(A) \otimes_{\mathbb{R}} \mathbb{C} \quad \text{and} \quad \mathcal{D}(A, \mathbb{C}) := \mathcal{D}(A) \otimes_{\mathbb{R}} \mathbb{C}.$$

By symmetry, we may confine our considerations to the first closed orthant $Q = Q_1 = [0, \infty)^m$. The quasianalytic function germs

$$\varphi_i(x^{1/r}) := \varphi_{i1}(x_1^{1/r}, \ldots, x_m^{1/r})$$

have representatives which belong to $\mathcal{C}([0,\delta]^m, \mathbb{C})$ with $\delta > 0$ small enough; denote by $\widehat{\varphi}_i(x^{1/r})$ their Puiseux series. Let ϵ be a primitive r-th root of unity. It is easy to check that each algebraic conjugate

$$\widehat{\varphi}_i(\epsilon^{\alpha_1} x_1^{1/r}, \ldots, \epsilon^{\alpha_m} x_m^{1/r}), \quad \alpha = (\alpha_1, \ldots, \alpha_m) \in \mathbb{N}^m,$$

of any $\widehat{\varphi}_i(x^{1/r})$ is the Puiseux series $\widehat{\varphi_j}(x^{1/r})$ of some $\varphi_j(x^{1/r})$. In other words, the Puiseux series $\widehat{\varphi}_i(x^{1/r})$, $i = 1, \ldots, n$, are preserved under algebraic conjugacy.

We call a monic polynomial

$$f(x;t) = t^n + a_{n-1}(x)t^{n-1} + \cdots + a_0(x) \in \mathbb{R}\langle x \rangle [t]$$

hyperbolic if, for each value of the parameters x, all its roots are real. This is a shortened name for "a quasianalytic family of hyperbolic polynomials".

Now, we can readily turn to the proof of the main theorem on splitting of hyperbolic quasiordinary polynomials. Since the splitting problem is local, we can confine ourselves to consider quasianalytic function germs $a_i(x)$, $i = 0, 1, \ldots, n-1$, at zero. Without loss of generality, we may assume that they have representatives which are quasianalytic in a neighbourhood of $[-\delta, \delta]^m$ for some $\delta > 0$ small enough, and that the discriminant $D(x)$ of the polynomial $f(x;t)$ is of the form $D(x) = x^\gamma \cdot u(x)$, where $\gamma \in \mathbb{N}^m$ and $u(x) \neq 0$ for $x \in [-\delta, \delta]^m$.

Keeping the foregoing notation, it is clear that the Puiseux series $\widehat{\varphi}_i(x^{1/r})$, $i = 1, \ldots, n$, of the roots $\varphi_i(x^{1/r})$ of the hyperbolic quasiordinary polynomial $f(x;t)$ are real series. Since they are preserved under algebraic conjugacy, we get $\widehat{\varphi}_i(x^{1/r}) \in \mathbb{R}[[x]]$.

The above reasoning about Puiseux series may be repeated at each point from $[0, \delta]^m$. Therefore it follows from Glaeser's composite function theorem (cf. [1]) that the functions $\psi_i(x) := \varphi_i(x^{1/r})$ are smooth:

$$\psi_i(x) = \varphi_i(x^{1/r}) \in \mathcal{D}([0,\delta]^m), \quad i = 1, \ldots, n.$$

Note that we applied, in fact, a very special case of Glaeser's theorem. Denote by $T_a \psi_i(x)$ the Taylor series at a point $a \in Q_1$ of the smooth function ψ_i, $i = 1, \ldots, n$.

For each closed orthant Q_k in $\mathbb{R}^m$, $k = 1, \ldots, 2^m$, we thus have in the vicinity of $0 \in \mathbb{R}^m$ a factorization of the form

$$f(x;t) = \prod_{i=1}^{n} (t - \psi_{ik}(x)) \quad \text{for all} \quad x \in Q_k \cap [-\delta, \delta]^m,$$

where $\psi_{ik} \in \mathcal{D}(Q_k \cap [-\delta, \delta]^m)$ with $\delta > 0$ small enough. But for every $k = 1, \ldots, 2^m$, the roots $\psi_{ik}(x)$ of the polynomial $f(x;t)$ determine common Taylor series $\widehat{\varphi}_i(x^{1/r}) \in \mathbb{R}[[x]]$, $i = 1, \ldots, n$. Consequently, those roots can be glued together to n smooth functions definable in a cube $[-\delta, \delta]^m$. Indeed, consider two adjacent orthants Q_k, Q_l with common face F, and next fix $i = 1, \ldots, n$ and put

$$F_j := \{a \in F : T_a \psi_{i,k}(x) = T_a \psi_{j,l}(x)\}, \quad j = 1, \ldots, n.$$

It is clear that $F_1, \ldots, F_n$ are closed, pairwise disjoint subsets of F such that $F = F_1 \cup \ldots \cup F_n$. Since F is a connected set, we get $F = F_{j(i)}$ for a unique $j(i) = 1, \ldots, n$. This means that the functions $\psi_{i,k}(x)$ and $\psi_{j(i),l}$ can be glued together to a smooth quasi-subanalytic function, which completes the proof.

Acknowledgements. This research was partially supported by Research Project No. N N201 372336 from the Polish Ministry of Science and Higher Education.

References

[1] G. Glaeser, *Fonctions composées différentiables*, Ann. Math. **77** (1963), 193–209.

[2] H. Hironaka, *Introduction to the theory of infinitely near singular points*, Memorias Math. Inst. Jorge Juan **28** (C.S.I.C., Madrid, 1974).

[3] K. Kiyek, J.L. Vicente, *On the Jung–Abhyankar theorem*, Arch. Math. **83** (2004), 123–134.

[4] K. Kurdyka, L. Paunescu, *Hyperbolic polynomials and multi-parameter real analytic perturbation theory*, Duke Math. J. **141** (2008), 123–149.

[5] I. Luengo, *A new proof of the Jung–Abhyankar theorem*, J. Algebra **85** (1983), 399–409.

[6] K.J. Nowak, *On the Abhyankar–Jung theorem for henselian $k[x]$-algebras of formal power series*, IMUJ Preprint **2**, 2009.

[7] —, *Hyperbolic polynomials and quasianalytic perturbation theory*, IMUJ Preprint **5**, 2009.

[8] —, *Quasianalytic perturbation of multi-parameter hyperbolic polynomials and symmetric matrices*, Ann. Polon. Math., to appear.

[9] A. Parusiński, G. Rond, *The Abhyankar–Jung theorem*, Eprint ArXiv: 1103.2559 [math.AC], (2011), v.1.

Institute of Mathematics
Jagiellonian University
ul. Profesora Łojasiewicza 6
30-348 Kraków, Poland
e-mail address: nowak@im.uj.edu.pl