Expansivity and cone-fields in metric spaces

Łukasz Struski and Jacek Tabor

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Abstract: Due to the results of Lewowicz and Tolosa expansivity can be characterized with the aid of Lyapunov function. In this paper we study a similar problem for uniform expansivity and show that it can be described using generalized cone-fields on metric spaces.
We say that a function $f\colon X \to X$ is uniformly expansive on a set $\Lambda \subset X$ if there exist $\varepsilon > 0$ and $\alpha \in (0,1)$ such that for any two orbits $\mathrm{x}\colon \{-N,\dots,N\} \to \Lambda$, $\mathrm{v}\colon \{-N,\dots,N\} \to X$ of f we have

$$\sup_{-N\le n\le N} d(\mathrm{x}_n,\mathrm{v}_n) \le \varepsilon \implies d(\mathrm{x}_0,\mathrm{v}_0) \le \alpha \sup_{-N\le n\le N} d(\mathrm{x}_n,\mathrm{v}_n).$$

It occurs that a function is uniformly expansive iff there exists a generalized cone-field on X such that f is cone-hyperbolic.

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1. Introduction

In 1892 A. M. Lyapunov [9] introduced the idea of Lyapunov functions to study stability of equilibria of differential equations. The Lyapunov approach allows to assess the stability of equilibrium points of a system without solving the differential equations that describe the system. This theory is widely used in qualitative theory of dynamical systems.

In [6, 8] J. Lewowicz proposed to use Lyapunov functions of two variables to study structural stability and similar concepts, such as topological stability and persistence. The method has been applied in particular to study hyperbolic diffeomorphisms on manifolds. For the survey of the results, methods and possible generalizations see [11].

Let us quote one of the most interesting results from [11]. Let $f\colon M \to M$ be a homeomorphism of a compact manifold M. For $U\colon M\times M \to \mathbb{R}$ we define

$$\Delta_f U(x,y) := U(f(x),f(y)) - U(x,y) \quad \text{for} \quad x,y \in M.$$

We say that U is a Lyapunov function for f if it is continuous, vanishes on the diagonal, and $\Delta_f U(x,y)$ is positive for (x,y) on a neighborhood of the diagonal, $x \neq y$.

The following result characterizes expansive homeomorphisms in terms of Lyapunov functions.

Theorem [11, Theorem 3.2]. *Let f be a homeomorphism of a compact manifold M. The following conditions are equivalent:*

i) f is expansive;
ii) there exists a Lyapunov function for f.

The proof of this result for diffeomorphisms f can be found in [6]; see Section 4 and Lemma 3.3 of that paper. Additional arguments required for the case of a homeomorphism are discussed in [7, Section 1]. See also [11], where J. Tolosa, basing on the results of J. Lewowicz, characterized the expansivity on metric spaces with the using Lyapunov functions.

In this paper we use a generalized notion of cone-fields on metric space to describe uniform expansivity. The notions of cone-fields and cone condition [3, 10] originally appeared in the late 60's in the works of Alekseev, Anosov, Moser and Sinai. Recently, Sheldon Newhouse [10] obtained new conditions for dominated and hyperbolic splittings on compact invariant sets with the use of cone-fields. It is also worth mentioning that the notion of cone-field can be very useful in the study of hyperbolicity [1, 2, 3, 10].

Let us briefly describe the contents of this paper. In Section 2 we discuss the notion of uniform expansivity. We show that if f is uniformly expansive then it is also expansive. In Section 3 we recall our generalization of cone-fields to metric spaces which we presented in paper [4] and show that the existence of hyperbolic cone fields guarantees uniform expansivity. In Section 4 we show how to construct functions c_s, c_u for a uniformly expansive f such that f is cone-hyperbolic with respect to the cone-field (c_s, c_u). The main result of the section can be summarized as follows:

(Theorem 4.1) *Let X be a metric space and let $f : X \rightharpoonup X$ be an L-bilipschitz map. Assume that $\Lambda \subset X$ is an invariant set for f such that f is uniformly expansive on Λ.*

Then there exists a cone-field on Λ such that f is cone-hyperbolic on Λ.

2. Uniform Expansivity

First we define uniform expansivity of f and show that this notion is stronger than the classical expansivity.

By $\mathrm{dom}(f)$ we denote the domain of a partial map $f : X \rightharpoonup Y$, and by $\mathrm{im}(f)$ we denote its inverse image. For a given $f : X \rightharpoonup X$ we say that a sequence $\mathrm{x}\colon I \to X$ defined on a subinterval[1] I of $\mathbb{Z}$ is an *orbit of f* if

$$\mathrm{x}_n \in \mathrm{dom}(f) \ \text{ and } \ \mathrm{x}_{n+1} = f(\mathrm{x}_n) \ \text{ for } \ n \in I \text{ such that } \ n+1 \in I.$$

We recall the classical definition of expansivity. We say that $f\colon X \rightharpoonup X$ is *expansive* on $\Lambda \subset X$ if there exists an $\varepsilon > 0$ such that for any two orbits $\mathrm{x}\colon \mathbb{Z} \to \Lambda$, $\mathrm{v}\colon \mathbb{Z} \to X$ if $\sup\limits_{n\in\mathbb{Z}} d(\mathrm{x}_n, \mathrm{v}_n) \le \varepsilon$ then $\mathrm{x} = \mathrm{v}$.

Definition 2.1. Let $N \in \mathbb{N}$, $\varepsilon > 0$ and $\alpha \in (0,1)$ be given. We say that $f\colon X \rightharpoonup X$ is *(N, ε, α)-uniformly expansive on a set* $\Lambda \subset X$ if for any two orbits $\mathrm{x}\colon \{-N, \dots, N\} \to \Lambda$, $\mathrm{v}\colon \{-N, \dots, N\} \to X$ we have

$$d_{\sup}(\mathrm{x}, \mathrm{v}) \le \varepsilon \implies d(\mathrm{x}_0, \mathrm{v}_0) \le \alpha d_{\sup}(\mathrm{x}, \mathrm{v}),$$

where

$$d_{\sup}(\mathrm{x}, \mathrm{v}) := \sup_{-N \le n \le N} d(\mathrm{x}_n, \mathrm{v}_n).$$

[1] We say the I is a subinterval of $\mathbb{Z}$ if $[k, l] \cap \mathbb{Z} \subset I$ for any $k, l \in I$.

This notion is more useful because it does not need an infinite trajectory.

One can easily verify that uniform expansivity implies classical expansivity (this result can also be easily deduced from Theorem 2.1 below).

Observation 2.1 ([4, Observation 4.1])**.** *Let $N \in \mathbb{N}$, $\varepsilon > 0$, $\alpha \in (0,1)$, $\Lambda \subset X$ and $f\colon X \rightharpoonup X$ be given. If f is (N,ε,α)-uniformly expansive on Λ, then it is also expansive on Λ.*

Given $L \geq 1$ and $f\colon X \rightharpoonup Y$ we call f *L-bilipschitz* if

$$L^{-1}d(x,y) \leq d(f(x),f(y)) \leq Ld(x,y) \quad \text{for} \quad x,y \in \mathrm{dom} f. \tag{2.1}$$

Note that if a function f is L-bilipschitz then it is injective.

For $\delta > 0$ and a set $A \subset X$ we define the *δ-neighbourhood of A* as

$$A_\delta := \bigcup_{x \in A} B(x,\delta).$$

Let an injective map $f\colon X \rightharpoonup X$ be given. We call $A \subset \mathrm{dom}(f)$ an *invariant set for f* if $f(x)$ and $f^{-1}(x) \in A$ for every $x \in A$.

Now we show how to change the metric so that the function f which is $(N,\cdot,\cdot)$-uniformly expansive becomes $(1,\cdot,\cdot)$-uniformly expansive.

Theorem 2.1. *Let $f\colon X \rightharpoonup X$ be an L-bilipschitz map for some $L > 1$. Let $\Lambda \subset X$ and $\delta > 0$ be such that $\Lambda_\delta \subset \mathrm{dom}(f) \cap \mathrm{im}(f)$. We assume that Λ is an invariant set for f and that f is (N,δ,α)-uniformly expansive on Λ.*

Then there exists a metric ρ on $\Lambda_{\delta L^{-N+1}}$ such that

$$d(x,v) \leq \rho(x,v) \leq L^{N-1}d(x,v) \quad \textit{for} \quad x,v \in \Lambda_{\delta L^{-N+1}}, \tag{2.2}$$

that f is $(1, \delta L^{-N+1}, \sqrt[N]{\alpha})$-uniformly expansive on $\Lambda_{\delta L^{-N+1}}$ and $\max\{\alpha^{-1/N}, L\}$-bilipschitz map with respect to the metric ρ.

Proof. Let $\beta = \sqrt[N]{\alpha}$. We put

$$\rho(x,v) := \max_{k \in \{-N+1,\ldots,N-1\}} \beta^{|k|} d(f^k(x), f^k(v)) \quad \text{for} \quad x,v \in \Lambda_{\delta L^{-N+1}}.$$

Inequalities (2.2) follow from the definition and (2.1). Note that for $k \in \{-N+1,\ldots,N-1\}$ we have

$$x,v \in \Lambda_{\delta L^{-N+1}} \Longrightarrow f^k(x), f^k(v) \in \Lambda_{\delta L^{-N+1+|k|}}.$$

This means that ρ is well defined on $\Lambda_{\delta L^{-N+1}}$.

First we show that f is $\max\{\beta^{-1}, L\}$-bilipschitz map with respect to the metric ρ. Since f is L-bilipschitz in the metric d, we know that $d(f^N(x), f^N(v)) \leq Ld(f^{N-1}(x), f^{N-1}(v))$

and finally we get

$$\begin{aligned}
\rho(f(x), f(v)) &= \max_{k\in\{-N+1,\dots,N-1\}} \beta^{|k|} d(f^k(f(x)), f^k(f(v))) \\
&= \max\{\beta^{|-N+1|} d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta^{N-1} d(f^N(x), f^N(v))\} \\
&= \max\{\beta^{|-N+1|}\beta^{-1}\beta d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta^1\beta^{-1}\beta d(x,v), \\
&\qquad \beta^0\beta\beta^{-1} d(f(x), f(v)), \dots, \beta^{N-2}\beta\beta^{-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&= \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta\beta^0 d(x,v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \dots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&\le \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta\beta^0 d(x,v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \dots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} L d(f^{N-1}(x), f^{N-1}(v))\} \\
&\le \max\{\beta, \beta^{-1}, L\}\cdot\rho(x,v) = \max\{\beta^{-1}, L\}\cdot\rho(x,v).
\end{aligned}$$

Similarly, as for the opposite inequality, we know that $L^{-1}d(f^{N-1}(x), f^{N-1}(v)) \le d(f^N(x), f^N(v))$ and $L^{-1}d(f^{-N+1}(x), f^{-N+1}(v)) \le d(f^{-N+2}(x), f^{-N+2}(v))$. Hence

$$\begin{aligned}
\rho(f(x), f(v)) &= \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta\beta^0 d(x,v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \dots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} d(f^N(x), f^N(v))\} \\
&\ge \max\{\beta\beta^{|-N+2|} d(f^{-N+2}(x), f^{-N+2}(v)), \dots, \beta\beta^0 d(x,v), \\
&\qquad \beta^{-1}\beta^1 d(f(x), f(v)), \dots, \beta^{-1}\beta^{N-1} d(f^{N-1}(x), f^{N-1}(v)), \\
&\qquad \beta^{N-1} L^{-1} d(f^{N-1}(x), f^{N-1}(v))\} \\
&\ge \min\{\beta, L^{-1}\}\cdot\rho(x,v).
\end{aligned}$$

Now we show that for $x \in \Lambda$ and $v \in \Lambda_{\delta L^{-N+1}}$ such that

$$\max\left\{\rho(f^{-1}(x), f^{-1}(v)), \rho(x,v), \rho(f(x), f(v))\right\} \le \delta L^{-N+1} \tag{2.3}$$

the following inequality holds:

$$\rho(x,v) \le \beta\max(\rho(f(x), f(v)), \rho(f^{-1}(x), f^{-1}(v))).$$

We have to show that for $k = -N+1, \dots, N-1$

$$\beta^{|k|} d(f^k(x), f^k(v)) \le$$
$$\beta\max\left(\max_{k=-N+1,\dots,N-1} \beta^{|k|} d(f^{k+1}(x), f^{k+1}(v)), \max_{k=-N+1,\dots,N-1} \beta^{|k|} d(f^{k-1}(x), f^{k-1}(v))\right).$$

For $k < 0$ or $k > 0$ it is straightforward. Consider the case $k = 0$. From (2.2) and (2.3) we get

$$\max\left\{d(f^{-1}(x), f^{-1}(v)), d(x,v), d(f(x), f(v))\right\} \le \delta L^{-N+1},$$

which together with (2.1) implies that $d(f^k(x), f^k(v)) \leq \delta$ for $k = -N, \ldots, N$. By the uniform expansivity and the fact that $\beta < 1$ we get

$$d(x, v) \leq \alpha \max_{|k|\leq N} d(f^k(x), f^k(v)) \leq \beta \max_{|k|\leq N} (\beta^{N-1} d(f^k(x), f^k(v)))$$

$$\leq \beta \max \left(\max_{|k|\leq N-1} \beta^{|k|} d(f^{k+1}(x), f^{k+1}(v)), \max_{|k|\leq N-1} \beta^{|k|} d(f^{k-1}(x), f^{k-1}(v)) \right).$$

□

3. Cone-fields and Cone-hyperbolic Maps

In this section, for the convenience of the reader, we recall basic definitions concerning generalization of cone-fields to metric spaces (for more information and motivation see [4, 5]).

Definition 3.1 ([4, Definition 3.1]). Let $\delta > 0$ and $\Lambda \subset X$ be nonempty. We say that a pair of functions $c_s, c_u \colon U \to \mathbb{R}_+$ for some $U \subset X \times X$ forms a δ*-cone-field on* Λ if

$$\{x\} \times B(x, \delta) \subset U \quad \text{for } x \in \Lambda.$$

We put $c(x, v) := \max\{c_s(x, v), c_u(x, v)\}$. If there exists $K > 0$ such that:

$$\frac{1}{K} d(x, v) \leq c(x, v) \leq K d(x, v) \text{ for } (x, v) \in U$$

then we call it (K, δ)*-cone-field on* Λ or *uniform* δ*-cone-field on* Λ.

For each point $x \in \Lambda$ we introduce *unstable* and *stable cones* by the formula

$$C_x^u(\delta) := \{v \in B(x, \delta) \,:\, c_s(x, v) \leq c_u(x, v)\},$$
$$C_x^s(\delta) := \{v \in B(x, \delta) \,:\, c_s(x, v) \geq c_u(x, v)\}.$$

We consider a partial map $f \colon X \rightharpoonup Y$ between metric spaces X and Y and $\Lambda \subset \mathrm{dom}(f)$. Assume that X is equipped with a uniform δ-cone-field on Λ and Y is equipped with a uniform δ-cone-field on a subset Z of Y such that $f(\Lambda) \subset Z$.

For every $x \in \mathrm{dom}(f)$ we put

$$B_f(x, \delta) := \{v \in B(x, \delta) \cap \mathrm{dom}(f) : f(v) \in B(f(x), \delta)\}.$$

Now we define $u_x(f; \delta)$ and $s_x(f; \delta)$, the expansion and the contraction rates of f, respectively. These rates are a modification of the classical definition from [10], but we do not assume that the function f is invertible (for more information see [4]).

Definition 3.2 ([4, Definition 3.2]). Let $x \in \mathrm{dom}(f)$ and $\delta > 0$ be given. We define

$$u_x(f; \delta) := \sup\{R \in [0, \infty] \mid c(f(x), f(v)) \geq Rc(x, v), v \in B_f(x, \delta); v \in C_x^u(\delta)\},$$
$$s_x(f; \delta) := \inf\{R \in [0, \infty] \mid c(f(x), f(v)) \leq Rc(x, v), v \in B_f(x, \delta); f(v) \in C_{f(x)}^s(\delta)\}.$$

Let $u_\Lambda(f; \delta) := \inf_{x\in\Lambda}\{u_x(f; \delta)\}$ and $s_\Lambda(f; \delta) := \sup_{x\in\Lambda}\{s_x(f; \delta)\}$.

Definition 3.3. We say that f is δ*-cone-hyperbolic* on Λ if

$$s_\Lambda(f; \delta) < 1 < u_\Lambda(f; \delta).$$

The next proposition is a simple analogue of [10, Lemma 1.1].

Proposition 3.1 ([4, Proposition 3.1]). *Every δ-cone-hyperbolic is δ-cone-invariant, i.e. for $x \in \Lambda$ and $v \in B_f(x,\delta)$ we have*

$$v \in C^u_x(\delta) \implies f(v) \in C^u_{f(x)}(\delta),$$

and

$$f(v) \in C^s_{f(x)}(\delta) \implies v \in C^s_x(\delta).$$

Since the article [4] is a preprint, we present the proof for completeness.

Proof. To prove the first implication, suppose that there exist $x \in \Lambda$ and $v \in C^u_x(\delta)$ such that $f(v) \notin C^u_{f(x)}(\delta)$. This implies that $f(x) \neq f(v)$ and therefore $c(x,v) > 0$. We also know that $f(v) \in C^s_{f(x)}(\delta)$. From Definition 3.2 we obtain

$$c(f(x), f(v)) \leq s_x(f;\delta)c(x,v) \leq s_\Lambda(f;\delta)c(x,v),$$

but on the other hand

$$c(f(x), f(v)) \geq u_x(f;\delta)c(x,v) \geq u_\Lambda(f;\delta)c(x,v).$$

Thus $s_\Lambda(f;\delta) \geq u_\Lambda(f;\delta)$, which leads to contradiction.

The second implication can be proved similarly. □

Theorem 3.1 ([4, Theorem 4.1]). *Suppose that for $K > 0$ and $\delta > 0$ we are given a (K,δ)-cone-field on $\Lambda \subset X$. Let $f \colon \Lambda_\delta \rightharpoonup X$ be δ-cone-hyperbolic on Λ and let $\lambda > 1$ be chosen such that*

$$s_\Lambda(f;\delta) \leq \lambda^{-1}, \; u_\Lambda(f;\delta) \geq \lambda.$$

Then f is $(N, \delta, K^2/\lambda^N)$-uniformly expansive on Λ for every $N \in \mathbb{N}$, $N > 2\log_\lambda K$.

As was the case for the previous proposition, we present the proof for the sake of completeness.

Proof. From Proposition 3.1 we know that f is δ-cone-invariant. Let us take two orbits $\mathrm{x}\colon \{-N,\dots,N\} \to \Lambda$, $\mathrm{v}\colon \{-N,\dots,N\} \to X$ such that

$$d_{\sup}(\mathrm{x},\mathrm{v}) \leq \delta.$$

Since $\mathrm{v}_0 \in B(\mathrm{x}_0,\delta) = C^s_{\mathrm{x}_0}(\delta) \cup C^u_{\mathrm{x}_0}(\delta)$, it is enough to consider two cases.

Let $\mathrm{v}_0 \in C^s_{\mathrm{x}_0}(\delta)$. From the cone-invariance we know that $\mathrm{v}_n \in C^s_{\mathrm{x}_n}(\delta)$ for $n < 0$. From Definition 3.2 we get $c(\mathrm{x}_0,\mathrm{v}_0) \leq \lambda^{-1}c(\mathrm{x}_{-1},\mathrm{v}_{-1}) \leq \dots \leq \lambda^{-N}c(\mathrm{x}_{-N},\mathrm{v}_{-N})$. Finally

$$d(\mathrm{x}_0,\mathrm{v}_0) \leq Kc(\mathrm{x}_0,\mathrm{v}_0) \leq K^2\lambda^{-N}d_{\sup}(\mathrm{x},\mathrm{v}).$$

If $\mathrm{v}_0 \in C^u_{\mathrm{x}_0}(\delta)$, then from the cone-invariance we obtain $\mathrm{v}_n \in C^u_{\mathrm{x}_n}(\delta)$ for $n > 0$ and consequently

$$d(\mathrm{x}_0,\mathrm{v}_0) \leq Kc(\mathrm{x}_0,\mathrm{v}_0) \leq K\lambda^{-N}c(\mathrm{x}_N,\mathrm{v}_N) \leq K^2\lambda^{-N}d_{\sup}(\mathrm{x},\mathrm{v}).$$

□

4. Expansivity and Cone-fields

In this section we show that uniform expansiveness of f on an invariant set Λ lets us construct a cone-field on Λ such that f is cone-hyperbolic on Λ. In our reasoning we will need the notion of ε-quasiconvexity.

Definition 4.1. Let I be a subinterval of $\mathbb{Z}$, and let $\varepsilon \geq 0$ be fixed. We call a sequence $\alpha\colon I \to \mathbb{R}$ *ε-quasiconvex* if

$$\alpha_n \leq \max\{\alpha_{n-1}, \alpha_{n+1}\} - \varepsilon \text{ for } n \in I : n-1, n+1 \in I.$$

Now we show some properties of ε-quasiconvex sequences, which will be used later.

Observation 4.1. *Let $\varepsilon \geq 0$ and $\alpha\colon I \to \mathbb{R}$ be an ε-quasiconvex sequence. Then*

i) if m, $m+2 \in I$ and $\alpha_{m+1} > \alpha_m - \varepsilon$ then

$$\alpha_{n+1} \geq \alpha_n + \varepsilon \text{ for } n \geq m+1 \text{ such that } n, n+1 \in I. \tag{4.1}$$

ii) if $m-1$, $m+1 \in I$ and $\alpha_{m+1} < \alpha_m + \varepsilon$ then

$$\alpha_{n+1} \leq \alpha_n - \varepsilon \text{ for } n < m \text{ such that } n, n+1 \in I. \tag{4.2}$$

Proof. The above statements are similar so we show the first one. The proof proceeds on induction. Suppose that $m, m+2 \in I$ and $\alpha_{m+1} > \alpha_m - \varepsilon$. Since α is ε-quasiconvex,

$$\alpha_{m+1} \leq \max\{\alpha_m, \alpha_{m+2}\} - \varepsilon = \max\{\alpha_m - \varepsilon, \alpha_{m+2} - \varepsilon\}.$$

But $\alpha_{m+1} > \alpha_m - \varepsilon$, so we get

$$\alpha_{m+1} \leq \alpha_{m+2} - \varepsilon,$$

and hence

$$\alpha_{m+2} \geq \alpha_{m+1} + \varepsilon.$$

It implies that (4.1) is valid for $n = m+1$. Suppose now that (4.1) holds for some $n \geq m+1$, i.e. that $n; n+1 \in I$ and $\alpha_{n+1} \geq \alpha_n + \varepsilon$. Assume additionally that $n+2 \in I$. Then we get

$$\alpha_{n+1} \leq \alpha_{n+2} - \varepsilon,$$

thus

$$\alpha_{n+2} \geq \alpha_{n+1} + \varepsilon,$$

which completes the proof. □

The following proposition will be a basic tool in the proof of our main result, Theorem 4.1.

Proposition 4.1. *Let $\varepsilon > 0, L > 1$, $\beta \in (0,1)$ and let (Y, ρ) be a metric space. Let $\Lambda \subset Y$ be given and $f\colon Y \rightharpoonup Y$ be an L-bilipschitz map such that $\Lambda_\varepsilon \subset \mathrm{dom}(f) \cap \mathrm{im}(f)$. Assume that Λ is an invariant set for f and that f is $(1, \varepsilon, \beta)$-uniformly expansive on Λ. Then*

$$\begin{aligned} c_s(x,v) &:= \inf\{\rho(f^k(x), f^k(v)) \,|\, k \in (-\infty, 0) \cap \mathbb{Z} : f^l(v) \in B(f^l(x), \varepsilon) \\ &\qquad\qquad \textit{for } l \in [k, 0] \cap \mathbb{Z}\}, \\ c_u(x,v) &:= \inf\{\rho(f^k(x), f^k(v)) \,|\, k \in [0, \infty) \cap \mathbb{Z} : f^l(v) \in B(f^l(x), \varepsilon) \\ &\qquad\qquad \textit{for } l \in [0, k] \cap \mathbb{Z}\}, \end{aligned} \tag{4.3}$$

define an $(L,\varepsilon/L)$ cone-field on Λ. Moreover, f is cone-hyperbolic on Λ and

$$s_\Lambda(f;\varepsilon/L) \le \beta < \frac{1}{\beta} \le u_\Lambda(f;\varepsilon/L). \tag{4.4}$$

Proof. First we show that $c_s(x,v)$ and $c_u(x,v)$ defined above are $(L,\varepsilon/L)$ cone-field on Λ, i.e.

$$\frac{1}{L}\rho(x,v) \le c(x,v) \le L\rho(x,v) \text{ for } (x,v) \in \big\{(x,v) : x \in \Lambda, v \in B(x,\varepsilon/L)\big\},$$

where $c(x,v) := \max\{c_s(x,v), c_u(x,v)\}$.

Choose an arbitrary point $x \in \Lambda$ and $v \in B(x,\varepsilon/L)$. We can assume that $x \neq v$, because the case $x = v$ is trivial ($c_s(x,v) = c_u(x,v) = 0 = \rho(x,v)$).

Let I be the biggest subinterval of $\mathbb{Z}$ containing 0 such that

$$\sup\{\rho(f^n(x), f^n(v)) : n \in I\} \le \varepsilon. \tag{4.5}$$

Since f is L-bilipschitz, we know that $f^{-1}(v) \in B(f^{-1}(x),\varepsilon)$, and therefore $\{-1,0\} \subset I$. This yields $c(x,v) < \infty$.

Now we define a sequence $\{a_n\}_{n\in I} \subset \mathbb{R}$ by the formula

$$a_n := \ln \rho(f^n(x), f^n(v)) \text{ for } n \in I. \tag{4.6}$$

Observe that a_n is well-defined because $\rho(f^n(x), f^n(v)) > 0$ for all $n \in I$.

Let

$$I_- := \{n \in I : n < 0\} \text{ and } I_+ := \{n \in I : n \ge 0\}.$$

We have the following relations:

$$c_s(x,v) = \exp\left(\inf_{n\in I_-}\{a_n\}\right) \text{ and } c_u(x,v) = \exp\left(\inf_{n\in I_+}\{a_n\}\right),$$

where we use the convention $\exp(-\infty) = 0$.

We show that the sequence $\{a_n\}$ is $\ln(1/\beta)$-quasiconvex. Let $n \in I$ be such that $n-1$, $n+1 \in I$. By (4.5) we observe that

$$\max\{\rho(f^{n-1}(x), f^{n-1}(v)), \rho(f^n(x), f^n(v)), \rho(f^{n+1}(x), f^{n+1}(v))\} \le \varepsilon.$$

Consequently, by $(1,\varepsilon,\beta)$-uniform expansivity of f we get

$$\rho(f^n(x), f^n(v)) \le \beta \max\{\rho(f^{n-1}(x), f^{n-1}(v)), \rho(f^{n+1}(x), f^{n+1}(v))\},$$

which implies that $a_n \le \max\{a_{n-1}, a_{n+1}\} - \ln(1/\beta)$.

Now we consider two cases. If $a_{-1} \le a_0$ then by Observation 4.1 i) we get

$$a_{n+1} \ge a_n + \ln\frac{1}{\beta} \text{ for } n \ge 0, n \in I,$$

which yields

$$\inf_{n\in I_-}\{a_n\} \le a_{-1} \le a_0 = \inf_{n\in I_+}\{a_n\},$$

Hence

$$c_s(x,v) \le c_u(x,v) = c(x,v) = e^{a_0} = \rho(x,v).$$

On the other hand if $a_{-1} \ge a_0$ then by Observation 4.1 ii) we get

$$a_{n+1} \le a_n - \ln\frac{1}{\beta} \text{ for } n < -1, n \in I.$$

Therefore
$$\inf_{n\in I_-}\{a_n\} = a_{-1} \geq a_0 \geq \inf_{n\in I_+}\{a_n\},$$
and consequently
$$c_u(x,v) \leq c_s(x,v) = c(x,v) = e^{a_{-1}} = \rho(f^{-1}(x), f^{-1}(v)).$$
Since f is L-bilipschitz, we get that c_s, c_u define an $(L, \varepsilon/L)$ cone-field on Λ.

Now we check that f is cone-hyperbolic on Λ. Let us take $x \in \Lambda$ and $v \in B_f(x, \varepsilon/L)$ such that $f(v) \in C^s_{f(x)}(\varepsilon/L)$. We define the sequence $\{a_n\}_{n\in I}$ as in (4.6).

We show that $a_0 \geq a_1$. Suppose that, on the contrary, $a_0 < a_1$. By Observation 4.1 i) we get
$$a_{n+1} \geq a_n \ \text{ for } \ n \geq 1, n \in I.$$
Hence
$$\ln(c_u(f(x), f(v))) = \inf_{n\geq 1, n\in I}\{a_n\} = a_1 > a_0 \geq \inf_{n<1, n\in I}\{a_n\} = \ln(c_s(f(x), f(v))),$$
which is a contradiction with $f(v) \in C^s_{f(x)}(\varepsilon/L)$. So we have $a_1 \leq a_0$. By the Observation 4.1 ii) we get
$$a_{n+1} \leq a_n - \ln(1/\beta) \text{ for } n < 0 \text{ such that } n, n+1 \in I.$$
In particular,
$$a_0 \leq a_{-1} - \ln(1/\beta). \tag{4.7}$$
Consequently,
$$\begin{aligned} c_u(f(x), f(v)) &= \exp\left(\inf_{n\geq 1, n\in I}\{a_n\}\right) \leq \exp(a_1) \leq \exp(a_0) \\ &= \exp\left(\inf_{n<1, n\in I}\{a_n\}\right) = c_s(f(x), f(v)) = \underline{c(f(x), f(v))} \\ &\overset{(4.7)}{\leq} \beta \exp(a_{-1}) = \beta \exp\left(\inf_{n\in I_-}\{a_n\}\right) \leq \underline{\beta c(x,v)}. \end{aligned}$$
Therefore
$$s_\Lambda(f; \varepsilon/L) = \sup_{x\in\Lambda}\{s_x(f; \varepsilon/L)\} \leq \beta < 1.$$

Now we consider an $x \in \Lambda$ and $v \in B_f(x, \varepsilon/L)$ such that $v \in C^u_x(\varepsilon/L)$. We show that $a_0 \geq a_{-1}$. Suppose the contrary, $a_0 < a_{-1}$. By Observation 4.1 ii) we get
$$a_{n+1} \geq a_n \ \text{ for } \ n < -1, n \in I.$$
Hence
$$\inf_{n\in I_-}\{a_n\} = a_{-1} > a_0 \geq \inf_{n\in I_+}\{a_n\},$$
which is contradiction with $v \in C^u_x(\varepsilon/L)$. So we have $a_0 \geq a_{-1}$. By the Observation 4.1 i) we get
$$a_{n+1} \geq a_n + \ln(1/\beta) \text{ for } n \geq 0 \text{ such that } n, n+1 \in I.$$
In particular,
$$a_1 \geq a_0 + \ln(1/\beta). \tag{4.8}$$

Finally

$$c_s(f(x),f(v)) = \exp\left(\inf_{n<1, n\in I}\{a_n\}\right) \le \exp(a_0) \le \exp(a_1)$$
$$= \exp\left(\inf_{n\ge 1, n\in I}\{a_n\}\right) = c_u(f(x),f(v)) = c(f(x),f(v)),$$

which yields

$$\exp(a_1) = \underline{c(f(x),f(v))} \overset{(4.8)}{\ge} \frac{1}{\beta}\exp(a_0) = \frac{1}{\beta}\exp\left(\inf_{n\in I_+}\{a_n\}\right) = \underline{\frac{1}{\beta}c(x,v)}.$$

This shows that

$$u_\Lambda(f;\varepsilon/L) = \inf_{x\in\Lambda}\{u_x(f;\varepsilon/L)\} \ge \frac{1}{\beta} > 1.$$

Therefore f is cone-hyperbolic on Λ. □

As a consequence of earlier results we obtain the following theorem.

Theorem 4.1. *Let $\varepsilon > 0, L > 1, N \in \mathbb{N}$, $\alpha \in (0,1)$ be fixed. Let (X,d) be a metric space and $\Lambda \subset X$ be given. Let $f\colon X \rightharpoonup X$ be an L-bilipschitz map such that $\Lambda_\varepsilon \subset \mathrm{dom}(f) \cap \mathrm{im}(f)$. Assume that Λ is an invariant set for f and that f is (N,ε,α)-uniformly expansive on Λ.*

Then there exists an $(\max\{\alpha^{-1/N}L^{N-1}, L^N\}, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}\})$ cone-field on Λ such that f is cone-hyperbolic on Λ and

$$s_\Lambda(f, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}) \le \sqrt[N]{\alpha} < \frac{1}{\sqrt[N]{\alpha}} \le u_\Lambda(f, \min\{\varepsilon L^{-2N+1}\sqrt[N]{\alpha}, \varepsilon L^{-2N}).$$

Proof. We will apply Proposition 4.1. By applying Theorem 2.1 (for $\delta = \varepsilon$) we obtain the metric ρ which is equivalent to d on $U = \{x : d(x,\Lambda) < \varepsilon L^{-N+1}\}$ and such that

i) $d(x,v) \le \rho(x,v) \le L^{N-1}d(x,v)$ for $x, v \in U$,
ii) f is $(1, \varepsilon L^{-N+1}, \sqrt[N]{\alpha})$-uniformly expansive on U with respect to the metric ρ,
iii) f is $\max\{\alpha^{-1/N}, L\}$-bilipschitz map on U with respect to the metric ρ.

Let $\widetilde{Y} = \{y : d(y,\Lambda) < L^{-N+1}\varepsilon\}$ and $\widetilde{L} = \max\{\alpha^{-1/N}, L\}$. We use Proposition 4.1 (for $\widetilde{\varepsilon} = \varepsilon L^{-N}$, $\widetilde{L}$, $\widetilde{\beta} = \sqrt[N]{\alpha}$, $\widetilde{f} = f|_{\{x:\, d(x,\Lambda)<\varepsilon L^{-N}\}}$) and construct functions $\widetilde{c_s}$, $\widetilde{c_u}$ which define an $(\widetilde{L},\widetilde{\delta})$ cone-field on U such that $\widetilde{f}$ is $\widetilde{\delta}$-cone-hyperbolic with respect to the metric ρ, where $\widetilde{\delta} = \varepsilon L^{-N}/\widetilde{L}$.

Now we need to "translate" the results from the metric ρ to the original metric d. For clarity of notation we use the subscript $(.)_d$ to denote objects with respect to the metric d and $(.)_\rho$ to denote objects with respect to the metric ρ.

By the definition of $(\widetilde{L},\widetilde{\delta})$ cone-field on U and i) we get

$$\frac{1}{\widetilde{L}L^{N-1}}d(x,v) \le \frac{1}{\widetilde{L}}d(x,v) \le \frac{1}{\widetilde{L}}\rho(x,v) \le c(x,v) \le \widetilde{L}\rho(x,y)$$
$$\le \widetilde{L}L^{N-1}d(x,y) \text{ for } (x,v) \in \{x \in U, v \in B(x,\widetilde{\delta})_\rho\}.$$

From i) we have

$$B(x,\widetilde{\delta}/L^{N-1})_d \subset B(x,\widetilde{\delta})_\rho, \quad B_f(x,\widetilde{\delta}/L^{N-1})_d \subset B_f(x,\widetilde{\delta})_\rho,$$

and

$$C^u_x(\widetilde{\delta}/L^{N-1})_d \subset C^u_x(\widetilde{\delta})_\rho, \quad C^s_x(\widetilde{\delta}/L^{N-1})_d \subset C^s_x(\widetilde{\delta})_\rho.$$

Consequently, from Definition 3.2 for an arbitrary $x \in U$ we get

$$u_x(f;\widetilde{\delta})_\rho \leq u_x(f;\widetilde{\delta}/L^{N-1})_d, \quad s_x(f;\widetilde{\delta})_\rho \geq s_x(f;\widetilde{\delta}/L^{N-1})_d.$$

Hence

$$u_U(f;\widetilde{\delta})_\rho \leq u_U(f;\widetilde{\delta}/L^{N-1})_d, \quad s_U(f;\widetilde{\delta})_\rho \geq s_U(f;\widetilde{\delta}/L^{N-1})_d.$$

From the above inequalities and (4.4)

$$s_U(f;\widetilde{\delta})_\rho \leq \widetilde{\beta} < 1 < \frac{1}{\widetilde{\beta}} \leq u_U(f;\widetilde{\delta})_\rho,$$

we obtain that f is $(\widetilde{\delta}/L^{N-1})$-cone-hyperbolic in metric d and

$$s_U(f;\widetilde{\delta}/L^{N-1})_d \leq \widetilde{\beta} < 1 < \frac{1}{\widetilde{\beta}} \leq u_U(f;\widetilde{\delta}/L^{N-1})_d.$$

Finally we conclude that $\widetilde{c_s}$ and $\widetilde{c_u}$ are $(\max\{\alpha^{-1/N}L^{N-1}, L^N\}, \widetilde{\delta}/L^{N-1})$-cone-field on Λ such that f is $(\widetilde{\delta}/L^{N-1})$-cone-hyperbolic on Λ with respect to metric d. □

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