CONE-FIELDS WITHOUT CONSTANT ORBIT CORE DIMENSION

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Abstract. As is well-known, the existence of a cone-field with constant orbit core dimension is, roughly speaking, equivalent to hyperbolicity, and consequently guarantees expansivity and shadowing.

In this paper we study the case when the given cone-field does not have the constant orbit core dimension. It occurs that we still obtain expansivity even in general metric spaces.

Main Result. *Let X be a metric space and let $f : X \rightharpoonup X$ be a given partial map. If there exists a uniform cone-field on X such that f is* cone-hyperbolic, *then f is* uniformly expansive, *i.e. there exists $N \in \mathbb{N}$, $\lambda \in [0, 1)$ and $\varepsilon > 0$ such that for all orbits* $\mathrm{x}, \mathrm{v} : \{-N, \dots, N\} \to X$

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \le \varepsilon \implies d(\mathrm{x}_0, \mathrm{v}_0) \le \lambda d_{\sup}(\mathrm{x}, \mathrm{v}).$$

We also show a simple example of a cone hyperbolic orbit in $\mathbb{R}^3$ which does not have the shadowing property.

1. **Introduction.** The notion of cone condition and cone-field [4, 8] originally appeared in the late 60's in the works of Alekseev, Anosov, Moser and Sinai. It can be well applied in the study of hyperbolic systems [2, 3, 8]. In particular Newhouse [8] gives conditions for existence of dominated and hyperbolic splittings on compact invariant sets for a diffeomorphism in terms of its induced action on a cone-field and its complement.

Precise definitions concerning cone-field are presented in the next sections. For the convenience of the reader we just recall that a cone-field C on a compact subset Λ of a finite dimensional Banach space E is constructed by a splitting

$$E = E^s_x \oplus E^u_x \text{ for every } x \in \Lambda.$$

We say that a diffeomorphism $f : U \to E$,where $\Lambda \subset U$, is *cone-hyperbolic* on Λ if it is both expanding and co-expanding on C. The cone-field C has the *constant orbit core dimension* on Λ if $\dim E^u_x = \dim E^u_{f(x)}$ for all $x \in \Lambda$. One of the main results from [8] is as follows:

Theorem N [8, Theorem 1.4]. *A necessary and sufficient condition for Λ to be a uniformly hyperbolic set for diffeomorphism f is that there are an integer $N > 0$ and a cone-field C with constant orbit core dimension over Λ such that f^N is cone-hyperbolic.*

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In [5] we have constructed a global metric analogue of a cone-field which allows to estimate the fractal dimension of the hyperbolic iterated functions systems. In this article we define and study its local version. It occurs that a classical cone-field can be seen as a limit version of our metric modification (see Section 5). Moreover, our approach is well-suited to examination of the case when we skip the constant orbit core dimension assumption – in our main result we show that the existence of a hyperbolic local metric cone-field guarantees a uniform version of expansivity. For more information about the expansivity in metric spaces we refer the reader to the results of Lewowicz [6, 7].

Main Result [Theorem 4.1]. *Suppose that we are given a cone-field on Λ, where Λ is a compact subset of a metric space X. Let $f : X \rightharpoonup X$ be cone-hyperbolic on Λ. Then f is uniformly expansive on Λ.*

However, the absence of a constant orbit core dimension eliminates the pseudoorbit tracking property (shadowing). In the last section of the paper we show a simple system consisting of two hyperbolic fixed points with heteroclinic, but not transversal, connection, which is cone-hyperbolic, but does not have the shadowing property.

2. **Cone-fields for Linear Maps.** In this section we generalize and adapt standard notation (see for example [8]) to our needs. At the beginning we give the definitions of pair of cones in the normed space through which we define our expansion and contraction rates of linear map.

We begin with the finite dimensional normed space E which is split as

$$E = E^s \oplus E^u$$

(E^u corresponds to the forward/unstable and E^s to the backward/stable directions). Given a vector $v \in E$, by v^s and v^u we denote its stable and unstable components that is $v^s \in E^s$, $v^u \in E^u$ are such that $v = v^s + v^u$. From now on we assume that the norm in E satisfies the condition $\|v\| = \|v^s + v^u\| := \max\{\|v\|_s, \|v\|_u\}$, where

$$\|v\|_s := \|v^s\| \text{ and } \|v\|_u := \|v^u\|.$$

Definition 2.1. We define the pair of cones corresponding to expanding and contracting directions

$$C^u_E := \{v \in E : \|v\|_s \leq \|v\|_u\},\ C^s_E := \{v \in E : \|v\|_s \geq \|v\|_u\}.$$

We modify the classical definition from [8] to allow the study of non-invertible maps.

Definition 2.2. Let a linear map $A : E^s \oplus E^u \to F^s \oplus F^u$ be given. We define $\mathbf{U}(A)$, the *expansion*, and $\mathbf{S}(A)$, the *contraction* rates of A by the formulas:

$$\begin{aligned} \mathbf{U}(A) &:= \sup\{R \in [0,\infty] \mid \|Av\| \geq R\|v\| \text{ for } v \in C^u_E\}, \\ \mathbf{S}(A) &:= \inf\{R \in [0,\infty] \mid \|Av\| \leq R\|v\| \text{ for } v : Av \in C^s_F\}^{1}. \end{aligned} \tag{1}$$

[1]Note that $\mathbf{U}(A)$ is called in Newhouse [8] the expansion rate but $1/\mathbf{S}(A)$ is exactly the co-expansion rate. This definitions of expansion and contraction rates are more suited to the hyperbolic situation , see Proposition 2.1 where mapping A need not to be invertible.

In the case when A is invertible one can easily transform formulas (1) into the commonly encountered form

$$\mathbf{U}(A)=\inf_{v\in C_E^u\setminus\{0\}}\frac{\|Av\|}{\|v\|},\quad \mathbf{S}(A)=\sup_{v\in C_F^s\setminus\{0\}}\frac{\|v\|}{\|A^{-1}v\|}. \tag{2}$$

Remark 2.1. Let $A: E^s\oplus E^u \to F^s\oplus F^u$ be a linear invertible map. We put $\tilde{E}^s := E^u$, $\tilde{E}^u := E^s$, $\tilde{F}^s := F^u$, $\tilde{F}^u := F^s$ and consider the map $\tilde{B}: \tilde{F}^s\oplus\tilde{F}^u \ni x \to A^{-1}x \in \tilde{E}^s\oplus\tilde{E}^u$. Then trivially

$$\mathbf{S}(A)=1/\mathbf{U}(\tilde{B}),\ \mathbf{U}(A)=1/\mathbf{S}(\tilde{B}).$$

The above equalities are useful, as for example one can directly obtain formula for $\mathbf{S}(A)$ in (2) from the formula for $\mathbf{U}(A)$.

A is called *dominating* [8] if

$$\mathbf{S}(A)<\mathbf{U}(A).$$

We say that A is *cone-hyperbolic*[2] if

$$\mathbf{S}(A)<1<\mathbf{U}(A).$$

Proposition 2.1. *Let E_{-1}, E_0, E_1, F_{-1}, F_0, F_1 be given. Consider a invertible linear map $A: E_{-1}\oplus E_0\oplus E_1 \to F_{-1}\oplus F_0\oplus F_1$ given in a block matrix form by*

$$A:=\begin{bmatrix} A_{-1} & 0 & 0\\ 0 & A_0 & 0\\ 0 & 0 & A_1\end{bmatrix}.$$

We assume that for $x=x_{-1}+x_0+x_1\in E$ we have $\|x\|=\max\{\|x_{-1}\|,\|x_0\|,\|x_1\|\}$ and that the same holds for F.

If $E^s:=E_{-1}\oplus E_0$, $E^u:=E_1$, $F^s:=F_{-1}$, $F^u:=F_0\oplus F_1$ then

$$\mathbf{U}(A)=\|A_1^{-1}\|^{-1},\ \mathbf{S}(A)=\|A_{-1}\|.$$

Proof. At first we prove that $\mathbf{S}(A)=\|A_{-1}\|$. Let $v=v_{-1}+v_0+v_1\in E_{-1}\oplus E_0\oplus E_1$ be such that $\|Av\|_u\le\|Av\|_s$. We know that

$$\|Av\|=\max\{\ \|Av\|_u,\|Av\|_s\}=\|Av\|_s=\|A_{-1}v_{-1}\|\le\|A_{-1}\|\cdot\|v_{-1}\|\le\|A_{-1}\|\|v\|.$$

Hence $\mathbf{S}(A)\le\|A_{-1}\|$.

Now let $\varepsilon>0$. We know that there exists $v_{-1}^{\varepsilon}\in E_{-1}\setminus\{0\}$ such that

$$\|A_{-1}v_{-1}^{\varepsilon}\|>(\|A_{-1}\|-\varepsilon)\|v_{-1}^{\varepsilon}\|.$$

Observe that $Av_{-1}^{\varepsilon}=A_{-1}v_{-1}^{\varepsilon}\in C_F^s$. Therefore we get $\mathbf{S}(A)\ge\|A_{-1}\|-\varepsilon$.

Now take $v=v_{-1}+v_0+v_1\in E$ such that $v\in C_E^u$. Observe that $\|v\|=\|v_1\|$. Now we get

$$\|Av\|\ge\|A_1v_1\|\ge\|A_1^{-1}\|^{-1}\|v_1\|=\|A_1^{-1}\|^{-1}\|v\|.$$

Therefore $\mathbf{U}(A)\ge\|A_1^{-1}\|^{-1}$.

Now again let $\varepsilon>0$. There exists $v_1^{\varepsilon}\in E_1\setminus\{0\}$ such that

$$\|A_1v_1^{\varepsilon}\|<(\|A_1\|-\varepsilon)^{-1}\|v_1^{\varepsilon}\|.$$

Obviously $v_1^{\varepsilon}\in C_E^u$ and $\|Av_1^{\varepsilon}\|<(\|A_1\|-\varepsilon)^{-1}\|v_1^{\varepsilon}\|$ which completes the proof. □

As a direct corollary of Proposition 2.1 putting $E_0=\{0\}=F_0$ we get the following.

[2]In the notation from [8] A is cone-hyperbolic iff A is both expanding and co-expanding.

Corollary 2.1. *Consider a invertible linear map $A : E_{-1} \oplus E_1 \to F_{-1} \oplus F_1$ given in a block matrix form by*

$$A := \begin{bmatrix} A_{-1} & 0 \\ 0 & A_1 \end{bmatrix}.$$

We assume that for $x = x_{-1} + x_1 \in E$ we have $\|x\| = \max\{\|x_{-1}\|, \|x_1\|\}$ and that the same holds for F.

If $E^s := E_{-1}$, $E^u := E_1$, $F^s := F_{-1}$, $F^u := F_1$ then

$$\mathbf{U}(A) = \|A_1^{-1}\|^{-1},\ \mathbf{S}(A) = \|A_{-1}\|.$$

We say that $A : E^s \oplus E^u \to F^s \oplus F^u$ has a *constant core dimension* if $\dim E^s = \dim F^s$, $\dim E^u = \dim F^u$. We show a cone-hyperbolic linear map A which does not have the constant core dimension.

Corollary 2.2. *Let*

$$E^s = \mathbb{R}^2 \times \{0\}\,,\ E^u = \{(0,0)\} \times \mathbb{R}$$

and

$$F^s = \mathbb{R} \times \{(0\,0)\}\,,\ F^u = \{0\} \times \mathbb{R}^2.$$

Consider map A given in a block matrix from by

$$A := \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

where $a \in \mathbb{R}$. Then A is cone-hyperbolic.

Clearly $\dim E^s = 2 \neq 1 = \dim F^s$ and $\dim E^u = 1 \neq 2 = \dim F^u$. Using the Proposition 2.1 we have $\mathbf{U}(A) = 2$ and $\mathbf{S}(A) = 1/2$. Therefore A is cone-hyperbolic but does not have the constant orbit core dimension.

After studying linear maps we proceed to diffeomorphisms.

Definition 2.3. Let $(E, \|\cdot\|)$ be a finite dimensional normed space and $\Lambda \subset E$ be nonempty. By a *splitting on* Λ we understand that for each $x \in \Lambda$ we are given a pair of subspaces $(E^s_x, \|\cdot\|^s_x)$,$(E^u_x, \|\cdot\|^u_x)$ of E such that

$$E = E^s_x \oplus E^u_x.$$

Note that we do not assume continuity in the above definition.

Definition 2.4. Let $(E, \|\cdot\|)$ be a finite dimensional normed space with splitting on $\Lambda \subset E$. If there exists $K > 0$ such that:

$$\frac{1}{K}\|v^s + v^u\| \le \max\{\|v^s\|^s_x, \|v^u\|^u_x\} \le K\|v^s + v^u\| \text{ for } x \in \Lambda,\ v^s + v^u \in E^s_x \oplus E^u_x;$$

then we call it *K-splitting* or *uniform splitting*.

Cones at $x \in \Lambda$ are defined as follows

$$C^u_x := \{v^s + v^u \in E^s_x \oplus E^u_x : \|v^s\|^s_x \le \|v^u\|^u_x\},$$

$$C^s_x := \{v^s + v^u \in E^s_x \oplus E^u_x : \|v^s\|^s_x \ge \|v^u\|^u_x\}.$$

Gathering together all such cones over $x \in \Lambda$ forms a cone-field on Λ.

Definition 2.5. Let U, V be open subsets of finite dimensional Banach spaces and $f \in C^1(U,V)$ and $\Lambda \subset U$ be nonempty. Assume that we are given splittings on Λ and $f(\Lambda)$.

For $x \in U$ we put

$$\begin{array}{rclrcl} \mathbf{U}_x(f) & := & \mathbf{U}(d_x f), & \mathbf{U}_\Lambda(f) & := & \inf_{x\in\Lambda}\{\mathbf{U}(d_x f)\} \\ \mathbf{S}_x(f) & := & \mathbf{S}(d_x f), & \mathbf{S}_\Lambda(f) & := & \sup_{x\in\Lambda}\{\mathbf{S}(d_x f)\}. \end{array}$$

Remark 2.2. Let $A : E^s \oplus E^u \to F^s \oplus F^u$ be a linear map and $\Lambda \subset E^s \oplus E^u$ be given. Spaces E^s, E^u and their norms are fixed with respect to $x \in \Lambda$. Then

$$\mathbf{U}_x(A) = \mathbf{U}_\Lambda(A) = \mathbf{U}_0(A) \quad \text{for } x \in \Lambda.$$

3. **Cone-fields on Metric Spaces.** Before we generalize the notion of cone-field to metric spaces, let us emphasise the benefits we get from it. First, we can study Lipschitz maps as we do not need differential structure. Moreover we have control over behavior of the orbits since we do not to work in tangent spaces but in the space itself.

Let us now explain how we define cone fields on metric spaces. The basic idea lies in "exchanging" the map $d_x f$ with $f|_{B(x,\delta)}$ for some small $\delta > 0$ where $B(x,\delta)$ denotes an open ball of radius δ centered at x.

Let (X, d) be a metric space and let Λ be a closed subset of X. For $\delta > 0$ we put

$$\Delta_\delta(\Lambda) := \bigcup_{x\in\Lambda} \{x\} \times B(x,\delta).$$

Definition 3.1. Let $\delta > 0$ and $\Lambda \subset X$ be nonempty. We say that a pair of functions $c_s, c_u : U \to \mathbb{R}_+$ for $U \subset X \times X$ form a *δ-cone-field on* Λ if $\Delta_\delta(\Lambda) \subset U$. If there exists K such that:

$$\frac{1}{K} d(x,v) \leq \max\{c_s(x,v), c_u(x,v)\} \leq K d(x,v) \text{ for } (x,v) \in U$$

then we call it (K,δ) *cone-field on* Λ or *uniform δ-cone-field on* Λ. We put

$$c(x,v) := \max\{c_s(x,v), c_u(x,v)\}.$$

For each point $x \in \Lambda$ we introduce *unstable* and *stable cones* by the formula

$$C^u_x(\delta) := \{v \in B(x,\delta) \,:\, c_s(x,v) \leq c_u(x,v)\},$$
$$C^s_x(\delta) := \{v \in B(x,\delta) \,:\, c_s(x,v) \geq c_u(x,v)\}.$$

Remark 3.1. Let E be a normed space, let $\Lambda \subset E$ and assume that we are given a uniform splitting on Λ. For $x \in \Lambda$ and $v \in E$ we put

$$c_s(x,v) := \|(v-x)^s\|^s_x,\; c_u(x,v) := \|(v-x)^u\|^u_x, \tag{3}$$

where $v - x = (v-x)^s + (v-x)^u \in E^s_x \oplus E^u_x$.
Then (3) defines a uniform δ-cone-field on Λ for any $\delta > 0$.

We consider a partial map $f : X \rightharpoonup Y$ between metric spaces X and Y and $\Lambda \subset \operatorname{dom} f$. Assume that X is equipped with uniform δ-cone-field on Λ and Y is equipped with uniform δ-cone-field on a closed subset Z of Y such that $f(\Lambda) \subset Z$.

For every $x \in \operatorname{dom} f$ we put

$$B_f(x,\delta) := \{v \in B(x,\delta) \cap \operatorname{dom} f : f(v) \in B(f(x),\delta)\}.$$

Definition 3.2. Let $x \in \mathrm{dom} f$ and $\delta > 0$ be given. We define

$$\begin{aligned} u_x(f;\delta) \quad &:= \quad \sup\{R \in [0,\infty] \mid c(f(x), f(v)) \geq Rc(x,v), v \in B_f(x,\delta); \\ &\qquad\qquad v \in C_x^u(\delta)\}, \\ s_x(f;\delta) \quad &:= \quad \inf\{R \in [0,\infty] \mid c(f(x), f(v)) \leq Rc(x,v), v \in B_f(x,\delta); \\ &\qquad\qquad f(v) \in C_{f(x)}^s(\delta)\}. \end{aligned}$$

Let $u_\Lambda(f;\delta) := \inf_{x\in\Lambda}\{u_x(f;\delta)\}$ and $s_\Lambda(f;\delta) := \sup_{x\in\Lambda}\{s_x(f;\delta)\}$.

Remark 3.2. Let $A : E^s \oplus E^u \to F^s \oplus F^u$ be a linear map and $\Lambda \subset E^s \oplus E^u$ be given. For $x_0 \in E^s \oplus E^u$ we define a function $f : E^s \oplus E^u \to F^s \oplus F^u$ by a formula

$$f(x) := Ax + x_0.$$

Spaces E^s, E^u and their norms are fixed with respect to $x \in \Lambda$ and the same holds for F^s, F^u whit respect to $y \in f(\Lambda)$. Let $\delta > 0$ and uniform δ-cone-fields on Λ and $f(\Lambda)$ be given by (3). Then for any $x \in \Lambda$ we have

$$u_x(f;\delta) = u_\Lambda(f;\delta) = \mathbf{U}_0(A)$$

and

$$s_x(f;\delta) = s_\Lambda(f;\delta) = \mathbf{S}_0(A).$$

Definition 3.3. We say that f is δ-*dominating* on Λ if

$$s_\Lambda(f;\delta) < u_\Lambda(f;\delta),$$

and f is δ-*cone-hyperbolic* on Λ if

$$s_\Lambda(f;\delta) < 1 < u_\Lambda(f;\delta).$$

Trivially, a δ-*cone-hyperbolic* mapping is δ-*dominating*. The next proposition shows a simple analogue of [8, Lemma 1.1].

Proposition 3.1. *Every δ-dominating mapping is* δ-cone-invariant, *i.e. for $x \in \Lambda$ and $v \in B_f(x,\delta)$ we have*

$$v \in C_x^u(\delta) \implies f(v) \in C_{f(x)}^u(\delta),$$

and

$$f(v) \in C_{f(x)}^s(\delta) \implies v \in C_x^s(\delta).$$

Proof. To prove the first implication, suppose that there exist $x \in \Lambda$ and $v \in C_x^u(\delta)$ such that $f(v) \notin C_{f(x)}^u(\delta)$. This implies that $f(x) \neq f(v)$ and therefore $c(x,v) > 0$. We also know that $f(v) \in C_{f(x)}^s(\delta)$. From Definition 3.2 we obtain

$$c(f(x), f(v)) \leq s_x(f;\delta)c(x,v) \leq s_\Lambda(f;\delta)c(x,v)$$

but on the other hand

$$c(f(x), f(v)) \geq u_x(f;\delta)c(x,v) \geq u_\Lambda(f;\delta)c(x,v).$$

Thus $s_\Lambda(f;\delta) \geq u_\Lambda(f;\delta)$ which leads to contradiction.

The second implication is proved similarly. □

4. **Cone-fields and Expansivity.** In this section we show that the cone-hyperbolicity implies uniform expansivity. First we show that cone structure allows to estimate the distance between orbits for cone-hyperbolic mappings.

Let a partial map $f : X \rightharpoonup X$ be given. We call a sequence $\mathrm{x} : I \to X$ defined on a subinterval[3] I of $\mathbb{Z}$ an *orbit of* f if

$$\mathrm{x}_n \in \mathrm{dom} f \text{ and } \mathrm{x}_{n+1} = f(\mathrm{x}_n) \text{ for } n, n+1 \in I.$$

Definition 4.1. Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in (0,1)$ be given. We say that $f : X \rightharpoonup X$ is (N, ε, α)*-uniformly expansive on a set* $\Lambda \subset X$ if for any two orbits $\mathrm{x} : \{-N, \ldots, N\} \to \Lambda$, $\mathrm{v} : \{-N, \ldots, N\} \to X$ we have

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \le \varepsilon \implies d(\mathrm{x}_0, \mathrm{v}_0) \le \alpha d_{\sup}(\mathrm{x}, \mathrm{v}),$$

where

$$d_{\sup}(\mathrm{x}, \mathrm{v}) := \sup_{-N \le n \le N} d(\mathrm{x}_n, \mathrm{v}_n).$$

As we will see uniform expansiveness is stronger than the classical expansiveness.

Observation 4.1. *Let $N \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0,1)$, $\Lambda \subset X$ and $f : X \rightharpoonup X$ be given. If f is (N, ε, α)-uniformly expansive on Λ it is also expansive on Λ.*

Proof. Take any two orbits $\mathrm{x} : \mathbb{Z} \to \Lambda$, $\mathrm{v} : \mathbb{Z} \to X$ such that

$$d(x_n, v_n) \le \varepsilon \text{ for } n \in \mathbb{Z}.$$

We can take any pair of points $(\mathrm{x}_n, \mathrm{v}_n)$ on which we will start iterate. From (N, ε, α)-uniform expansiveness we get

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha d_{\sup}(\mathrm{x}, \mathrm{v}) \text{ for } n \in \mathbb{Z}.$$

Thus

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \le \alpha d_{\sup}(\mathrm{x}, \mathrm{v})$$

and as a consequence

$$d_{\sup}(\mathrm{x}, \mathrm{v}) = 0 \implies \mathrm{x} = \mathrm{v}.$$

□

Observation 4.2. *Let $k, N \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0,1)$, $\Lambda \subset X$ and $f : X \rightharpoonup X$ be given. If f is (N, ε, α)-uniformly expansive on Λ then for any two orbits $\mathrm{x} : \{-kN, \ldots, kN\} \to \Lambda$ i $\mathrm{v} : \{-kN, \ldots, kN\} \to X$ such that $d_{\sup}(\mathrm{x}, \mathrm{v}) \le \varepsilon$ we have*

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha\varepsilon \text{ for } n \in \{-(k-1)N, \ldots, (k-1)N\},$$
$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha^2\varepsilon \text{ for } n \in \{-(k-2)N, \ldots, (k-2)N\},$$
$$\vdots$$
$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha^{k-1}\varepsilon \text{ for } n \in [-N, N].$$

Proof. Take any two orbits $\mathrm{x} : \{-kN, \ldots, kN\} \to \Lambda$, $\mathrm{v} : \{-kN, \ldots, kN\} \to X$ such that

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \varepsilon \text{ for } n \in \{-kN, \ldots, kN\}.$$

We can take any pair of points $(\mathrm{x}_n, \mathrm{v}_n)$ for $n \in \{-(k-1)N, \ldots, (k-1)N\}$ on which we will start iterate. From (N, ε, α)-uniform expansiveness we get

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha\varepsilon \text{ for } n \in \{-(k-1)N, \ldots, (k-1)N\}.$$

[3]We say the I is a subinterval of $\mathbb{Z}$ if $[k, l] \cap \mathbb{Z} \subset I$ for any $k, l \in I$.

Now again we use uniform expansiveness to obtain

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha^2\varepsilon \text{ for } n \in \{-(k-2)N, \dots, (k-2)N\}.$$

Continuing this way we eventually get

$$d(\mathrm{x}_n, \mathrm{v}_n) \le \alpha^{k-1}\varepsilon \text{ for } n \in \{-N, \dots, N\}.$$

□

Given a set $\Lambda \subset X$ we define δ *neighborhood of* Λ as

$$\Lambda_\delta := \bigcup_{x\in\Lambda} B(x, \delta).$$

Theorem 4.1. *Suppose that for $K > 0$ and $\delta > 0$ we are given a (K, δ) cone-field on $\Lambda \subset X$. Let $f : \Lambda_\delta \rightharpoonup X$ be δ-cone-hyperbolic on Λ and let $\lambda > 1$ be chosen such that*

$$s_\Lambda(f; \delta) \le \lambda^{-1},\ u_\Lambda(f; \delta) \ge \lambda.$$

Then f is $(N, \delta, K^2/\lambda^N)$-uniformly expansive on Λ for every $N \in \mathbb{N}$, $N > 2\log_\lambda K$.

Proof. From Proposition 3.1 we know that f is δ-cone-invariant. Let us take two orbits $\mathrm{x} : \{-N, \dots, N\} \to \Lambda$, $\mathrm{v} : \{-N, \dots, N\} \to X$ such that

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \le \delta.$$

Since $\mathrm{v}_0 \in B(\mathrm{x}_0, \delta) = C^s_{\mathrm{x}_0}(\delta) \cup C^u_{\mathrm{x}_0}(\delta)$ it is enough to consider two cases.

Let $\mathrm{v}_0 \in C^s_{\mathrm{x}_0}(\delta)$. From the cone-invariance we know that $\mathrm{v}_n \in C^s_{\mathrm{x}_n}(\delta), n < 0$. From Definition 3.2 we get $c(\mathrm{x}_0, \mathrm{v}_0) \le \lambda^{-1}c(\mathrm{x}_{-1}, \mathrm{v}_{-1}) \le \dots \le \lambda^{-N}c(\mathrm{x}_{-N}, \mathrm{v}_{-N})$. Finally

$$d(\mathrm{x}_0, \mathrm{v}_0) \le Kc(\mathrm{x}_0, \mathrm{v}_0) \le K^2\lambda^{-N}d_{\sup}(\mathrm{x}, \mathrm{v}).$$

If $\mathrm{v}_0 \in C^u_{\mathrm{x}_0}(\delta)$ then from the cone-invariance we obtain $\mathrm{v}_n \in C^u_{\mathrm{x}_n}(\delta), n > 0$ and consequently

$$d(\mathrm{x}_0, \mathrm{v}_0) \le Kc(\mathrm{x}_0, \mathrm{v}_0) \le K\lambda^{-N}c(\mathrm{x}_N, \mathrm{v}_N) \le K^2\lambda^{-N}d_{\sup}(\mathrm{x}, \mathrm{v}).$$

□

5. **Limiting Case.** Let us return to the function $f : E \rightharpoonup F$ between finite dimensional Banach spaces E and F and $\Lambda \subset \mathrm{dom}f$. We show that for diffeomorphism f constants $u_\Lambda(f; \delta)$, $s_\Lambda(f; \delta)$ converge to $\mathbf{U}_\Lambda(f)$, $\mathbf{S}_\Lambda(f)$ as $\delta \to 0$. Let us begin with the following observation.

Observation 5.1. *Let $E = E^s \oplus E^u$, $F = F^s \oplus F^u$, $\delta > 0$ and a linear map $A : E \to F$ be given. Assume that $\Lambda \subset E$ and the uniform δ-cone-field on Λ is defined by (3). Then for any $x \in \Lambda$ we have*

$$u_\Lambda(A; \delta) = u_x(A; \delta) = \mathbf{U}_0(A),$$

$$s_\Lambda(A; \delta) = s_x(A; \delta) = \mathbf{S}_0(A).$$

Proposition 5.1. *Let $\delta > 0$, $A : E^s \oplus E^u \to F^s \oplus F^u$ be a linear map, $x \in E^s \oplus E^u$ and $p : B(x, \delta) \rightharpoonup F^s \oplus F^u$ be Lipschitz.*

Then

$$u_x(A + p; \delta) \in [\mathbf{U}_0(A) - \mathrm{lip}(p), \mathbf{U}_0(A) + \mathrm{lip}(p)].$$

Proof. From the equation (2), Remark 2.2 and the above observation we have

$$\begin{aligned}
u_x(A+p;\delta) &= \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|A(v-x)+p(v-x)\|}{\|v-x\|} \\
&\leq \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|A(v-x)\|+\|p(v-x)\|}{\|v-x\|} \\
&= \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|A(v-x)\|}{\|v-x\|} + \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|p(v-x)\|}{\|v-x\|} \\
&\leq u_x(A;\delta) + \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\mathrm{lip}(p)\|v-x\|}{\|v-x\|} \\
&= \mathbf{U}_0(A) + \mathrm{lip}(p)
\end{aligned}$$

and

$$\begin{aligned}
u_x(A+p;\delta) &= \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|A(v-x)+p(v-x)\|}{\|v-x\|} \\
&\geq \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{\|A(v-x)\|-\|p(v-x)\|}{\|v-x\|} \\
&\geq u_x(A;\delta) + \inf_{v\in C_E^u\setminus\{x\}\cap B(x,\delta)} \tfrac{-\mathrm{lip}(p)\|v-x\|}{\|v-x\|} \\
&= \mathbf{U}_0(A) - \mathrm{lip}(p).
\end{aligned}$$

□

Proposition 5.2. *Let $\delta > 0$, E and F be Banach spaces, U be an open subset of E, Λ be such that $\Lambda_\delta \subset U$ and $f \in C^1(U,F)$. Assume that $x \mapsto d_x f$ is uniformly continuous on Λ_δ and its modulus of continuity on Λ is equal to $\omega(\delta)$. Then*

$$u_\Lambda(f;\delta) \in [\mathbf{U}_\Lambda(f) - \omega(\delta), \mathbf{U}_\Lambda(f) + \omega(\delta)].$$

Proof. Let $x \in \Lambda$. We put

$$p_x^\delta(v) := f(v) - f(x) - d_x f(v-x) \text{ for } v \in B(x,\delta).$$

By uniform continuity of $x \mapsto d_x f$ on Λ_δ we have

$$\begin{aligned}
\|p_x^\delta(v)\| &= \|f(v) - f(x) - d_x f(v-x)\| \\
&\leq \sup\{\|d_\xi f - d_x f\| \,:\, \xi \in B(x,\delta)\}\|v-x\| \\
&= \omega(\delta)\|v-x\|.
\end{aligned}$$

Using the Proposition 5.1 we get

$$u_x(f;\delta) = u_x(d_x f + p_x^\delta;\delta) \in [\mathbf{U}_x(f) - \omega(\delta), \mathbf{U}_x(f) + \omega(\delta)].$$

□

Observation 5.2. *Let $f : X \to Y$ be an invertible map, $\Lambda \subset X$ and $\delta > 0$. Assume that functions c_s, c_u create uniform δ-cone-field on Λ and C_s, C_u yield uniform δ-cone-field on $\tilde{\Lambda} = \overline{f(\Lambda)}$.*

Then

$$s(f;\delta) = 1/u(\tilde{f};\delta),$$

$$u(f;\delta) = 1/s(\tilde{f};\delta),$$

where $\tilde{f} := f^{-1} : Y \to X$ with $\tilde{c}_s := C_u$, $\tilde{c}_u := C_s$ which form uniform δ-cone-field on $\tilde{\Lambda}$ and $\tilde{C}_s := c_u$, $\tilde{C}_u := c_s$ which form uniform δ-cone-field on $\tilde{f}(\tilde{\Lambda})$.

Theorem 5.1. *Let $U \subset E$, $V \subset F$ be open, Λ be a compact subset of U and $f \in C^1(U,V)$.*

Then

$$u_\Lambda(f;\delta) \nearrow \mathbf{U}_\Lambda(f) \quad as \quad \delta \to 0,$$

and

$$s_\Lambda(f;\delta) \searrow \mathbf{S}_\Lambda(f) \quad as \quad \delta \to 0.$$

Proof. Directly from the definition of $u_x(f;\delta)$ it is non-decreasing as δ tends to zero and therefore $u_\Lambda(f;\delta)$ is also non-decreasing. The first convergence follows from Proposition 5.2, the second one is a consequence of Observation 5.2. $\square$

6. **Cone-hyperbolic Orbit Without POTP.** We are going to show an example of a cone-hyperbolic connection between two hyperbolic fix points which does not have the shadowing property. The idea is based on the Corollary 2.2.

Let $p_{-1} = (0,0,-1)$, $p_0 = (0,0,0)$ and $p_1 = (1,0,0)$. We are going to define a function in neighborhood of these points. In $\mathbb{R}^3$ we consider the maximum metric.

Let

$$Q_{-1} := \left[-\frac{1}{5}, \frac{1}{5}\right]^2 \times \left[-\frac{6}{5}, -\frac{2}{5}\right]$$

and

$$Q_1 := \left[-\frac{1}{5}, \frac{6}{5}\right] \times \left[-\frac{1}{5}, \frac{1}{5}\right]^2$$

be pairwise disjoint cuboids, $X = Q_{-1} \cup Q_1$ and $F : X \to \mathbb{R}^3$ be given by a formula

$$F(x) := \begin{cases} A_{-1}(x - p_{-1}) + p_{-1} & \text{for} \quad x \in Q_{-1}, \\ A_1(x - p_1) + p_1 & \text{for} \quad x \in Q_1, \end{cases} \tag{4}$$

where

$$A_{-1} = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} \text{ and } A_1 = \begin{bmatrix} \frac{1}{2} & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

(see Figure 1(a)). By the partial map $f : X \rightharpoonup X$ we denote the restriction of F to the set $\mathrm{dom}(f) := \{x : F(x) \in X\}$.

Two-dimensional stable or unstable subspaces of p_{-1} and p_1 are marked by double arrows while one-dimensional subspaces by single ones.

Observation 6.1. *The partial map f has the following properties:*

1. *points p_{-1} and p_1 are hyperbolic fixed points;*
2. *for $k \in \mathbb{Z}$ $f^k(p_0) \in X$; consequently $\mathcal{O}(p_0) := \{f^k(p_0) \in X : k \in \mathbb{Z}\} \subset \mathrm{dom} f$;*
3. *point p_0 belongs to the unstable manifold of p_{-1} and to the stable manifold of p_1 (see Figure 1(b))*

$$p_0 \in W^u(p_{-1}) := \{x : f^k(x) \to p_{-1} \ as \ k \to -\infty\},$$

$$p_0 \in W^s(p_1) := \{x : f^k(x) \to p_1 \ as \ k \to +\infty\};$$

As one can see from the Figure 1(a) tangent spaces to $W^u(p_{-1})$ and $W^s(p_1)$ at point p_0 generate a two-dimensional space and not a three-dimensional one. Thus we have a non-transversal heteroclinic connection between two hyperbolic fixed points.

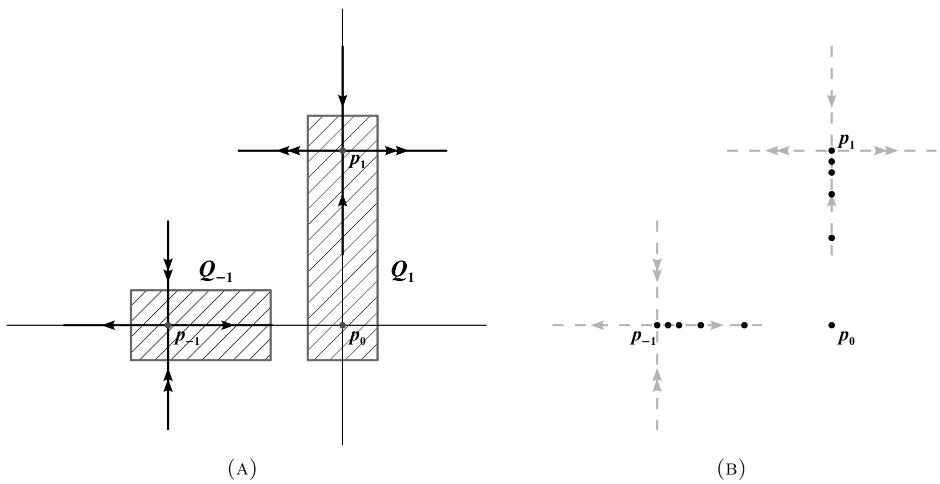

FIGURE 1

Let as define a splitting on X. For $x \in Q_{-1}$ we put $E^s_x = \mathbb{R}^2 \times \{0\}$, $E^u_x = \{0\}^2 \times \mathbb{R}$ and for $x \in Q_1$ we put $E^s_x = \mathbb{R} \times \{0\}^2$, $E^u_x = \{0\} \times \mathbb{R}^2$. Using Remark 3.1 and formulas (3) we define a cone structure on X. For $v = (v_1, v_2, v_3) \in \mathbb{R}^3$ we put

$$c_s(x, v) := \begin{cases} \max\{|x_1 - v_1|, |x_2 - v_2|\} & \text{for} \quad x = (x_1, x_2, x_3) \in Q_{-1}, \\ |x_1 - v_1| & \text{for} \quad x = (x_1, x_2, x_3) \in Q_1, \end{cases}$$

and

$$c_u(x, v) := \begin{cases} |x_3 - v_3| & \text{for} \quad x = (x_1, x_2, x_3) \in Q_{-1}, \\ \max\{|x_2 - v_2|, |x_3 - v_3|\} & \text{for} \quad x = (x_1, x_2, x_3) \in Q_1. \end{cases}$$

Proposition 6.1. *Let $\delta \in (0, \frac{1}{10})$. Mapping f defined by (4) is δ-cone-hyperbolic on* $\mathrm{dom}(f)$.

Proof. It holds that

$$f^{-1}(Q_{-1}) \cap Q_{-1} \subset Q^{-1}_{-1},$$

and

$$f^{-1}(Q_1) \cap Q_{-1} \subset Q^{1}_{-1},$$

where

$$Q^{-1}_{-1} := \mathbb{R}^2 \times \left[-\frac{6}{5}, -\frac{7}{10}\right] \cap Q_{-1},$$

and

$$Q^{1}_{-1} := \mathbb{R}^2 \times \left[-\frac{6}{10}, -\frac{2}{5}\right] \cap Q_{-1}$$

(see Figure 2). Therefore $\mathrm{dom}(f) \subset Q^{-1}_{-1} \cup Q^1_{-1} \cup Q_1$.

Not formally the idea is to first show that f restricted to each set $Q \in \{Q^{-1}_{-1}, Q^1_{-1}, Q_1\}$ has a fixed formula and the splittings are alse fixed on the δ-neighborhoods of Q and $f(Q)$, and then to use Remark 3.2.

Let $x \in \mathrm{dom}(f) \cap Q^{-1}_{-1}$. Note that $f(x) \in Q_{-1}$ and the function $f_{|B(x,\delta)} : B(x, \delta) \rightharpoonup B(f(x), \delta) \subset Q_{-1}$ is given by the formula $f_{|B(x,\delta)}(v) = A_{-1}(v - p_{-1}) + p_{-1}$ and therefore is affine. Moreover splitting are the same for each $v \in B(x, \delta)$ and

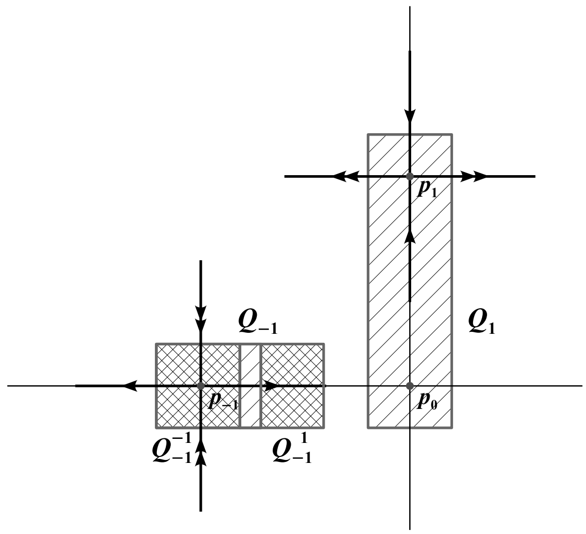

FIGURE 2

$v' \in B(f(x),\delta)$. Then using Remark 3.2 we get $s_x(f;\delta) = \mathbf{S}_0(A_{-1})$ and $u_x(f;\delta) = \mathbf{U}_0(A_{-1})$ where $A_{-1} : \mathbb{R}^2 \times \{0\} \oplus \{0\}^2 \times \mathbb{R} \to \mathbb{R}^2 \times \{0\} \oplus \{0\}^2 \times \mathbb{R}$ and finally by Corollary 2.1 it follows that $s_x(f;\delta) = \frac{1}{2}$ and $u_x(f;\delta) = 2$.

The case when $x \in \mathrm{dom}(f) \cap Q_1$ is analogous since $f(Q_1) \cap \mathrm{dom}(f) \subset Q_1$.

Now let $x \in \mathrm{dom}(f) \cap Q_{-1}^1$. It holds that $f(x) \in Q_1$ and the function $f_{|B(x,\delta)} : B(x,\delta) \rightharpoonup B(f(x),\delta) \subset Q_1$ is given by the formula $f_{|B(x,\delta)}(v) = A_{-1}(v - p_{-1}) + p_{-1}$ and therefore is affine. The splittings are constant for $v \in B(x,\delta)$ and $v' \in B(f(x),\delta)$, respectively. By Remark 3.2 we get $s_x(f;\delta) = \mathbf{S}_0(A_{-1})$ and $u_x(f;\delta) = \mathbf{U}_0(A_{-1})$ where $A_{-1} : \mathbb{R}^2 \times \{0\} \oplus \{0\}^2 \times \mathbb{R} \to \mathbb{R} \times \{0\}^2 \oplus \{0\} \times \mathbb{R}^2$. From Corollary 2.2 it follows that $s_x(f;\delta) = \frac{1}{2}$ and $u_x(f;\delta) = 2$.

Therefore f is δ-cone-hyperbolic on $\mathrm{dom}(f)$. $\square$

Using Theorem 4.1 we get the following.

Corollary 6.1. *Map f is $(1,\delta,\frac{1}{2})$-uniformly expansive on* $\mathrm{dom}(f)$ *for any* $\delta \in (0, \frac{1}{10})$.

Now let us recall the notion of shadowing.

Definition [4, Definition 18.1.1]**.** Let (X,d) be a metric space, $f : X \rightharpoonup X$. Let I be a subinterval of $\mathbb{Z}$ and $\delta > 0$. We say that a sequence $\mathrm{x} : I \to X$ is a *δ-pseudo-orbit* for f if

$$x_n \in \mathrm{dom} f \quad \text{and} \quad d(\mathrm{x}_{n+1}, f(\mathrm{x}_n)) \le \delta \quad \text{for all} \quad n \in I : n+1 \in I.$$

A δ-pseudo-orbit $\mathrm{x} : I \to X$ for f is said to be *ε-shadowed* by the orbit $\mathrm{y} : I \to X$ of f if

$$d(\mathrm{x}_n, \mathrm{y}_n) \le \varepsilon \text{ for all } n \in I.$$

The following definition can be extracted from [4, Theorem 18.1.2].

Definition 6.1. Let (X,d) be a metric space, $U \subset X$ open, $\Lambda \subset U$ and $f : U \to X$. We say that f has the *pseudo orbit tracing property on* Λ (abbr. POTP[4]) if there

[4]POTP may be called shadowing.

exists $r > 0$ such that $\Lambda_r \subset U$ and whenever $\varepsilon > 0$ there is a $\delta > 0$ such that every δ-pseudo-orbit in Λ_r is ε-shadowed by an orbit of f.

Shadowing Lemma [4, Theorem 18.1.2]**.** Let $U \subset \mathbb{R}^N$ be open, $f : U \to \mathbb{R}^N$ be a diffeomorphism, and $\Lambda \subset U$ be a compact hyperbolic set for f. Then f has POTP on Λ.

The aim of this section is to show that the function f defined by the equation (4) does not have shadowing property.

Theorem 6.1. *Let f be given by the formula* (4) *and $\Lambda = \{p_{-1}\} \cup \mathcal{O}(p_0) \cup \{p_1\}$. Then f does not have POTP on Λ.*

Proof. For an indirect proof assume that f has POTP on Λ. We know that there exists $r > 0$, as in the Definition 6.1.

Fix $\varepsilon = \frac{r}{4}$. From the equation (4) we know that the trajectory of any point $w = (w_1, w_2, w_3)$ in $[-\frac{1}{5}, \frac{1}{5}]^3$ is given by

$$\begin{array}{lll} f^k(w) = (1 + 2^{-k}(w_1 - 1), 2^k w_2, 2^k w_3) & \text{for} & k \geq 0, \\ f^k(w) = (2^{-k} w_1, 2^{-k} w_2, -1 + 2^k(w_3 + 1)) & \text{for} & k < 0. \end{array} \tag{5}$$

Take $\delta \in (0, 1)$ and let us construct the following δ-pseudo-orbit (see Figure 3). We start with forward iterating p_0 until $\|f^k(p_0) - p_1\| \leq \delta$. This happens for $k > -\log_2 \delta$. Let us fix such k_0. Then we jump into $p_1 = (1, 0, 0)$ and jump out to a point $(1, \delta, 0)$. Then we again iterate the point $(1, \delta, 0)$ using map f to get obtaining consecutive points $(1, 2^k\delta, 0)$ until $2^k\delta > r/2$ say for some $k_1 > \log_2 \frac{r}{2\delta}$. This defines the positive half of δ-pseudo-orbit v as follows

$$\mathrm{v}_k = \begin{cases} (1 - \frac{1}{2^k}, 0, 0) & \text{for } 0 \leq k \leq k_0 \\ (1, 0, 0) & \text{for } k = k_0 + 1 \\ (1, 2^l\delta, 0) & \text{for } k = k_0 + 1 + l,\ 0 < l \leq k_1. \end{cases}$$

In the similar way we define the negative half

$$\mathrm{v}_k = \begin{cases} (0, 0, -1 + \frac{1}{2^{-k}}) & \text{for } -k_0 \leq k < 0 \\ (0, 0, -1) & \text{for } k = -k_0 - 1 \\ (0, -2^l\delta, -1) & \text{for } k = -k_0 - 1 - l,\ 0 < l < k_1. \end{cases}$$

Put $N := k_0 + k_1 + 1$. Notice that the first point of the constructed δ-pseudo-orbit v is $\mathrm{v}_{-N} = (0, -2^{k_1}, -1)$ and the last point is $\mathrm{v}_N = (1, 2^{k_1}, 0)$. Since f has POTP there exists $w \in X$ such that the orbit of w is close to v i.e.

$$\|f^k(w) - \mathrm{v}_k\| < \varepsilon \text{ for } k \in \{-N, \ldots, N\}.$$

In particular using (5) we have

$$|2^N w_2 + 2^{k_1}\delta| \leq \|f^{-N}(w) - \mathrm{v}_{-N}\| < \varepsilon \ \text{ and } \ |2^N w_2 - 2^{k_1}\delta| \leq \|f^N(w) - \mathrm{v}_N\| < \varepsilon.$$

Now since $2^k\delta > r/2$ and $\varepsilon = r/4$ we get

$$2^N w_2 < -2^{k_1}\delta + \varepsilon < -r/4 \ \text{ and } \ 2^N w_2 > 2^{k_1}\delta - \varepsilon > r/4,$$

which is a contradiction because $2^N w_2$ is both positive and negative. $\square$

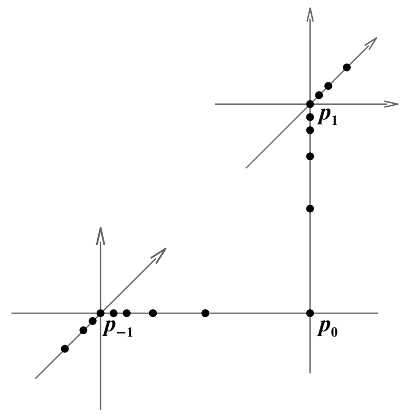

FIGURE 3

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