Numerical verification of condition for approximately midconvex functions

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Abstract. Let X be a normed space and V be a convex subset of X. Let $\alpha\colon \mathbb{R}_+ \to \mathbb{R}_+$. A function $f\colon V \to \mathbb{R}$ is called *α-midconvex* if

$$f(\frac{x+y}{2}) - \frac{f(x)+f(y)}{2} \leq \alpha(\|x-y\|) \quad \text{for } x, y \in V.$$

It can be shown that every continuous α-midconvex function satisfies the following estimation:

$$f(tx+(1-t)y)-tf(x)-(1-t)f(y) \leq \sum_{k=0}^{\infty} \frac{1}{2^k}\alpha(d(2^{kt}\|x-y\|)) \quad \text{for } t \in [0,1]$$

where $d(t) := 2\text{dist}(t, \mathbb{Z})$ for $t \in [0,1]$.

An important problem lies in verifying for which functions α the above estimation is optimal. The conjecture of Zs. Páles that this is the case for functions of type $\alpha(r) = r^p$ for $p \in (0,1)$, was proved by J. Mako and Zs. Páles in [Approximate convexity of Takagi type function, JMAA, 545–554 2010].

In this paper we present a computer assisted method to verifying optimality of this estimation in the class of piecewise linear functions α.

Mathematics Subject Classification (2010). Primary 26A51; Secondary 39B62, 65G40.

Keywords. midconvex function, convexity, numerical verification.

1. Introduction

Let V be a convex subset of a normed space X. The function $f\colon V \to \mathbb{R}$ is convex, if

$$Cf(x,y;t) := f(tx+(1-t)y) - tf(x) - (1-t)f(y) \leq 0 \quad \text{for } x, y \in V, t \in [0,1]$$

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and $f\colon V \to \mathbb{R}$ is midconvex if

$$Jf(x,y) := f\left(\frac{x+y}{2}\right) - \frac{f(x)+f(y)}{2} \le \varepsilon\|x-y\|^p \quad \text{for } x,y \in V.$$

One of the natural generalization of convexity is (ε,p)-midconvexity [5, 7].

Definition 1.1. Let X be a normed space, V be a convex subset of X. Let $\varepsilon \ge 0$ and $p \ge 0$ be fixed. A function $f\colon V \to \mathbb{R}$ is (ε,p)-midconvex if

$$Jf(x,y) \le \varepsilon\|x-y\|^p \quad \text{for } x,y \in V.$$

The relation between approximate midconvexity and convexity is one of the most important questions in the study of generalized convexity. Some results in this direction should be mentioned. We start with the Bernstein-Doetsch Theorem [8, Chapter 6.4], which tells that every locally bounded midconvex function is convex. In all results quoted below we assume that f is locally bounded.

In 1979 S. Rolewicz [12] proved that each (ε,p)-midconvex function for $p \in (2,\infty)$ is convex. C.T. Ng and K. Nikodem [10] found that the optimal bound for $Cf(x,y;t)$, when $p=0$ is given by the following inequality:

$$Cf(x,y;t) \le \varepsilon \sum_{k=0}^{\infty} \frac{1}{2^k} \mathbb{1}_{\mathbb{Z}}(2^k t) \quad \text{for } t \in [0,1],$$

where $\mathbb{1}_{\mathbb{Z}}$ denotes the characteristic function of $\mathbb{Z}$.

A. Hazy and Zs. Páles considered first the case $p=1$ in [11], and later in [7] the case $p \in [0,1]$ and proved that

$$Cf(x,y;t) \le \varepsilon \sum_{k=0}^{\infty} \frac{d^p(2^k t)}{2^k} \|x-y\|^p \quad \text{for } t \in [0,1], \tag{1.1}$$

where

$$d(t) := 2\mathrm{dist}(t;\mathbb{Z}) \text{ for } t \in [0,1].$$

In the case where $p=1$ Z. Boros [3] showed that inequality (1.1) is optimal. Note that for $p=0$ (1.1) reduces to the estimation obtained by C. T. Ng and K. Nikodem. The case $p \in [1,2]$ was completely solved in [13], where authors showed that the optimal estimation for the convexity differences of (ε,p)-midconvex functions is given by

$$Cf(x,y;t) \le \varepsilon \sum_{k=0}^{\infty} \frac{d^p(2^k t)}{2^k} \|x-y\|^p \quad \text{for } r \in [0,1].$$

To present the following results we need the generalization of the notion of (ε,p)-midconvexity:

Definition 1.2. Let X be a normed space, V be a convex subset of X and let $\alpha : [0, \mathrm{diam}\, V] \to \mathbb{R}_+$ be a given function. A function $f\colon V \to \mathbb{R}$ is *α-midconvex* if

$$Jf(x,y) \le \alpha(\|x-y\|) \quad \text{for } x,y \in V.$$

In [14] the authors showed that to check optimality of estimation (1.1) in the class of α-midconvex functions it suffices to check two inequalities. To quote this result it is convenient to formulate Condition T.

Definition 1.3 (Condition T). Let $\alpha\colon [0,1] \to \mathbb{R}_+$ be a non-decreasing function and let $\omega = \alpha \circ d$. We say that function α satisfies the *Condition T* if

$$\omega\left(\frac{x+y}{2}\right) - \frac{\omega(x)+\omega(y)}{2} + \frac{\omega(x+y) - \frac{\omega(2x)+\omega(2y)}{2}}{2} + \frac{\omega(2x-2y)}{4} \le \omega\left(\frac{x-y}{2}\right) \tag{1.2}$$

for $(x,y) \in B := \mathrm{conv}\{\left(0,\frac{1}{2}\right), \left(\frac{1}{2},\frac{1}{2}\right), (0,1)\}$, and

$$\omega\left(\frac{x+y}{2}\right) - \frac{\omega(x)+\omega(y)}{2} + \frac{1}{2}\omega(x-y) \le \omega\left(\frac{x-y}{2}\right) \tag{1.3}$$

for $(x,y) \in D := \mathrm{conv}\{(0,0), (\frac{1}{2},\frac{1}{2}), (0,1), \left(-\frac{1}{2},\frac{1}{2}\right)\} \setminus B$.

Now the main result from [14] (see Proposition 3.1) can be reformulated in the following form:

Theorem TT ([13]). Let $\alpha\colon [0,1] \to \mathbb{R}_+$ be a non-decreasing function.
We assume that the function α satisfies the Condition T. Then the estimation

$$Cf(x,y;t) \le \sum_{k=0}^{\infty} \frac{1}{2^k}\alpha(d(2^k t)\|x-y\|) \quad \text{for } t \in [0,1] \tag{1.4}$$

is optimal in class of α-midconvex functions.

A crucial result in this direction was obtained in 2010 J. Mako and Zs. Páles [9, Theorem 9] who showed that the estimation (1.4) is optimal for the class of functions $\alpha(r) = r^p$ for $p \in (0,1)$. In fact the even showed the optimality in the large class of C^1 functions satisfying some additional assumptions. The idea was based on verifying a nontrivial condition similar in nature to Condition T, which the authors checked analytically for α.

In this paper, a different approach to this problem will be presented, which allows to verify the optimality in the class of continuous piecewise linear functions. In order of do that, a theorem from [14] and an algorithm based on interval arithmetic will be used.

In the next chapter we demonstrate how to numerically verify Condition T. In the last chapter of the paper results similar to those in [9] for continuous piecewise linear functions will be achieved.

2. Optimality in the class of continuous piecewise linear functions

In this chapter we show how to numerically verify optimality. Let us first introduce some notation.

For $[a,b] \subset \mathbb{R}$ and $N \in \mathbb{N}$ let

$$[a,b]_N \colon = \{a + \frac{k(b-a)}{N} : k = 0, \ldots, N\}$$

and let

$$\mathrm{Aff}_N([a,b]) \colon = \{f \in C([a,b],\mathbb{R}) \colon f|_{[a+\frac{k(b-a)}{N}, a+\frac{(k+1)(b-a)}{N}]} \text{ is affine for } k = 0, \ldots, N-1\}.$$

Remark 2.1. In this article for simplicity the segment $[a,b]$ is divided into equal is but analogous reasoning can be performed on any finite division of the interval $[a,b]$.

We say that $P \subset \mathbb{R}^2$ is a *convex polygon* if there exist $n \in \mathbb{N}$ ($n > 3$) and $W = \{w_i\}_{i \in \mathbb{Z}_n} \subset \mathbb{R}^2$ such that P is convex hull of W ($P = \mathrm{conv}(W)$) and for all $i \in \mathbb{Z}_n$ (where $\mathbb{Z}_n$ cyclic group) points w_{i-1}, w_i, w_{i+1} are not collinear. The set W will be called the set of vertices of the polygon P. Moreover, from now on we assume that the sequence W is chosen so that W is minimal (which means $w_i \neq w_j$ for $i \neq j$) and

$$[w_i, w_{i+1}] \colon = \{tw_i + (1-t)w_{i+1} \colon t \in [0,1]\} \subset \partial P \quad \text{for } i \in \mathbb{Z}_n.$$

The collection of intervals

$$\overline{\partial} P := \{[w_i, w_{i+1}]\}_{i \in \mathbb{Z}_n}$$

will be called *algebraic border* of P. For a convex polygon P we consider restrictions of affine and linear functions to P:

$$\begin{aligned} \mathrm{Aff}_P &:= \{f|_P \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ is affine}\}, \\ \mathrm{Lin}_P &:= \{f|_P \mid f \colon \mathbb{R}^2 \to \mathbb{R} \text{ is linear and } f \neq 0\}. \end{aligned}$$

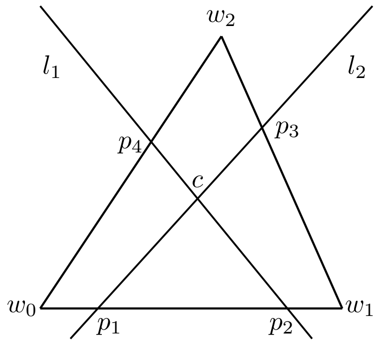
For $\mathcal{F} \subset \mathrm{Lin}_P$, S finite subset of $\mathbb{R}$ and convex polygon P let

$$\mathit{Lines}_P(\mathcal{F}, S) \colon = \overline{\partial} P \cup \{f^{-1}(s) \cap P \colon f \in \mathcal{F},\ s \in S\}.$$

Observe that $\mathit{Lines}_P(\mathcal{F}, S)$ is a finite collection of line-segments with ends in ∂P. Let

$$\mathit{Points}_P(\mathcal{F}, S) = \bigcup \{k \cap l : k, l \in \mathit{Lines}_P(\mathcal{F}, S), k \neq l\}.$$

Example. Let T be a triangle with vertices w_0, w_1, w_2. Let functions f_1, f_2 and points a_1, a_2 be such that lines $l_1 := f_1^{-1}(a_1)$, $l_2 := f_2^{-1}(a_2)$ intersect triangle T as in the picture below.



Then $Lines_T(\{f_1, f_2\}, \{a_1, a_2\}) = \{[w_0, w_1], [w_1, w_2], [w_2, w_0], [p_1, p_3], [p_2, p_4]\}$ and $Points_T(\{f_1, f_2\}, \{a_1, a_2\}) = \{w_0, w_1, w_2, p_1, p_2, p_3, p_4, c\}$.

Given any family of sets $\mathcal{P}$ by its support we understand

$$\operatorname{supp}(\mathcal{P}) = \bigcup_{P \in \mathcal{P}} P.$$

To proceed further we will need some technical results.

Proposition 2.2. *Assume that $P \subset \mathbb{R}^2$ is a convex polygon and $[a, b] \subset \mathbb{R}$, $N \in \mathbb{N}$. Let $\omega_i \in \operatorname{Aff}_N([a, b])$ and $f_i \in \operatorname{Lin}_P$ be such that $f_i(P) \subset [a, b]$ for $i \in \{1, \ldots, m\}$.*

Let $F\colon P \to \mathbb{R}$ be defined by the formula

$$F(x) = \sum_{i=0}^{m} \omega_i(f_i(x)) \quad \text{for } x \in P.$$

Then for every connected component U of $P \setminus \operatorname{supp}(Lines_P(\{f_i\}_{i=1}^m, [a, b]_N))$

(a) $\overline{U}$ is a convex polygon;
(b) the set of vertices of $\overline{U}$ is a subset of $Points_P(\{f_i\}_{i=1}^m, [a, b]_N)$;
(c) $F|_{\overline{U}} \in \operatorname{Aff}_{\overline{U}}$.

Proof. Let U be a connected component of $P \setminus \operatorname{supp}(Lines_P(\{f_i\}_{i=1}^m, [a, b]_N))$.

(a) For every $i \in \{1, \ldots, m\}$ there exists $k_i \in \{1, \ldots, N-1\}$ such that

$$f_i(x) \in (a + \frac{k_i}{N}(b-a), a + \frac{k_i+1}{N}(b-a)) \quad \text{for } x \in U.$$

Then

$$U = \bigcap_{i=1}^{N-1} f_i^{-1}\left((a + \frac{k_i}{N}, a + \frac{k_i+1}{N})\right).$$

Clearly $f_i^{-1}\left([a + \frac{k_i}{N}, a + \frac{k_i+1}{N}]\right) \cap P$ is a convex polygon. Hence, $\overline{U}$ is a convex polygon as an intersection of finite family of convex polygons.

(b) Let w be a point from set of vertices of $\overline{U}$. Then w is an intersection of two edges e_1, e_2 of $\overline{U}$. Then e_i (for $i = 1, 2$) is either a subinterval of an edge of P or e_i is subinterval of $f_k^{-1}(c)$, for a certain $k \in \{1, \ldots, N-1\}$ and $c \in [a, b]_N$. Consequently we obtain that

$$w \in Points_P(\{f_i\}_{i=1}^m, [a, b]_N).$$

(c) For every $i \in \{1, \ldots, m\}$ there exists $k_i \in \{1, \ldots, N-1\}$ such that

$$f_i(x) \in (a + \frac{k_i}{N}(b-a), a + \frac{k_i+1}{N}(b-a)) \quad \text{for } x \in U.$$

Since for $i \in \{1, \ldots, m\}$ functions $\omega_i \in \operatorname{Aff}_N([a, b])$, so $\omega_i(f_i|_U) \in \operatorname{Aff}_N([a, b])$. We obtain that $F|_{\overline{U}} \in \operatorname{Aff}_{\overline{U}}$ as a sum of affine functions.

$\square$

Remark 2.3. Let $P \subset \mathbb{R}^2$ be a convex polygon. It is well-known that an affine function $F\colon P \to \mathbb{R}$ attains its maximum (and minimum) in one of the vertices of P.

Now we can prove the theorem:

Theorem 2.4. *Let $P \subset \mathbb{R}^2$ be a convex polygon and $[a,b] \subset \mathbb{R}$. Let $N \in \mathbb{N}$, $\omega_i \in \mathrm{Aff}_N([a,b])$ and let $f_i \in \mathrm{Lin}_P$ be such that $f_i(P) \subset \mathrm{dom}(\omega_i)$ for $i = 1,\dots,m$. Let $F\colon P \to \mathbb{R}$ be given by:*

$$F(x) = \sum_{i=0}^{m} \omega_i(f_i(x)).$$

Then the following conditions are equivalent:

(a) $F(x) \le 0$ for $x \in P$,
(b) $F(x) \le 0$ for $x \in Points_P(\{f_i\}_{i=1}^m, [a,b]_N)$.

Proof. Implication $a) \Rightarrow b)$ is obvious. We show the opposite one.

Assumptions of Proposition 2.2 are met. To check condition $a)$, it is sufficient to show that the maximum of the function $F(x)$ is attained on

$$Points_P(\{f_i\}_{i=1}^m, [a,b]_N).$$

Let $\{U_i\}_{i=1}^l$ for $l \in \mathbb{N}$ be the collection of connected components of the set $P \setminus \mathrm{supp}(Lines_P(\{f_i\}_{i=1}^m, [a,b]_N))$. Thank to Proposition 2.2 $a)$ each element of $\{U_i\}_{i=1}^l$ is convex polygon. By Proposition 2.2 $c)$ we obtain that $F|_{\overline{U}_i} \in \mathrm{Aff}_{\overline{U}_i}$ for all $i \in \{1,\dots,l\}$. Thanks to Remark 2.3 we get that the maximum of the function F is attained on the one of the vertices of $\overline{U}_i$ for some $i \in \{1,\dots,l\}$. To finish this proof it is enough to notice that vertices of $\overline{U}_i$ for $i \in \{1,\dots,l\}$ are included in $Points_P(\{f_i\}_{i=1}^m, [a,b]_N)$, see Proposition 2.2 $b)$. $\square$

As a corollary from the above theorem we get the following result.

Theorem 2.5. *Let $\alpha \in \mathrm{Aff}_N([-1,1])$, $N \in \mathbb{N}$ and let $\omega = \alpha \circ d$. Then the function α satisfies the Condition T if and only if*

$$\omega\left(\frac{x+y}{2}\right) - \frac{\omega(x)+\omega(y)}{2} + \frac{\omega(x+y) - \frac{\omega(2x)+\omega(2y)}{2}}{2} + \frac{\omega(2x-2y)}{4} \le \omega\left(\frac{x-y}{2}\right) \tag{2.1}$$

for $(x,y) \in \left(\frac{\mathbb{Z}}{4N} \times \frac{\mathbb{Z}}{4N}\right) \cap ([-1,1] \times [0,1]) \cap B$ where

$$B := \mathrm{conv}\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0,1)\},$$

and

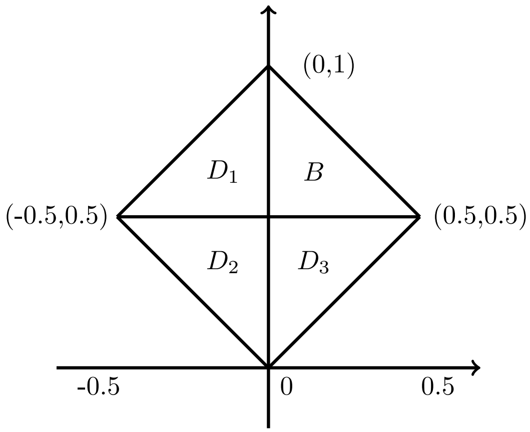
$$\omega\left(\frac{x+y}{2}\right) - \frac{\omega(x)+\omega(y)}{2} + \frac{1}{2}\omega(x-y) \le \omega\left(\frac{x-y}{2}\right) \tag{2.2}$$

for $(x,y) \in \left(\frac{\mathbb{Z}}{2N} \times \frac{\mathbb{Z}}{2N}\right) \cap ([-1,1] \times [0,1]) \cap D$ where

$$D := \mathrm{conv}\{(0,0), (\frac{1}{2}, \frac{1}{2}), (0,1), \left(-\frac{1}{2}, \frac{1}{2}\right)\} \setminus B.$$

Proof. The implication Condition T $\Rightarrow$ inequality (5, 6) is obvious. We show the opposite implication.

To check the Condition T we need to verify two inequalities (5) and (6) in two sets D and B. Set B is itself a convex polygon and D is union of convex polygons $D_1\colon = \operatorname{conv}\{\left(-\frac{1}{2},\frac{1}{2}\right),\left(0,\frac{1}{2}\right),(0,1)\}$, $D_2\colon = \operatorname{conv}\{\left(-\frac{1}{2},\frac{1}{2}\right),(0,0),(0,\frac{1}{2})\}$, $D_3\colon = \operatorname{conv}\{(0,0),\left(\frac{1}{2},\frac{1}{2}\right),(0,\frac{1}{2})\}$.



We have to show that (2.1) holds for all $(x,y)\in B$ and that (2.2) holds for all $(x,y)\in D_1$, $(x,y)\in D_2$ $(x,y)\in D_3$. We show that (2.1) is valid for all $(x,y)\in B$, the other inequalities can be shown analogically.

Thanks to Theorem 2.4 we know that to check the inequality (1.2) from Condition T it is enough to verify

$$\omega\left(\frac{x+y}{2}\right)-\frac{\omega(x)+\omega(y)}{2}+\frac{\omega(x+y)-\frac{\omega(2x)+\omega(2y)}{2}}{2}+\frac{\omega(2x-2y)}{4}-\omega\left(\frac{x-y}{2}\right)\le 0 \tag{2.3}$$

for $(x,y)\in Points_B(\{f_i\}_{i=1}^8,[-1,1]_N)$. Let

$$\begin{array}{llll}
f_1(x,y)=\frac{x+y}{2}, & \omega_1(x)=\omega(x), & f_2(x,y)=x, & \omega_2(x)=-\frac{1}{2}\omega(x),\\
f_3(x,y)=y, & \omega_3(x)=-\frac{1}{2}\omega(x), & f_4(x,y)=x+y, & \omega_4(x)=\frac{1}{2}\omega(x),\\
f_5(x,y)=2x, & \omega_5(x)=-\frac{1}{4}\omega(x), & f_6(x,y)=2y, & \omega_6(x)=-\frac{1}{4}\omega(x),\\
f_7(x,y)=2x-2y, & \omega_7(x)=\frac{1}{4}\omega(x), & f_8(x,y)=\frac{x-y}{2}, & \omega_8(x)=-\omega(x).
\end{array}$$

Then inequality (2.3) can be rewritten as

$$\sum_{i=1}^{8}\omega_i(f_i(x,y))\le 0.$$

Thanks to equality

$$Points_B(\{f_i\}_{i=1}^8,[-1,1]_N)=\bigcup_{i,j=1,i\neq j}^{8} Points_B(\{f_i,f_j\},[-1,1]_N)$$

and the fact that $\overline{\partial}B\subset Lines_B(\{f_i\}_{i=1}^8,[-1,1]_N)$, to calculate $Points_B(\{f_i\}_{i=1}^8,[-1,1]_N)$ it is enough to solve for $i,j\in\{1,\ldots,8\}$ and $a,b\in[-1,1]_N$ the equations

$$\begin{cases} f_i(x,y)=a,\\ f_j(x,y)=b,\end{cases}$$

and take the union of points which are unique solutions of above equations. Thus our aim is to show that all unique solutions of the equations

$$\begin{cases} Ax + By = a, \\ Cx + Dy = b, \end{cases} \tag{2.4}$$

for $(A, B), (C, D) \in \{(\frac{1}{2}, \frac{1}{2}), (1, 0), (0, 1), (1, 1), (0, 2), (2, 0), (2, -2), (1, -1)\}$ and $a, b \in [-1, 1]_N$, lie in the set

$$\frac{\mathbb{Z}}{4N} \times \frac{\mathbb{Z}}{4N}.$$

Without loss of generality we can consider $(A, B), (C, D) \in \{(2, -2), (0, 2), (2, 0), (1, 1)\}$, because for given $r_1, r_2 \in \mathbb{R}$ and $n \in \mathbb{Z}, n \neq 0$ we have

$$\{(x, y) \colon r_1 x + r_2 y \in \mathbb{Z}\} \subset \{(x, y) \colon nr_1 x + nr_2 y \in \mathbb{Z}\}).$$

We consider only coefficients A, B, C, D such that (2.4) has unique solution; so $AD - BC \neq 0$. Then

$$x = \frac{Ab - Ca}{AD - BC},\ y = \frac{aD - bB}{AD - BC}.$$

Since $AD - BC$ is an element of $\{-4, -2, -1, 1, 2, 4\}$ and $a = \frac{k}{N}$, $b = \frac{l}{N}$ (for some $l, k \in \{1, \dots, N\}$) we obtain that $Points_B(\{f_i\}_{i=1}^8, [-1, 1]_N) \subset \frac{\mathbb{Z}}{4N} \times \frac{\mathbb{Z}}{4N}$. □

We can see that to verify Condition T it is enough to check finite number of inequalities. In order to do this, a computer program will be used.

3. Strict numerical verification

3.1. Interval arithmetic and reduction of similar terms

In this chapter it will be shown how to strictly verify Condition T using a computer assisted approach.

One of the greatest problems concerning strict numerical calculations is caused by the rounding errors connected with the computer representation of real numbers. Interval arithmetic helps to deal with them, unfortunately it also causes some complications.

For the convenience of the reader we give a short description of the interval arithmetic. Interval arithmetic [4, chapter 2.5.3] is based on the operations on segments. Let $\mathcal{P}$ be a finite subset of $\mathbb{R}$ consisting of numbers which we interpret as representable. Let $\overline{\mathcal{P}} = \mathcal{P} \cup \{-\infty, +\infty\}$. For $x \in \mathbb{R}$ let

$$x_{\mathcal{P}} \colon = \sup\{a \in \mathcal{P} \colon a \leq x\} \text{ and } x^{\mathcal{P}} \colon = \inf\{a \in \mathcal{P} \colon a \geq x\}.$$

Interval approximation of a real numbers we define by

$$[x]_{\mathcal{P}} \colon = [x_{\mathcal{P}}, x^{\mathcal{P}}]. \tag{3.1}$$

Directly from definition we have $x \in [x]_{\mathcal{P}}$, but for $x \notin \mathcal{P}$ we have $\{x\} \subsetneq [x]_{\mathcal{P}}$. It is obvious that the use of interval arithmetic does not usually give accurate results.

The collection of representable intervals we denote as

$$\mathcal{I}_{\mathcal{P}} \colon = \{[x, y] \colon x, y \in \overline{\mathcal{P}}\}.$$

Example. Let $\mathcal{P} = \{-1, 0, 1, 2\}$, then $[-6]_{\mathcal{P}} = [-\infty, -1]$, $[0.5]_{\mathcal{P}} = [0, 1]$.

Example. If $\mathcal{P} = \{x \in \mathbb{R} \colon 10^3 x \in \mathbb{Z}\} \cap [-2, 2]$, then $[\sqrt{2}]_{\mathcal{P}} = [1.414, 1.415]$.

In the collection of segments some operations can be defined.

Definition 3.1. Let $f \colon \mathbb{R}^n \rightharpoonup \mathbb{R}$ where "$\rightharpoonup$" means that f is a partial map. We say that the operation $f_{\mathcal{P}} \colon \mathcal{I}_{\mathcal{P}} \to \mathcal{I}_{\mathcal{P}}$ is a $\mathcal{P}$–extension of $f \colon \mathbb{R}^n \rightharpoonup \mathbb{R}$ iff

$$f(x_1, \ldots, x_n) \in f_{\mathcal{P}}([x_1]_{\mathcal{P}}, \ldots, [x_n]_{\mathcal{P}}) \text{ for } (x_1, \ldots, x_n) \in \mathrm{dom} f.$$

From now on by $f_{\mathcal{P}}$ we denote extension of $f \colon \mathbb{R}^n \to \mathbb{R}$ for $n \in \mathbb{N}$.

Example. Let $\mathcal{P}$ be given and let $\mathcal{I}_{\mathcal{P}}$ be the set of intervals. Operation $+_{\mathcal{P}} \colon \mathcal{I}_{\mathcal{P}} \times \mathcal{I}_{\mathcal{P}} \to \mathcal{I}_{\mathcal{P}}$ defined by the formula

$$[x]_{\mathcal{P}} +_{\mathcal{P}} [y]_{\mathcal{P}} \colon = [(x + y)_{\mathcal{P}}, (x + y)^{\mathcal{P}}]$$

is a well-defined $\mathcal{P}$–extension of $+ \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}$.

Let $\mathcal{P}$ be given and let $+_{\mathcal{P}}$, $\cdot_{\mathcal{P}}$ be $\mathcal{P}$–extantion of standard operation on real numbers. Instead of writing $+_{\mathcal{P}}$, $\cdot_{\mathcal{P}}$ for the shortness of notation we will write $+$, $\cdot$. Let $n \in \mathbb{N}$ and $\omega \colon \mathbb{R} \rightharpoonup \mathbb{R}$ be given. Let R be a finite subset of $\mathbb{R}$. Interval arithmetic will be used to obtain estimation of the value of:

$$\sum_{i=1}^{n} k_i \omega(r_i) \tag{3.2}$$

for $r_i \in R$ and $k_i \in \mathbb{Z}$.

By direct use of interval arthmetic we get the following approximation of (3.2):

$$\sum_{i=1}^{n} k_i \omega(r_i) \in \sum_{i=1}^{n} [k_i]_{\mathcal{P}} \cdot \omega_{\mathcal{P}}([r_i]_{\mathcal{P}}). \tag{3.3}$$

This approach, as will be shown in the next example, is insufficient for our needs.

Example. Let $\overline{r} = \frac{1}{3}$, $\mathcal{P} = \{x \in \mathbb{R} \colon 10^3 x \in \mathbb{Z}\} \cap [-2, 2]$. Then $[r_0]_{\mathcal{P}} = \left[\frac{1}{3}\right]_{\mathcal{P}} = [0.333, 0.334]$ and $R = \{\frac{1}{3}\}$. Let $\omega : \mathbb{R} \to \mathbb{R}$ be such that $\omega(x) = x$ and $k_1 = 1$, $k_2 = -1$, $r_1 = r_2 = \overline{r}$. By (3.3) we obtain

$$r_1 - r_2 \in [r_1]_{\mathcal{P}} - [r_2]_{\mathcal{P}} = \left[\frac{1}{3}\right]_{\mathcal{P}} - \left[\frac{1}{3}\right]_{\mathcal{P}} =$$

$$[0.333, 0.334] - [0.333, 0.334] = [-0.001, 0.001] \neq [0]_{\mathcal{P}}.$$

Consequently $[r_1]_{\mathcal{P}} - [r_2]_{\mathcal{P}}$ is not always zero.

To obtain better approximation of (3.2) reduction of similar terms can be used. By simple change of summation in (3.2) we get

$$\sum_{i=1}^{n} k_i\omega(r_i) = \sum_{r\in R}\left(\sum_{i\,:\,r_i=r} k_i\right)\omega(r) \in \sum_{r\in R}\left(\sum_{i\,:\,r_i=r} [k_i]_{\mathcal{P}}\right)\omega_{\mathcal{P}}(r). \tag{3.4}$$

As we can see in the following examples, by this simple operation we obtain better estimation. In the next example we assume that $\{-1,0,1\}\subset\mathcal{P}$.

Example. Let $R=\{\frac{1}{3},\frac{1}{4},\frac{1}{6}\}$, $\mathcal{P}=\{x\in\mathbb{R}\colon 10^3x\in\mathbb{Z}\}\cap[-2,2]$ and $\omega(x)=x$. We put

$$\begin{array}{llll} r_1=\frac{1}{3}, & k_1=1, & r_2=\frac{1}{4}, & k_2=-1,\\ r_3=\frac{1}{6}, & k_3=1, & r_4=\frac{1}{3}, & k_4=-1,\\ r_5=\frac{1}{4}, & k_5=1, & r_6=\frac{1}{6}, & k_6=1. \end{array}$$

By (3.3) we get

$$\sum_{i=1}^{n} k_i\omega(r_i) = \frac{1}{3}-\frac{1}{4}+\frac{1}{6}-\frac{1}{3}+\frac{1}{4}+\frac{1}{6}\in$$

$$\in\left[\frac{1}{3}\right]_{\mathcal{P}}-\left[\frac{1}{4}\right]_{\mathcal{P}}+\left[\frac{1}{6}\right]_{\mathcal{P}}-\left[\frac{1}{3}\right]_{\mathcal{P}}+\left[\frac{1}{4}\right]_{\mathcal{P}}+\left[\frac{1}{6}\right]_{\mathcal{P}}=$$

$$=\left[\frac{1}{3}\right]_{\mathcal{P}}-\left[\frac{1}{3}\right]_{\mathcal{P}}+[0.25]_{\mathcal{P}}-[0.25]_{\mathcal{P}}+\left[\frac{1}{6}\right]_{\mathcal{P}}+\left[\frac{1}{6}\right]_{\mathcal{P}}=$$

$$=\left[\frac{1}{3}\right]_{\mathcal{P}}-\left[\frac{1}{3}\right]_{\mathcal{P}}+[2]_{\mathcal{P}}\cdot\left[\frac{1}{6}\right]_{\mathcal{P}}=$$

$$=[-0.001,0.001]_{\mathcal{P}}+[2]_{\mathcal{P}}\cdot\left[\frac{1}{6}\right]_{\mathcal{P}}.$$

while the (3.4) gives

$$\sum_{i=1}^{n} k_i\omega(r_i) = \frac{1}{3}-\frac{1}{4}+\frac{1}{6}-\frac{1}{3}+\frac{1}{4}+\frac{1}{6}=$$

$$=(1-1)\frac{1}{3}+(1-1)\frac{1}{4}+(1+1)\frac{1}{6}\in\sum_{r\in\mathcal{P}}\left(\sum_{i\,:\,r_i=r}[k_i]_{\mathcal{P}}\right)\omega_{\mathcal{P}}(r)=[2]_{\mathcal{P}}\cdot\left[\frac{1}{6}\right]_{\mathcal{P}}.$$

Example. Let us come back to situation from Example 3.1. By use of (3.4) we get

$$\frac{1}{3}-\frac{1}{3}\in([1]_{\mathcal{P}}-[1]_{\mathcal{P}})\cdot\left[\frac{1}{3}\right]_{\mathcal{P}}=[0]_{\mathcal{P}}\cdot\left[\frac{1}{3}\right]_{\mathcal{P}}=[0]_{\mathcal{P}}.$$

3.2. Algorithm

To implement the algorithm the boost interval library [2] and map standard library from C++ will be used(map standard library is used to reduce similar terms).

Let $N \in \mathbb{N}$. The algorithm will check the Condition T for $f \in \mathrm{Aff}_N([-1,1])$. Thanks to Theorem 2.5 it is enough to show inequalities (2.1) and (2.2) from assertion of this theorem. We present algorithm only for inequality (2.2) (inequality (2.1) can be verified analogically). For the convenience of the reader we recall inequality (2.2). Let $\omega = f \circ d$ and let

$$G_\omega(x,y) := \omega\left(\frac{x+y}{2}\right) - \frac{\omega(x)+\omega(y)}{2} + \frac{1}{2}\omega(x-y) - \omega\left(\frac{x-y}{2}\right).$$

Our aim is to show that

$$2G_\omega(x,y) \le 0 \quad \text{for } (x,y) \in \left(\frac{\mathbb{Z}}{2N} \times \frac{\mathbb{Z}}{2N}\right) \cap ([-1,1] \times [0,1]) \cap D,$$

where

$$D = \mathrm{conv}\{(0,0), (\frac{1}{2}, \frac{1}{2}), (0,1), \left(-\frac{1}{2}, \frac{1}{2}\right)\} \setminus \mathrm{conv}\{\left(0, \frac{1}{2}\right), \left(\frac{1}{2}, \frac{1}{2}\right), (0,1)\}.$$

We consider the inequality $2G_\omega(x,y) \le 0$ instead of $G_\omega(x,y) \le 0$ since the first has integer coefficients, and consequently we do not have errors caused by division.

The algorithm consists of two parts. First, we reduce similar terms in $2G_\omega$. Then we calculate segments containing strict values of G_ω by using interval arithmetic. At the end we verify the above inequality. The source code is available on website [1].

```
for i j from 0 to 2*N do
  if ( i/2*N , j/2*N ) in D
    reduce_similar_terms( 2G ( i/2*N , j/2*N ) )
    if upper_bound of interval( 2G ) > 0 then
      print( "The Condition was not verified" )
      break
print( "The Condition was verified" )
```

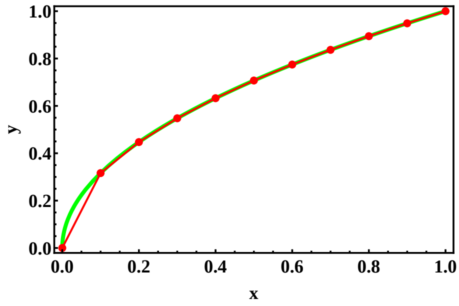
3.3. Verification for affine approximation of the function $\alpha_p(x) = x^p$

In this chapter we show similar result to that from [9] but for continuous piecewise linear approximations of function $\alpha_p = x^p$ for $p \in (0,1)$.

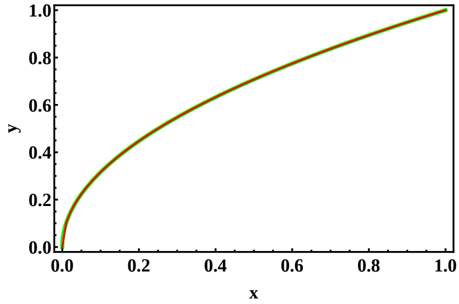
Definition 3.2. For continuous function $\alpha\colon [0,1] \to \mathbb{R}$ let $[\alpha]_N \in \mathrm{Aff}_N([0,1])$ be the unique function satisfying

$$[\alpha]_N|_{[\frac{k}{N}, \frac{k+1}{n}]} \text{ is affine for } k = 0, \ldots, N.$$

Example. Let $\alpha_{\frac{1}{2}}(x) = x^{\frac{1}{2}}$. On the picture we can see comparison of function $\alpha_{\frac{1}{2}}$ and $[\alpha_{\frac{1}{2}}]_N$ for $N = 10$ and $N = 100$.



(a) $N = 10$



(b) $N = 100$

FIGURE 1. Above plots show graphs of $\alpha_{\frac{1}{2}}$ and $[\alpha_{\frac{1}{2}}]_N$.

Observe that the functions α and $[\alpha]_N$ are close, for N sufficiently large. Thanks to Theorem 2.5 and above algorithm we can verify optimality of function $[\alpha_p]_N$, where as we recall $\alpha_p(r) = r^p$ for $p \in [0, 1]$.

Example. We show that for $N \in \mathbb{N}$ we have

$$\|\alpha_{\frac{1}{2}} - [\alpha_{\frac{1}{2}}]_N\|_{\sup} = \frac{1}{4\sqrt{N}}. \tag{3.5}$$

To prove that, we consider the interval $[k/N, (k+1)/N]$. Clearly

$$\|\alpha_{\frac{1}{2}}|_{[k/N,(k+1)/N]} - [\alpha_{\frac{1}{2}}]_N|_{[k/N,(k+1)/N]}\|_{\sup} = \sup_{x \in [k/N,(k+1)/N]} |q_k(x)|,$$

where

$$q_k(x) := \sqrt{x} - \frac{\sqrt{N}}{\sqrt{k} + \sqrt{k+1}} x - \frac{\sqrt{k}\sqrt{k+1}}{\sqrt{N}(\sqrt{k} + \sqrt{k+1})}$$

To find the desired extreme let's calculate the derivative of the function q_k. Comparing it to zero and solving the equation $\frac{1}{2\sqrt{x}} - \frac{\sqrt{N}}{\sqrt{k}+\sqrt{k+1}} = 0$, we get that q_k has the maximum at $x_k = \frac{1}{4}\frac{(\sqrt{k}+\sqrt{k+1})^2}{N}$, which equals

$$q_k(x_k) = \frac{(\sqrt{k} - \sqrt{k+1})^2}{4\sqrt{N}(\sqrt{k} + \sqrt{k+1})}.$$

Since the function $x \to \frac{(\sqrt{x}-\sqrt{x+1})^2}{(\sqrt{x}+\sqrt{x+1})}$ is decreasing for $x \in [0,\infty)$, to find $\|\alpha - [\alpha]_N\|$ we take $k=0$ and get $q_0(\frac{1}{4N}) = \frac{1}{2}\sqrt{\frac{1}{N} - \frac{1}{4\sqrt{N}}}$, which proves (3.5).

Thanks to our algorithm [1] we can get optimality of $[\alpha_p]_{1000}$ for every $p \in [0.1, 0.9]$. To do so we verify that inequalities (2.1) and (2.2) from Theorem 2.5 are valid for every $p \in P$, where $P \in [0.1, 0.9]_{1000}$ is arbitrary:

Corollary 3.3. *Let $N = 1000$ and let $p \in [0.1, 0.9]$ be arbitrary. Then the estimation (1.4) is optimal in the class of $[\alpha_p]_N$-midconvex functions.*

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