Quasianalytic structures revisited: quantifier elimination, valuation property and rectilinearization of functions

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IMUJ PREPRINT 2012/02

This paper continues our previous article devoted to quantifier elimination and the valuation property for the expansion of the real field by restricted quasianalytic functions. A basic tool developed there was the concept of active and non-active infinitesimals, whose study relied on transformation to normal crossings by blowing up, and the technique of special cubes and modifications, introduced in our earlier papers. However, the theorem on an active infinitesimal, being one of the crucial results, was proved not in full generality (covering, nevertheless, the classical case of analytic functions). The main purpose of this paper is to provide a proof of the general quasianalytic case and, consequently, to legitimize the results of our previous article. Also given is yet another approach to quantifier elimination and a description of definable functions by terms (in the language augmented by the names of rational powers), which is much shorter and more natural with regard to the techniques applied. Finally, we present some theorems on the rectilinearization of definable functions, which are counterparts of those from our paper about functions definable by a Weierstrass system.

2010 Mathematics Subject Classification: Primary 03C10, 14P15, 32S45; Secondary 03C64, 26E10, 32B20.

Key words: quasianalytic structures, quantifier elimination, active infinitesimals, special cubes and modifications, valuation property, rectilinearization of quasi-subanalytic functions.

1. Introduction. This paper is a continuation of our article [10], devoted to quasianalytic structures, i.e. the expansions $\mathcal{R}_Q$ of the real field by restricted quasianalytic functions. That article was intended to establish quantifier elimination and a description of definable functions by terms in the language $\mathcal{L}$ augmented by the names of rational powers, and the valuation property. It was achieved by a study of non-standard models $\mathcal{R}$ of the universal diagram T of the structure $\mathcal{R}_Q$. Our research made no appeal to the Weierstrass preparation theorem, because it is not at our disposal in quasianalytic geometry. Instead, the fundamental tools applied there were the concepts of active and non-active infinitesimals (*op.cit.*) and of special cubes and modifications (developed in [9, 10]). These techniques relied, to a great extent, on the method of transformation to normal crossings by blowing up. We often made use of the following sharper version of this method: a finite number of Q-analytic function germs $f_1(x), \dots, f_N(x)$ can be simultaneously transformed to normal crossings by a finite sequence φ of blowings-up so that

$$f_n^{\varphi}(x') = x'^{\beta_n}\, v_n(x'),\ v_n(0) \neq 0 \quad \text{for} \quad n = 1, \dots, N,$$

and the monomials $x'^{\beta_1}, \dots, x'^{\beta_N}$ are totally ordered by the divisibility relation. This idea, being frequently used by mathematicians (see e.g. [1, 3]), goes back at least as far as Zariski's paper [15], § 2.

However, the theorem on an active infinitesimal (Theorem 4.4 from [10]), being one of the crucial results, was proven not in full generality (covering, nevertheless, the classical case of analytic functions). The main purpose of this paper is to provide a proof of the general quasianalytic case and, consequently, to legitimize the results of that article. In the analytic case, we established the theorem by means of diagonal series, which are apparently unavailable in quasianalytic function theory. The general proof, given here in Section 2, consists in an induction procedure which embraces the theorem itself and several other conclusions concerning the infinitesimals under consideration, as — in particular — the valuation property and exchange property for $\mathcal{L}$-terms.

We should emphasize that the valuation property is valid in the case of arbitrary, polynomially bounded, o-minimal structures, as proven in a different way by van den Dries–Speissegger [6]. From this property, it can be drawn, through model-theoretic compactness (cf. [5, 8], the preparation theorem in the sense of Parusiński–Lion–Rolin (cf. [13, 7]).

Section 3 gives an approach to quantifier elimination and a description of definable functions by terms in the language augmented by the names of rational powers, which is much shorter and more natural (in comparison with the one in [10]) with regard to the techniques applied. Here we make use of immersion cubes (introduced in [9]) and their sections given piecewise by terms. In the last section, we present some theorems on the rectilinearization of $\mathcal{L}$-terms and definable functions, which are counterparts of those from our paper [11] about functions definable by a Weierstrass system.

2. Proof of the theorem on an active infinitesimal. We begin with some basic notions introduced in our article [10]. For a model $\mathcal{R}$ of the universal diagram T, v stands for the standard valuation on the field $\mathcal{R}$, i.e. the valuation induced by the convex hull of the real field $\mathbb{R}$ in $\mathcal{R}$. A sequence $\lambda = (\lambda_1, \dots, \lambda_m)$ of infinitesimals is regular with main part $\lambda_1, \dots, \lambda_k$, i.e. the valuations

$$v(\lambda_1), \dots, v(\lambda_k) \in \Gamma_{\langle \lambda_1, \dots, \lambda_m \rangle}$$

form a basis over $\mathbb{Q}$ of the valuation group $\Gamma_{\langle \lambda \rangle}$ of the structure $\langle \lambda \rangle$ generated by the λ's (*op.cit.*, Section 4).

We shall prove the theorem on an active infinitesimal by induction with respect to the number of infinitesimals λ. Actually, this theorem will be combined with several results from our article [10] in one induction procedure, presented below.

Theorem 1. $(\mathbf{I_n})$ (*op.cit.*, Theorem 4.4)

For any $m \leq n$, consider a regular sequence $\mu, \lambda_1, \dots, \lambda_m$ of infinitesimals with main part $\mu, \lambda_1, \dots, \lambda_k$ and an $\mathcal{L}$-term $t(y, x)$, $x = (x_1 \dots, x_m)$, such that

$$\nu := t(\mu, \lambda) \notin \langle \lambda \rangle$$

is an infinitesimal. If $v(\mu) \notin \Gamma_{\langle \lambda \rangle}$, then ν is active over the infinitesimals λ.

$(\mathbf{II_n})$ (*op.cit.*, Proposition 4.7)

For any $m \leq n$, consider a regular sequence $\lambda_1, \dots, \lambda_m$ of infinitesimals with main part $\lambda_1, \dots, \lambda_k$ and an infinitesimal μ with $v(\mu) \notin \Gamma_{\langle \lambda \rangle}$. Then $\dim \Gamma_{\langle \mu, \lambda \rangle} = k+1$ whence $\mu, \lambda_1, \dots, \lambda_m$ is a regular sequence of infinitesimals with main part $\mu, \lambda_1, \dots, \lambda_k$.

Remark 1. The proof of Proposition 4.7 (*loc.cit.*) applies Theorem 4.4 indicating the inference

$$(\mathbf{I_n}) \Longrightarrow (\mathbf{II_{n+1}}).$$

We recall the following two assertions about analytically independent infinitesimals $\mu, \lambda_1, \ldots, \lambda_m$.

$(\mathbf{III_n})$ Valuation Property for $\mathcal{L}$-terms (*op.cit.*, Corollary 4.8):

Whenever $m \leq n$, we have the following dichotomy: either
- *μ is non-active over λ, and then $\Gamma_{\langle\lambda,\mu\rangle} = \Gamma_{\langle\lambda\rangle}$; or*
- *μ is active over λ, and then $\dim \Gamma_{\langle\lambda,\mu\rangle} = \dim \Gamma_{\langle\lambda\rangle} + 1$.*

In the latter case, one can find an $\mathcal{L}$-term $t(x)$ such that

$$v(\mu - t(\lambda)) \notin \Gamma_{\langle\lambda\rangle} \quad \textit{and} \quad \Gamma_{\langle\lambda,\mu\rangle} = \Gamma_{\langle\lambda\rangle} \oplus \mathbb{Q} \cdot v(\mu - t(\lambda)).$$

$(\mathbf{IV_n})$ Exchange Property (*op.cit.*, Corollary 4.9):

Whenever $m \leq n$, if $\nu \in \langle\lambda,\mu\rangle$ and $\nu \notin \langle\lambda\rangle$, then $\mu \in \langle\lambda,\nu\rangle$.

The exchange property $(\mathbf{IV_n})$ allows one to define the concept of rank and basis for substructures generated by $\leq n+1$ infinitesimals (*op.cit.*, Section 5). We still need some results referring to this topic, recalled below. Consider two sequences of analytically independent infinitesimals $\lambda = (\lambda_1, \ldots, \lambda_m)$ and $\lambda' = (\lambda'_1, \ldots, \lambda'_m)$, and an infinitesimal μ.

$(\mathbf{V_n})$ On Analytically Independent Infinitesimals (*op.cit.*, Proposition 5.1 and Corollary 5.2):

Whenever $m \leq n$, if $\langle\lambda\rangle \subset \langle\lambda'\rangle$, then $\langle\lambda\rangle = \langle\lambda'\rangle$. The infinitesimals $(\lambda_1, \ldots, \lambda_m, \mu)$ are analytically independent iff $\mu \notin \langle\lambda\rangle$.

Remark 2. The proofs of the foregoing theorems (*op.cit.*) indicate the following inferences:

$$(\mathbf{II_n}) \Longrightarrow (\mathbf{III_n}) \quad \text{and} \quad (\mathbf{I_n}) \wedge (\mathbf{III_n}) \Longrightarrow (\mathbf{IV_{n+1}}) \Longrightarrow (\mathbf{V_{n+1}}).$$

Having disposed of this pattern of induction, we can readily turn to the proof of the theorem on an active infinitesimal.

PROOF. When $m = 0$, then $\langle\lambda\rangle = \langle\emptyset\rangle = \mathbb{R}$ and the conclusion is evident. So take $m > 0$ and assume the theorem holds provided that the number of infinitesimals λ is smaller than m.

In [10], we have reduced the problem to the case where

$$\nu := t(\mu, \lambda) = f(\mu, \lambda_1/\mu, \tilde{\lambda}),$$

where $f(u, v, \tilde{x})$ is a function Q-analytic at $0 \in \mathbb{R}^{m+1}$; here $\tilde{x} = (x_2, \ldots, x_m)$ and $\tilde{\lambda} = (\lambda_2 \ldots, \lambda_m)$.

The valuation group $\Gamma_{\langle \lambda, \mu \rangle}$ is a vector space over $\mathbb{Q}$ of dimension $\leq (m+1)$ (*op.cit.*, Corollary 3.4). It is a direct sum of finitely many archimedean subgroups

$$\Gamma_{\langle \lambda, \mu \rangle} = G_1 \oplus \ldots \oplus G_r \quad \text{with} \quad G_1^+ > \ldots > G_r^+,$$

where G_i^+ stands for the semigroup of all positive elements of G_i. It is well-known that every archimedean ordered abelian group is isomorphic to a subgroup of the ordered additive group $\mathbb{R}$ of real numbers.

Let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_p)$ denote those infinitesimals from among λ for which

$$v(\varepsilon_1), \ldots, v(\varepsilon_p) > G_2 \oplus \ldots \oplus G_r$$

and $\delta = (\delta_1, \ldots, \delta_q)$ the remaining λ's; obviously, $p+q = m$. The valued field $\langle \lambda, \mu \rangle$ can be completed with respect to the standard valuation v, and the completion has the same valuation group. Notice that the topology induced by v is metrizable with basis of zero neighbourhoods consisting of sets of the form

$$\{t(\mu, \delta, \varepsilon) : \; v(t(\mu, \delta, \varepsilon)) > \gamma\}, \quad \gamma \in G_1.$$

In the completion, one can deal with formal power series in the infinitesimals ε with Q-analytic coefficients taken on the infinitesimals δ.

We encounter two cases:

Case A, where λ_1 is one of the ε's, say $\lambda_1 = \varepsilon_1$;

or

Case B, where λ_1 is one of the δ's, say $\lambda_1 = \delta_1$.

CASE A. Consider the Taylor coefficients

$$\frac{1}{i!j!} \cdot \frac{\partial^{i+j} f}{\partial u^i \partial v^j}(0, 0, \tilde{x}) =: a_{ij}(\tilde{x}), \quad i, j \in \mathbb{N},$$

which are Q-analytic functions at zero. A crucial role is played by the following

Lemma 1. *Under the assumptions of Theorem 1, we must have*

$$\sum_{i=0}^{\infty} a_{i,i+s}(\tilde{\lambda}) \cdot \varepsilon_1^i \neq 0 \quad \textit{or} \quad \sum_{j=0}^{\infty} a_{j+s,j}(\tilde{\lambda}) \cdot \varepsilon_1^j \neq 0$$

for some $s \in \mathbb{N} \setminus \{0\}$.

Suppose Lemma 1 were false. Then

$$\sum_{i=0}^{\infty} a_{i,i+s}(\tilde{\lambda}) \cdot \varepsilon_1^i = 0 \quad \text{and} \quad \sum_{j=0}^{\infty} a_{j+s,j}(\tilde{\lambda}) \cdot \varepsilon_1^j = 0$$

for all $s \in \mathbb{N} \setminus \{0\}$.

But we can find a model of the universal diagram T with the infinitesimals λ and an infinitesimal μ^* such that they are analytically independent and

$$v(\mu^*), v(\varepsilon_1/\mu^*) > G_2 \oplus \ldots \oplus G_r.$$

Indeed, by Remarks 1 and 2, it follows from the induction hypothesis that the assertion $(\mathbf{V_m})$ holds. The infinitesimals λ and μ^* are therefore analytically independent iff $\mu^* \notin \langle \lambda \rangle$. Consequently, it suffices to find a model of the universal diagram T along with the diagram of the structure $\langle \lambda \rangle$ and the sentences of the form

$$c \neq t(\lambda) \quad \text{where} \quad t(x) \ \text{ are } \ \mathcal{L}\text{-terms}$$

and the sentence

$$\frac{1}{2}\sqrt{\varepsilon_1} < c < 2\sqrt{\varepsilon_1}\,;$$

here c denotes a new constant construed as μ^*. Its existence can be immediately deduced through model-theoretic compactness.

Under the conditions stated above, we have

$$t(\mu^*, \lambda) = f(\mu^*, \lambda_1/\mu^*, \tilde{\lambda}) = \sum_{i=0}^{\infty} a_{i,i}(\tilde{\lambda}) \cdot \varepsilon_1^i +$$

$$+ \sum_{s=1}^{\infty} (\varepsilon_1/\mu^*)^s \cdot \sum_{i=0}^{\infty} a_{i,i+s}(\tilde{\lambda}) \cdot \varepsilon_1^i + \sum_{s=1}^{\infty} (\mu^*)^s \cdot \sum_{j=0}^{\infty} a_{j+s,j}(\tilde{\lambda}) \cdot \varepsilon_1^i =$$

$$= \sum_{i=0}^{\infty} a_{i,i}(\tilde{\lambda}) \cdot \varepsilon_1^i,$$

and thus we get

$$\partial t/\partial u\,(\mu^*, \lambda) = 0.$$

Therefore the Q-analytic function

$$\partial t/\partial u\,(u, x) = \partial/\partial u\, f(u, x_1/u, \tilde{x})$$

vanishes on an open special cube in $\mathbb{R}_u \times \mathbb{R}_{x_1} \times \mathbb{R}_{\tilde{x}}^{m-1}$. By the identity principle for quasianalytic functions, this function vanishes identically, and thus

$$t(u, x) = f(u, x_1/u, \tilde{x}) = g(x),$$

where, for some $r > 0$ small enough, $g(x) := f(r, x_1/r, \tilde{x})$ is a Q-analytic function at $0 \in \mathbb{R}_x^m$. Hence

$$\nu := t(\mu, \lambda) = g(\lambda) \in \langle \lambda \rangle,$$

and this contradiction completes the proof of Lemma 1.

Since $\mu \cdot \varepsilon_1/\mu = \varepsilon_1$, we see that

$$v(\mu) \geq 1/2\, v(\varepsilon_1) \quad \text{or} \quad v(\varepsilon_1/\mu) \geq 1/2\, v(\varepsilon_1).$$

By symmetry, we may assume that the former condition holds. Then the series

$$\nu_0 := \sum_{i=j=0}^{\infty} a_{ij}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu}\right)^j = \sum_{i=0}^{\infty} a_{ii}(\tilde{\lambda}) \cdot \varepsilon_1^i,$$

$$\nu_+ := \sum_{i=0}^{\infty} \sum_{j<i} a_{ij}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu}\right)^j = \sum_{i=0}^{\infty} \sum_{j<i} a_{ij}(\tilde{\lambda}) \cdot \mu^{i-j} \cdot \varepsilon_1^j$$

and

$$\nu_- := \sum_{i=0}^{\infty} \left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, \frac{\varepsilon_1}{\mu}, \tilde{\lambda}) \cdot \mu^i - \sum_{j \leq i} a_{ij}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu}\right)^j \right) =$$

$$= \sum_{i=0}^{\infty} \left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, \frac{\varepsilon_1}{\mu}, \tilde{\lambda}) - \sum_{j \leq i} a_{ij}(\tilde{\lambda}) \cdot \left(\frac{\varepsilon_1}{\mu}\right)^j \right) \cdot \mu^i$$

are well defined elements of the completion of the valued field $\langle \lambda, \mu \rangle$ with respect to the standard valuation v. We have $\nu = \nu_+ + \nu_0 + \nu_-$. Observe that the Q-analytic function which occurs in the i-th summand of the last series is of the form

$$\left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, v, \tilde{x}) - \sum_{j \leq i} a_{ij}(\tilde{x}) \cdot v^j \right) \cdot u^i = g_i(v, \tilde{x}) \cdot v^{i+1} \cdot u^i$$

for some function g_i q-analytic at zero.

Lemma 2. *If, for some $s \in \mathbb{N} \setminus \{0\}$,*

$$\sum_{i=0}^{\infty} a_{i,i+s}(\tilde{\lambda}) \cdot \varepsilon_1^i \neq 0 \quad \textit{or} \quad \sum_{j=0}^{\infty} a_{j+s,j}(\tilde{\lambda}) \cdot \varepsilon_1^j \neq 0,$$

then, respectively, we have

$$\nu_- \neq 0 \quad \textit{and} \quad v(\nu_-) \in \Gamma_{\langle \lambda \rangle} - (\mathbb{N} \setminus \{0\}) \cdot v(\mu)$$

or

$$\nu_+ \neq 0 \quad \textit{and} \quad v(\nu_+) \in \Gamma_{\langle \lambda \rangle} + (\mathbb{N} \setminus \{0\}) \cdot v(\mu).$$

Consider first the latter case. Clearly,

$$\nu_+ = \sum_{l=1}^{\infty} \left(\sum_{j=0}^{\infty} a_{j+l,j}(\tilde{\lambda}) \cdot \varepsilon_1^j \right) \cdot \mu^l,$$

and the values of (the valuation v taken on) the l-th summands of the above series are pairwise distinct, unless they are infinity. Consequently, $v(\nu_+)$ is the minimum of the values of those summands, because some summands are non-vanishing (for instance, the s-th one). Hence

$$v(\nu_+) < \infty \quad \text{and} \quad v(\nu_+) \in \Gamma_{\langle \lambda \rangle} + (\mathbb{N} \setminus \{0\}) \cdot v(\mu),$$

as desired.

In the former, take n large enough so that

$$v(\mu^n) > v \left(\sum_{i=0}^{\infty} a_{i,i+s}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu} \right)^{i+s} \right),$$

and write down ν_- as follows:

$$\nu_- = \sum_{l=1}^{s} \left(\sum_{i=0}^{\infty} a_{i,i+l}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu} \right)^{i+l} \right) +$$

$$+ \sum_{i=0}^{n-1} \left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, \frac{\varepsilon_1}{\mu}, \tilde{\lambda}) - \sum_{j \le i+s} a_{ij}(\tilde{\lambda}) \cdot \left(\frac{\varepsilon_1}{\mu} \right)^j \right) \cdot \mu^i +$$

$$+ \sum_{i=n}^{\infty} \left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, \frac{\varepsilon_1}{\mu}, \tilde{\lambda}) - \sum_{j \le i+s} a_{ij}(\tilde{\lambda}) \cdot \left(\frac{\varepsilon_1}{\mu} \right)^j \right) \cdot \mu^i.$$

Observe again that the Q-analytic functions which occur in the i-th summands of the second series above are of the form

$$\left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, v, \tilde{x}) - \sum_{j \le i+s} a_{ij}(\tilde{x}) \cdot v^j \right) \cdot u^i = h_i(v, \tilde{x}) \cdot v^{i+s+1} \cdot u^i$$

for some functions h_i Q-analytic at zero, $i = 0, \ldots, n$. Hence

$$\sum_{i=0}^{n-1} \left(\frac{1}{i!} \cdot \frac{\partial^i f}{\partial u^i}(0, \frac{\varepsilon_1}{\mu}, \tilde{\lambda}) - \sum_{j \le i+s} a_{ij}(\tilde{\lambda}) \cdot \left(\frac{\varepsilon_1}{\mu} \right)^j \right) \cdot \mu^i =$$

$$= \left(\frac{\varepsilon_1}{\mu} \right)^{s+1} \cdot \sum_{i=0}^{n} h_i(\frac{\varepsilon_1}{\mu}, \tilde{\lambda}) \cdot \varepsilon_1^i.$$

By Corollary 2.11, *op.cit.*, we have

$$v(h_i(\frac{\varepsilon_1}{\mu}, \tilde{\lambda})) \in \Gamma_{\langle \lambda \rangle} \oplus \mathbb{N} \cdot v(\frac{\varepsilon_1}{\mu}).$$

Since

$$\sum_{i=0}^{\infty} a_{i,i+l}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu} \right)^{i+l} = \left(\frac{\varepsilon_1}{\mu} \right)^{l} \cdot \sum_{i=0}^{\infty} a_{i,i+l}(\tilde{\lambda}) \cdot \varepsilon_1^i \cdot, \quad l = 1, \ldots, s,$$

the values

$$v \left(\sum_{i=0}^{\infty} a_{i,i+l}(\tilde{\lambda}) \cdot \mu^i \cdot \left(\frac{\varepsilon_1}{\mu} \right)^{i+l} \right), \quad l = 1, \ldots, s,$$

and
$$v\left(\sum_{i=0}^{n-1}\left(\frac{1}{i!}\cdot\frac{\partial^i f}{\partial u^i}(0,\frac{\varepsilon_1}{\mu},\tilde{\lambda})-\sum_{j\leq i+s} a_{ij}(\tilde{\lambda})\cdot\left(\frac{\varepsilon_1}{\mu}\right)^j\right)\cdot\mu^i\right)$$
are pairwise distinct, unless they are infinity. Consequently, $v(\nu_-)$ is the minimum of the above $(q+1)$ values. Hence
$$v(\nu_-)<\infty \quad\text{and}\quad v(\nu_-)\in\Gamma_{\langle\lambda\rangle}-(\mathbb{N}\setminus\{0\})\cdot v(\mu),$$
which completes the proof of Lemma 2.

Now, take $n\in\mathbb{N}$ large enough so that
$$n\cdot v(\varepsilon_1)>\min\{\,v(\nu_-),v(\nu_+)\,\}.$$
Then
$$v\left(\nu-\sum_{i=0}^{n-1} a_{i,i}(\tilde{\lambda})\cdot\varepsilon_1^i\right)\notin\Gamma_{\langle\lambda\rangle}.$$
This means that ν is active over the infinitesimals λ, concluding the proof in Case A.

CASE B. Let us rename the coordinates in $\mathbb{R}^m$ in the following fashion: the coordinates $x=(x_1,\dots,x_q)$ correspond to the infinitesimals δ and the coordinates $y=(y_1,\dots,y_p)$ correspond to the infinitesimals ε. In the new variables, the function f can be written down as $f(u,v,\tilde{x},y)$, $\tilde{x}=(x_2,\dots,x_q)$. We first establish the following

Reduction Step. *We are able to assume that the infinitesimal ν is of the form $\nu=f(\mu,\delta_1/\mu,\tilde{\delta},\varepsilon)$, where f is a function Q-analytic at zero, and*
$$v(\varepsilon)>G_2\oplus\ldots\oplus G_r \quad\textit{and}\quad \Gamma_{\langle\delta,\mu\rangle}<G_1^+.$$

Indeed, we shall first recursively attach the old infinitesimals $\delta_2,\dots,\delta_q$, after performing suitable special modifications, either the new infinitesimals δ' or to the new infinitesimals ε', so as to fulfil the conditions
$$v(\varepsilon')>G_2\oplus\ldots\oplus G_r \quad\text{and}\quad \Gamma_{\langle\delta'\rangle}<G_1^+. \tag{$*$}$$

At the beginning, take as new infinitesimals δ' those infinitesimals from $\delta_2,\dots,\delta_q$, which lie in the main part of the regular sequence $\mu,\lambda_1,\dots,\lambda_k$.

Having constructed a sequence $\delta'_2, \ldots, \delta'_i$, consider an infinitesimal δ_j from $\delta_2, \ldots, \delta_q$ which has not yet considered in the process. If

$$\Gamma_{\langle \delta'_2, \ldots, \delta'_i, \delta_j \rangle} < G_1^+,$$

attach $\delta_j =: \delta'_{i+1}$ to the new infinitesimals δ'. Otherwise δ_j is active over $\delta'_2, \ldots, \delta'_i$. By the valuation property ($\mathbf{III_m}$), which holds by the induction hypothesis, there is an $\mathcal{L}$-term $\tau(\delta'_2, \ldots, \delta'_i)$ such that

$$v(\delta_j - \tau(\delta'_2, \ldots, \delta'_i)) > G_2 \oplus \ldots \oplus G_r.$$

Via desingularization of $\mathcal{L}$-terms (*op.cit.*, Corollary 2.6), we can assume, after a suitable change of the infinitesimals $\delta'_2, \ldots, \delta'_i$ by special modification, that

$$\tau(\delta'_2, \ldots, \delta'_i) = \varphi(\delta'_2, \ldots, \delta'_i),$$

where φ is a function Q-analytic at zero. Then we attach the infinitesimal

$$\omega := \delta_j - \varphi(\delta'_2, \ldots, \delta'_i)$$

to the new infinitesimals ε. By substitution $\omega + \varphi(\delta'_2, \ldots, \delta'_i)$ for δ_j, we are done. We continue this process until all infinitesimals $\delta_2, \ldots, \delta_q$ have been considered.

Next, consider the infinitesimal δ_1. If

$$\Gamma_{\langle \delta'_2, \ldots, \delta'_t, \delta_1 \rangle} < G_1^+,$$

we are reduced by putting $\delta'_1 := \delta_1$. Otherwise δ_1 is active over $\delta'_2, \ldots, \delta'_t$. As before, by the valuation property ($\mathbf{III_m}$) and via desingularization of $\mathcal{L}$-terms, we can assume that

$$v(\omega) > G_2 \oplus \ldots \oplus G_r \quad \text{with} \quad \omega := \delta_1 - \varphi(\delta'_2, \ldots, \delta'_t),$$

where φ is a function Q-analytic at zero. Let $\delta' = (\delta'_2, \ldots, \delta'_t)$. Then, similarly to *op.cit.*, Section 4, we can replace the function f by some other Q-analytic functions as follows:

$$\nu = f(\mu, (\varphi(\delta') + \omega)/\mu, \delta', \varepsilon') = f_1(\mu, \varphi(\delta')/\mu, \omega/\mu, \delta', \varepsilon') =$$

$$= f_2(\mu, \varphi(\delta')/\mu, \omega/\varphi(\delta'), \delta', \varepsilon').$$

We can thus attach the infinitesimal $\varepsilon'_{s+1} := \omega/\varphi(\delta')$ to the infinitesimals ε', and then

$$\nu = f_2(\mu, \varphi(\delta')/\mu, \varepsilon'_{s+1}, \delta', \varepsilon').$$

For simplicity, we drop the sign of apostrophe over the name of infinitesimals and renumber the infinitesimals $\delta = (\delta_1, \dots, \delta_q)$. Via transformation the function φ to normal crossings, we may assume that

$$\nu = f_3(\mu, \delta^\alpha/\mu, \delta, \varepsilon)$$

for some $\alpha \in \mathbb{N}^q$. Replacing the infinitesimals μ and δ by their suitable roots, we may assume that $\delta^\alpha = \delta_1 \cdot \ldots \cdot \delta_k$ for some $k \leq q$. Now, we can successively lower the number k of these factors as follows. Since $v(\mu) \notin \Gamma_{\langle\delta\rangle}$, exactly one of the two fractions $\delta_1 \cdot \ldots \cdot \delta_{k-1}/\mu$ or $\mu/\delta_1 \cdot \ldots \cdot \delta_{k-1}$ is an infinitesimal. In the former case, we get

$$\nu = f_3(\mu, \delta_1 \cdot \ldots \cdot \delta_k/\mu, \delta, \varepsilon) = f_4(\mu, \delta_1 \cdot \ldots \cdot \delta_{k-1}/\mu, \delta, \varepsilon);$$

and in the latter

$$\nu = f_3(\mu, \delta_1 \cdot \ldots \cdot \delta_k/\mu, \delta, \varepsilon) = f_4(\mu, \mu/\delta_1 \cdot \ldots \cdot \delta_{k-1}, \delta_1 \cdot \ldots \cdot \delta_k/\mu, \delta, \varepsilon) =$$

$$= f_5(\mu/\delta_1 \cdot \ldots \cdot \delta_{k-1}, \delta_1 \cdot \ldots \cdot \delta_k/\mu, \delta, \varepsilon),$$

and thus, replacing μ by $\mu' := \mu/\delta_1 \cdot \ldots \cdot \delta_{k-1}$, we get

$$\nu = f_5(\mu', \delta_k/\mu', \delta, \varepsilon).$$

Again, we drop the sign of apostrophe. Eventually, we can assume that

$$\nu = f_6(\mu, \delta_1/\mu, \delta_1, \tilde{\delta}, \varepsilon) = f_7(\mu, \delta_1/\mu, \tilde{\delta}, \varepsilon),$$

which is the desired result.

Summing up, the above construction yields a new regular sequence of infinitesimals δ, ε which satisfy conditions $(*)$. To conclude the reduction step, we must still show that $\Gamma_{\langle\delta,\mu\rangle} < G_1^+$. But this follows immediately from the valuation property $(\mathbf{III_m})$, which is at our disposal by the induction hypothesis, applied to the infinitesimals δ and μ.

For the rest of the proof of Theorem 1, we shall keep the conditions established in the reduction step. In the completion of the valued field $\langle\lambda, \mu\rangle$

with respect to the standard valuation v, we can present the infinitesimal ν in the following form:

$$\nu = \sum_{\alpha \in \mathbb{N}^p} \varepsilon^\alpha \cdot f_\alpha(\mu, \delta_1/\mu, \tilde{\delta}),$$

where

$$f_\alpha(u, v, \tilde{x}) := \frac{1}{\alpha!} \cdot \frac{\partial^{|\alpha|} f}{\partial y^\alpha}(u, v, \tilde{x}, 0), \quad \alpha \in \mathbb{N}^p.$$

We need an elementary fact about the standard valuation v.

Lemma 3. *Consider a finite number of elements $g_i, h_i \in \langle \lambda, \mu \rangle$, $i = 1, \ldots, n$, such that*

$$v(h_1), \ldots, v(h_n) > G_2 \oplus \ldots \oplus G_r$$

and

$$v\left(\sum_{i=1}^n c_i \cdot g_i\right) < G_1^+, \quad c_i \in \mathbb{R},\ i = 1, \ldots, n,$$

for all real linear combinations of the elements $g_1, \ldots, g_n$. Then there exist n real linear combinations

$$G_j = \sum_{i=1}^n c_{ji} g_i, \quad H_j = \sum_{i=1}^n d_{ji} h_i, \quad c_{ji}, d_{ji} \in \mathbb{R}, \quad i, j = 1, \ldots, n,$$

such that

$$\sum_{i=1}^n h_i \cdot g_i = \sum_{i=1}^n H_i \cdot G_i,$$

and that the valuations $v(H_1), \ldots, v(H_n)$ are pairwise distinct; then, of course, the valuations $v(H_1 \cdot G_1), \ldots, v(H_n \cdot G_n)$ are pairwise distinct.

The proof is by induction with respect to n. We show the case where $n = 2$. If $v(h_1) \neq v(h_1)$, we are done. Otherwise there are real numbers $d_1, d_2 \neq 0$ such that

$$v(d_1 h_1 - d_2 h_2) > v(h_1) = v(h_2).$$

We get

$$h_1 \cdot g_1 + h_2 \cdot g_2 = d_1 h_1 \cdot d_1^{-1} g_1 + d_2 h_2 \cdot d_2^{-1} g_2 =$$

$$= d_1h_1 \cdot (d_1^{-1}g_1 + d_2^{-1}g_2) + (d_2h_2 - d_1h_1) \cdot d_2^{-1}g_2,$$

which is the required result.

We shall apply Lemma 3 to the elements

$$h_\alpha = \varepsilon^\alpha \quad \text{and} \quad g_\alpha = f_\alpha(\mu, \delta_1/\mu, \tilde{\delta})g_\alpha.$$

We are now going to recursively define an increasing sequence of positive integers (N_k) and three sequences of infinitesimals (f_k), (ϕ_k) and (ψ_k), $k \in \mathbb{N}$. Let $\gamma \in G_1^+$, N_0 be any positive integer and

$$A_0 := \{\alpha \in \mathbb{N}^p : \; v(\varepsilon^\alpha) < N_0\gamma\}, \quad n_0 := \sharp\, A_0.$$

With the notation of Lemma 3, we get

$$f_0 := \sum_{\alpha \in A_0} \varepsilon^\alpha \cdot g_\alpha = \sum_{i=1}^{n_0} H_i \cdot G_i;$$

put

$$\psi_0 := 0, \quad \varphi_0 := \sum \{H_i \cdot G_i : \; v(H_i) < N_0\gamma\} \quad \text{and} \quad \psi_1 := f_0 - \varphi_0.$$

Take N_1 so large that $v(H_i) < N_1\gamma$ for all $i = 1, \ldots, n_0$. Let

$$A_1 := \{\alpha \in \mathbb{N}^p : \; N_0\gamma \le v(\varepsilon^\alpha) < N_0\gamma\}$$

and n_1 be the sum of $\sharp\, A_1$ and the number of the summands of ψ_1. Again, with the notation of Lemma 3, we get

$$f_1 := \sum_{\alpha \in A_1 \setminus A_0} \varepsilon^\alpha \cdot g_\alpha + \psi_1 = \sum_{i=1}^{n_1} H_i \cdot G_i;$$

clearly, $v(H_i) \ge N_0\gamma$ for $i = 1, \ldots, n_1$; put

$$\varphi_1 := \sum \{H_i \cdot G_i : \; v(H_i) < N_1\gamma\} \quad \text{and} \quad \psi_2 := f_1 - \varphi_1.$$

We continue this process recursively. By construction, each φ_k is a finite sum of the form

$$\varphi_k := \sum_i H_i \cdot G_i\,, \quad \text{where} \;\; N_{k-1}\gamma \le v(H_i) < N_k\gamma \;\; \text{for all } i,$$

and the values $v(H_i)$ are pairwise distinct. It is easy to check that

$$\nu = \sum_{\alpha \in \mathbb{N}^p} \varepsilon^\alpha \cdot g_\alpha = \sum_{k=0}^{\infty} f_k = \sum_{k=0}^{\infty} \varphi_k.$$

We encounter two possibilities: either $G_i \in \langle \delta \rangle$ for each $k \in \mathbb{N}$ and every i, or there is a $k \in \mathbb{N}$ such that $G_i \not\in \langle \delta \rangle$ for some i. The former leads to a contradiction. Indeed, G_i are of the form

$$G_i(\mu, \delta_1/\mu, \tilde{\delta}) \quad \text{or} \quad G_i(\mu, \varphi(\delta)/\mu, \delta),$$

where $G_i(u, v, \tilde{x})$ or $G_i(u, v, x)$, respectively, are functions Q-analytic in a common neighbourhood of zero. Like in Case A, we thus have the equivalence: $G_i \in \langle \delta \rangle$ iff the function

$$\partial/\partial u \, G_i(u, x_1/u, \tilde{x})$$

vanishes for $u = \mu$ and $x = \delta$. Therefore, the former possibility implies that the function

$$\partial/\partial u \, f(u, x_1/u, \tilde{x}, y)$$

vanishes for $u = \mu$, $x = \delta$ and $y = \varepsilon$. But then, the above partial derivative with respect to the variable u would vanish on a special cube containing the infinitesimals μ, δ and ε. By the identity principle for quasianalytic functions, this partial derivative would vanish identically, because those infinitesimals are analytically independent. Consequently, the function

$$f(u, x_1/u, \tilde{x}, y) \quad \text{or} \quad f(u, \varphi(x)/u, x, y),$$

respectively, coincides with a function $g(x, y)$ Q-analytic at zero. Hence

$$\nu = g(\delta, \varepsilon) \in \langle \delta, \varepsilon \rangle = \langle \lambda \rangle,$$

contrary to the assumption of Theorem 1.

In this fashion, we may assume the latter possibility. Then the set

$$J := \{i \in \mathbb{N} : \ G_i \not\in \langle \delta \rangle \neq \emptyset$$

is non-empty. There is a unique $i_0 \in J$ such that

$$v(H_{i_0}) = \min \{v(H_i) : \ i \in J\},$$

because the values $v(H_i)$ are pairwise distinct and for any $\gamma \in G_1$ there are only finitely many i for which $v(H_i) < \gamma$. Let

$$I := \{i \in \mathbb{N} : \ v(H_i) < v(H_{i_0})\}.$$

Now, it follows from the induction hypothesis that there is an element $\tau(\delta) \in \langle\delta\rangle$ such that

$$v(G_{i_0} - \tau(\delta)) \notin \Gamma_{\langle\delta\rangle}.$$

Then

$$\Lambda := H_{i_0} \cdot \tau(\delta) + \sum_{i \in I} H_i \cdot G_i \in \langle\delta, \varepsilon\rangle = \langle\lambda\rangle,$$

because $H_i \in \langle\varepsilon\rangle$ and $G_i \in \langle\delta\rangle$ for all $i \in I$. It is not difficult to check that

$$v(\nu - \Lambda) = v(H_{i_0}) + v(G_{i_0} - \tau(\delta)) \notin \Gamma_{\langle\delta,\varepsilon\rangle} = \Gamma_{\langle\lambda\rangle}.$$

This means that ν is active over the infinitesimals λ, which completes the proof of Theorem 1.

3. Quantifier elimination and description of definable functions by terms. In this section we are going to develop an approach to quantifier elimination and a description of definable functions by terms in the language augmented by the names of rational powers, which is much shorter and more natural than the one in [10]. Observe first that one can introduce a well-defined notion of the dimension of sets defined piecewise by $\mathcal{L}$-terms. Indeed, every such set E is a finite union of special cubes S_i (*op.cit.*, Theorem 2.1 and Corollary 2.3), and one can put

$$\dim E := \max \dim S_i.$$

Remark 3. It is easy to check that the dimension of a set E does not depend on the decomposition into special cubes S_i.

We now establish a generalization of *op.cit.*, Proposition 5.4.

Proposition 1. *Consider a mapping $f : \mathbb{R}^d \longrightarrow \mathbb{R}^m$ given piecewise by $\mathcal{L}$-terms and such that for every special cube $S \subset \mathbb{R}^m$ or, equivalently, for every subset E of $\mathbb{R}^m$ given piecewise by $\mathcal{L}$-terms, we have*

$$\dim f^{-1}(S) \leq \dim S \ \textit{or} \ \ \dim f^{-1}(E) \leq \dim E.$$

Then f admits a section given piecewise by $\mathcal{L}$-terms, i.e. there is a function $\xi : \mathbb{R}^m \longrightarrow \mathbb{R}^d$ given piecewise $\mathcal{L}$-terms such that $f(\xi(y)) = y$ for every point $y \in \mathbb{R}^m$.

We may, of course, assume that $f : (0,1)^d \longrightarrow (0,1)^m$. We first show that there exists a family $(t_\iota(y))_{\iota \in I}$ of $\mathcal{L}$-terms, $t_\iota(y) = (t_{\iota,1}(y), \ldots, t_{\iota,d}(y))$, such that the infinite disjunction

$$\bigvee_{\iota \in I} \left[\left(b = f(a) \wedge a \in (0,1)^d \right) \implies b = f(t_\iota(b)) \right]$$

holds for any tuples $a \in (0,1)^d$ and $b \in (0,1)^m$ in an arbitrary model $\mathcal{R}$ of the theory T. So take any elements $a \in (0,1)^d$ and $b \in (0,1)^m$ for which $b = \varphi(a)$. We may, of course, confine our analysis to the case where $a = \lambda$ and $b = \mu$ are infinitesimals. Let $k := \mathrm{rk}\,\langle \lambda \rangle$; then the infinitesimals λ lie on a special cube of dimension k, but lie on no special cube of dimension $< k$. Obviously,

$$\langle \mu \rangle \subset \langle \lambda \rangle \quad \text{and} \quad \mathrm{rk}\,\langle \mu \rangle \leq \mathrm{rk}\,\langle \lambda \rangle.$$

Were $\mathrm{rk}\,\langle \mu \rangle < k = \mathrm{rk}\,\langle \lambda \rangle$, then the infinitesimals μ would lie on a special cube S of dimension $< k$, and thus it follows from the assumption that the infinitesimals λ would lie on the set $f^{-1}(S)$ of dimension $< k$, which is impossible. Consequently,

$$\mathrm{rk}\,\langle \mu \rangle = \mathrm{rk}\,\langle \lambda \rangle \quad \text{and} \quad \langle \mu \rangle = \langle \lambda \rangle.$$

Therefore the infinitesimals λ can be expressed by $\mathcal{L}$-terms taken on the infinitesimals μ, and the assertion follows.

Now, through model-theoretic compactness, one can find a finite set $\iota_1, \ldots, \iota_n \in I$ of indices for which the finite disjunction

$$\bigvee_{k=1,\ldots,n} \left[\left(b = f(a) \wedge a \in (0,1)^d \right) \implies b = f(t_{\iota_k}(b)) \right]$$

holds for any tuples a and b in an arbitrary model $\mathcal{R}$ of the theory T. In particular, this finite disjunction is satisfied in the standard model $\mathbb{R}$, concluding the proof of Proposition 1.

Remark 4. The assumption of Proposition 1 is satisfied by every function f given piecewise by $\mathcal{L}$-terms which is an immersion. More generally, consider a decomposition of $\mathbb{R}^d$ into finitely many leaves F_j given piecewise by $\mathcal{L}$-terms, and a function f given piecewise by $\mathcal{L}$-terms whose restriction to each F_j is an immersion. Then f satisfies that assumption too.

Corollary 1. *Under the assumptions of Proposition 1, the image $f(\mathbb{R}^d)$ is given piecewise by $\mathcal{L}$-terms.*

Indeed, suppose the section ξ is given by a finite number of $\mathcal{L}$-terms

$$\tau_i(y) = (\tau_{i1}(y), \ldots, \tau_{id}(y)), \quad i = 1, \ldots, s, \quad y = (y_1, \ldots, y_m),$$

i.e.

$$x = \xi(y) \quad \Leftrightarrow \quad \bigvee_{i=1}^{s} x = \tau_i(y).$$

Then

$$y \in f(\mathbb{R}^d) \quad \Leftrightarrow \quad \bigvee_{i=1}^{s} f(\tau_i(y)) = y,$$

and thus the image $f(\mathbb{R}^d)$ is given by $\mathcal{L}$-terms $f(\tau_i(y)) - y$, $i = 1, \ldots, s$, as desired.

By an immersion cube $C \subset \mathbb{R}^m$ we mean image $\varphi((0,1)^d$ where φ is a Q-mapping in a neighbourhood of the compact cube $[0,1]^d$, whose restriction to $(0,1)^d$ is an immersion. As demonstrated in our paper [9], the theorem on decomposition into special cubes along with the technique of fiber cutting make it possible to decompose every bounded Q-subanalytic set into finitely many immersion cubes (*op. cit.*, Corollary 1). This, in turn, and the above corollary, immediately yield quantifier elimination for the expansion $\mathcal{R}_Q$ of the real field with restricted quasianalytic functions in the language $\mathcal{L}$ augmented by the names of rational powers (cf. [10], Theorem 5.8):

Theorem 2. (Quantifier Elimination) *Every set definable in the structure $\mathcal{R}_Q$ is given piecewise by a finite number of $\mathcal{L}$-terms.*

A fortiori, the structure $\mathcal{R}_Q$ is model complete, and we can recover, via e.g. the decomposition of Q-semianalytic sets into finitely many special cubes, a well-known result that it is a polynomially bounded, o-minimal structure which admits Q-analytic cell decomposition (cf. [14, 9]).

We now wish to turn to the problem of a description of definable function by $\mathcal{L}$-terms.

Theorem 3. *Each definable function* $f : \mathbb{R}^m \longrightarrow \mathbb{R}$ *is piecewise given by a finite number of* $\mathcal{L}$*-terms.*

Indeed, consider the graph $F \subset \mathbb{R}^{m+1} = \mathbb{R}^m_x \times \mathbb{R}_y$ of the function f and denote by

$$p : F \longrightarrow \mathbb{R}^m_x \quad \text{and} \quad q : F \longrightarrow \mathbb{R}_y$$

the canonical projections. Via cell decomposition (of class $\mathcal{C}^1$), the graph F can be partitioned into finitely many cells C_i defined by $\mathcal{L}$-terms such that the restriction of p to each cell C_i is an immersion and, in fact, a diffeomorphism onto the image $p(C_i)$. It follows from Proposition 1 (cf. Remark 4) that each inverse

$$p^{-1} : p(C_i) \longrightarrow C_i, \quad i = 1, \ldots, s,$$

is given piecewise by finitely many $\mathcal{L}$-terms, and thus so is the restriction of $f = q \circ p^{-1}$ to each set $p(C_i)$. This completes the proof.

We immediately obtain the following three corollaries.

Corollary 2. *The structure* $\mathcal{R}_Q$ *admits cell decompositions defined piecewise by* $\mathcal{L}$*-terms, and hence Skolem functions (of choice) given piecewise by* $\mathcal{L}$*-terms.*

Corollary 3. *The structure* $\mathcal{R}_Q$ *is universally axiomatizable. Hence its universal diagram* T *admits quantifier elimination (in the language* $\mathcal{L}$*) and* $\mathcal{R}_Q$ *can be embedded as a prime model into each model of* T*. Consequently, in every model of* T*, each definable function is defined piecewise by a finite number of* $\mathcal{L}$*-terms.*

Corollary 4. (Valuation Property for Definable Functions) *Consider a simple (with respect to definable closure) extension* $\mathcal{R} \subset \mathcal{R}\langle a \rangle$ *of substructures in a fixed model of the theory* T*. Then we have the following dichotomy:*

$$\textit{either} \quad \dim \Gamma_{\mathcal{R}\langle a \rangle} = \dim \Gamma_{\mathcal{R}} \quad \textit{or} \quad \dim \Gamma_{\mathcal{R}\langle a \rangle} = \dim \Gamma_{\mathcal{R}} + 1.$$

In the latter case, one can find an element $r \in \mathcal{R}$ *such that*

$$v(a - r) \notin \Gamma_{\mathcal{R}} \quad \textit{and} \quad \Gamma_{\mathcal{R}\langle a \rangle} = \Gamma_{\mathcal{R}} \oplus \mathbb{Q} \cdot v(a - r).$$

4. Rectilinearization of definable functions. By induction with respect to the complexity of terms, we can establish the rectilinearization of $\mathcal{L}$-terms in quasianalytic structures exactly in the same way as it was done in our paper [11] about functions definable by a Weierstrass system. Therefore, since functions definable in $\mathcal{R}_Q$ are given piecewise by $\mathcal{L}$-terms (Theorem 3), we immediately obtain several results about Q-subanalytic functions, presented in this section. We begin with some suitable terminology. By a quadrant in $\mathbb{R}^m$ we mean a subset of $\mathbb{R}^m$ of the form:

$$\{x = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i = 0, x_j > 0, x_k < 0 \quad \text{for} \ \ i \in I_0, j \in I_+, k \in I_-\},$$

where $\{I_0, I_+, I_-\}$ is a disjoint partition of $\{1, \ldots, m\}$; its trace Q on the cube $[-1, 1]^m$ shall be called a bounded quadrant. The interior $\mathrm{Int}\,(Q)$ of the quadrant Q is its trace on the open cube $(-1, 1)^m$. A bounded closed quadrant is the closure $\overline{Q}$ of a bounded quadrant Q, i.e. a subset of $\mathbb{R}^m$ of the form:

$$\overline{Q} := \{x \in [-1, 1]^m : x_i = 0, x_j \geq 0, x_k \leq 0 \quad \text{for} \ \ i \in I_0, j \in I_+, k \in I_-\}.$$

In this section, by a normal crossing on a bounded quadrant Q in $\mathbb{R}^m$ we mean a function g of the form

$$g(x) = x^\alpha \cdot u(x),$$

where $\alpha \in \mathbb{N}^m$ and u is a function Q-analytic near $\overline{Q}$ which vanishes nowhere on $\overline{Q}$. The proposition stated below is a quasianalytic counterpart of *op.cit.*, Theorem 1.

Proposition 2. (Simultaneous Rectilinearization of $\mathcal{L}$-terms) *If*

$$f_1, \ldots, f_s : \mathbb{R}^m \longrightarrow \mathbb{R}$$

are functions given piecewise by a finite number of $\mathcal{L}$-terms, and K is a compact subset of $\mathbb{R}^m$, then there exists a finite collection of modifications

$$\varphi_i : [-1, 1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \ldots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1, 1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) the union of the images $\varphi_i((-1,1)^m)$, $i = 1,\ldots,p$, is a neighbourhood of K.

3) for every bounded quadrant Q_j, $j = 1,\ldots,3^m$, the restriction to Q_j of each function $f_k \circ \varphi_i$, $k = 1,\ldots,s$, $i = 1,\ldots,p$, either vanishes or is a normal crossing or a reciprocal normal crossing on Q_j.

Remark 5. Observe that, if the functions $f_1,\ldots,f_s$ are given piecewise by terms in the language of restricted Q-analytic functions augmented by the reciprocal function $1/x$, then one can require that the modifications φ_i, $i = 1,\ldots,p$, be composite of finitely many blowings-up with smooth centers. The same refers to the results stated below.

Now let us recall some consequences of Proposition 2. Let U be a definable bounded open subset in $\mathbb{R}^m$, ∂U its frontier, ρ_1, ρ_2 be the distance functions from the sets U, ∂U, respectively, and $f : U \longrightarrow \mathbb{R}$ a definable function. Since definable functions are piecewise given by a finite number of $\mathcal{L}$-terms (Theorem 3 from Section 3), Proposition 2, applied to the functions f, ρ_1, ρ_2, yields the following (*op.cit.*, Theorem 2):

Proposition 3. (Rectilinearization of a Definable Function) *Let $U \subset \mathbb{R}^m$ be a bounded open subset and $f : U \longrightarrow \mathbb{R}$ be a definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1,1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1,\ldots,p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1,1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m-1$;

4) U is the union of the images $\varphi_i(\mathrm{Int}\,(Q))$ with Q ranging over the bounded quadrants contained in $\varphi_i^{-1}(U)$, $i = 1,\ldots,p$;

5) for every bounded quadrant Q, the restriction to Q of each function $f \circ \varphi_i$ either vanishes or is a normal crossing or a reciprocal normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$.

Remark 6. One can formulate Proposition 3, similarly to Proposition 2, for several definable functions $f_1,\ldots,f_s$.

It follows from points 1) and 2) that every bounded quadrant of dimension $< m$ contained in $\varphi_i^{-1}(U)$ is adjacent to a bounded quadrant of dimension m (a bounded orthant) contained in $\varphi_i^{-1}(U)$. Hence

$$\varphi_i^{-1}(\overline{U}) = \overline{\varphi_i^{-1}(U)},$$

and therefore point 4) implies that $\overline{U}$ is the union of the images $\varphi_i(\overline{Q})$ of the closures of those bounded quadrants of dimension m (bounded orthants) Q for which $\varphi_i(Q) \subset U$, $i = 1, \dots, p$.

For a bounded orthant Q contained in $\varphi_i^{-1}(U)$, denote by $\mathrm{dom}_i\,(Q)$ the union of Q and all those bounded quadrants that are adjacent to Q and disjoint with $\varphi_i^{-1}(\partial U)$; it is, of course, an open subset of the closure $\overline{Q}$. Moreover, the open subset $\varphi_i^{-1}(U)$ of the cube $[-1, 1]^m$ coincides with the union of $\mathrm{dom}_i\,(Q)$, where Q range over the bounded orthants that are contained in $\varphi_i^{-1}(U)$, and with the union of those bounded quadrants that are contained in $\varphi_i^{-1}(U)$. Consequently, the union of the images $\varphi_i(\mathrm{Int}\,(Q))$, where Q range over the bounded quadrants that are contained in $\varphi_i^{-1}(U)$, coincides with the union of the images

$$\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1, 1)^m),$$

where Q range over the bounded orthants Q that are contained in $\varphi_i^{-1}(U)$.

Corollary 5. (Rectilinearization of a Continuous Definable Function) *Let U be a bounded open subset in $\mathbb{R}^m$ and $f : U \longrightarrow \mathbb{R}$ be a continuous definable function. Then there exists a finite collection of modifications*

$$\varphi_i : [-1, 1]^m \longrightarrow \mathbb{R}^m, \qquad i = 1, \dots, p,$$

such that

1) each φ_i extends to a Q-analytic mapping in a neighbourhood of the cube $[-1, 1]^m$, which is a composite of finitely many local blowings-up with smooth centers and power substitutions;

2) each set $\varphi_i^{-1}(U)$ is a finite union of bounded quadrants in $\mathbb{R}^m$;

3) each set $\varphi_i^{-1}(\partial U)$ is a finite union of bounded closed quadrants in $\mathbb{R}^m$ of dimension $m - 1$;

4) U is the union of the images $\varphi_i(\mathrm{dom}_i\,(Q) \cap (-1, 1)^m)$ with Q ranging over the bounded orthants Q contained in $\varphi_i^{-1}(U)$, $i = 1, \dots, p$;

5) for every bounded orthant Q, the restriction to $\mathrm{dom}_i\,(Q)$ *of each function $f \circ \varphi_i$ either vanishes or is a normal crossing or a reciprocal normal crossing on Q, unless $\varphi_i^{-1}(U) \cap Q = \emptyset$.*

Remark 7. In the classical case, the structure $\mathbb{R}_{an}$ admits, as proven by Denef–van den Dries [4], quantifier elimination in the language of restricted analytic functions augmented by the reciprocal function $1/x$. In our next article [12], we demonstrate that this is no longer true for quasianalytic structures.

Acknowledgements. This research was partially supported by Research Project No. N N201 372336 from the Polish Ministry of Science and Higher Education.

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