A counter-example concerning quantifier elimination in quasianalytic structures

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IMUJ PREPRINT 2012/04

Abstract

This paper provides an example of a quasianalytic structure which, unlike the classical analytic structure, does not admit quantifier elimination in the language of restricted quasianalytic functions augmented by the reciprocal function $1/x$. Our construction applies rectilinearization of terms as well as some theorems on power substitution for Denjoy–Carleman classes and on non-extendability of quasianalytic germs.

1. Introduction. In our earlier papers [15, 17], we established, for a quasianalytic structure, quantifier elimination as well as rectilinearization and description of definable functions by terms, in the language augmented by rational powers. Full generality was achieved in the latter paper, which constitutes a basis for a new, self-contained article being currently prepared.

The main purpose of this paper is to provide a negative answer to the problem, posed in [17], whether a quasianalytic structure admits quantifier

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2010 Mathematics Subject Classification. Primary: 03C10, 26E10; Secondary: 14P15, 03C64.

Key words: quasianalytic structures, quantifier elimination, Denjoy–Carleman classes.

elimination in the language augmented merely by the reciprocal function $1/x$. In the case of the classical structure $\mathbb{R}_{an}$, the affirmative answer was given by J. Denef and L. van den Dries [6]. The construction of a counter-example, given in Section 4, is based on the rectilinearization of terms as well as on two function-theoretic results concerning Denjoy–Carleman classes. The first, presented in Section 2, refers to power substitution for those classes. The other is a non-extendability theorem, which is a refinement of a theorem by V. Thilliez [22].

Let us recall (cf. [14, 15, 18]) that a quasianalytic structure is the expansion of the real field by restricted quasianalytic functions determined by a quasianalytic system $\mathcal{Q} = (\mathcal{Q}_m)_{m\in\mathbb{N}}$ of sheaves of local $\mathbb{R}$-algebras of smooth functions on $\mathbb{R}^m$, submitted to the following six conditions:

1. each algebra $\mathcal{Q}(U)$ contains the restrictions of polynomials;
2. $\mathcal{Q}$ is closed under composition, i.e. the composition of Q-mappings is a Q-mapping (whenever it is well defined);
3. $\mathcal{Q}$ is closed under inverse, i.e. if $\varphi : U \longrightarrow V$ is a Q-mapping between open subsets $U, V \subset \mathbb{R}^n$, $a \in U$, $b \in V$ and if $\partial\varphi/\partial x(a) \neq 0$, then there are neighbourhoods U_a and V_b of a and b, respectively, and a Q-diffeomorphism $\psi : V_b \longrightarrow U_a$ such that $\varphi \circ \psi$ is the identity mapping on V_b;
4. $\mathcal{Q}$ is closed under differentiation;
5. $\mathcal{Q}$ is closed under division by a coordinate, i.e. if $f \in \mathcal{Q}(U)$ and $f(x_1, \dots, x_{i-1}, a_i, x_{i+1}, \dots, x_m) = 0$ as a function in the variables x_j, $j \neq i$, then $f(x) = (x_i - a_i)g(x)$ with some $g \in \mathcal{Q}(U)$;
6. $\mathcal{Q}$ is quasianalytic, i.e. if $f \in \mathcal{Q}(U)$ and the Taylor series $\widehat{f_a}$ = of f at a point $a \in U$ vanishes, then f vanishes in the vicinity of a.

Q-mappings give rise, in the ordinary manner, to the category Q of Q-manifolds, which is a subcategory of that of smooth manifolds and smooth mappings. Similarly, Q-analytic, Q-semianalytic and Q-subanalytic sets can be defined. The above conditions ensure some (limited) resolution of singularities in the category Q, including transformation to normal crossings by blowing up (cf. [2, 19]), upon which the geometry of quasianalytic structures relies; especially, in the absence of their good algebraic properties (cf. [19, 14, 15]).

The examples of such categories are provided by quasianalytic Denjoy–Carleman classes $Q_{M'}$, where $M' = (M'_n)_{n\in\mathbb{N}}$ are log-convex sequences. The class $Q_{M'}$ consists of smooth functions $f(x) = f(x_1, \ldots, x_m)$ in m variables, $m \in \mathbb{N}$, which are locally submitted to the following growth condition for their derivatives:

$$\left|f^{(n)}(x)\right| \leq AB^n n! M'_n$$

with some constants $A, B > 0$ depending only on the vicinity of a given point. Often this growth condition is formulated in a slightly different way:

$$\left|f^{(n)}(x)\right| \leq AB^n M_n,$$

where $M_n = n! M'_n$. Obviously, the class Q_M contains the real analytic functions. It is quasianalytic iff

$$\sum_{n=0}^{\infty} \frac{M'_n}{(n+1)M'_{n+1}} = \infty$$

(the Denjoy–Carleman theorem). It is closed under composition (Roumieu [20]) and under inverse (Komatsu [8]); furthermore, it is closed under differentiation and under division by a coordinate iff

$$\sup_n \sqrt[n]{\left(\frac{M'_{n+1}}{M'_n}\right)} < \infty$$

(cf. [12, 21]). On the other hand, every polynomially bounded, o-minimal structure $\mathcal{R}$ determines a quasianalytic system of sheaves of germs of smooth functions that are locally definable in $\mathcal{R}$.

It is well-known (cf. [4, 5, 21]) that, given two log-convex sequences M' and N', the inclusion $\mathcal{E}'(M') \subset \mathcal{E}'(N')$ holds iff there is a constant $C > 0$ such that $M'_n \leq C^n N'_n$ for all $n \in \mathbb{N}$ or, equivalently,

$$\sup \left\{ \sqrt[n]{\frac{M'_n}{N'_n}} : \; n \in \mathbb{N} \right\} < \infty.$$

2. Power substitution for Denjoy–Carleman classes. Let $M' = (M'_n)$ be an increasing sequence of real numbers with $M'_0 = 1$. Let I be an interval (open or closed) contained in $\mathbb{R}$. By $\mathcal{E}'(I, M')$ (resp. $\mathcal{E}(I, M)$) we denote the class of functions on I that satisfy the above conditions. Denote by $\mathcal{E}'(M')$ (resp. $\mathcal{E}(M)$) the set of germs at zero of smooth functions from the Denjoy-Carleman class corresponding to these sequences. In order to ensure some important algebraic and analytic properties of $\mathcal{E}(M)$, it suffices to assume that the sequence M or M' is log-convex. The former implies that the set $\mathcal{E}(M)$ is a ring; it is closed under multiplication by the Leibniz formula. The latter assumption is stronger, and implies, moreover, that $\mathcal{E}'(M')$ is closed under composition (as proven by Roumieu [20] and Cartan [4] in particular cases; see also [2]).

The main purpose of this section is Theorem 1, which is valid for an arbitrary sequence M.

Theorem 1. *Let $p > 1$ be an integer and I the interval $[0, 1]$ or $[-1, 1]$ according as p is even or odd. Consider power substitution $x = \xi^p$, which is a bijection of I onto itself. Let $f : I \longrightarrow \mathbb{R}$ be a smooth function. If*

$$F(\xi) := (f \circ \varphi)(\xi) = f(\xi^p) \in \mathcal{E}(I, M),$$

then $f(x) \in \mathcal{E}(I, M^{(p)})$, where the sequence $M^{(p)}$ is defined by putting $M_n^{(p)} := 1/n^{(p-1)n} \cdot M_{pn}$.

In terms of the corresponding sequences M', one can rephrase the theorem as follows. If

$$F(\xi) := (f \circ \varphi)(\xi) = f(\xi^p) \in \mathcal{E}'(I, M'),$$

then $f(x) \in \mathcal{E}'(I, M'^{(p)})$, where the sequence $M'^{(p)}$ is defined by putting $M_n'^{(p)} := M'_{pn}$.

Remark 1. Using a function constructed by Bang, we shall show at the end of this section that, in the case where $p = 2$ and the sequence M is log-convex, $\mathcal{E}(I, M^{(2)})$ is the smallest class containing all those functions $f(x)$. As communicated to us in written form by E. Bierstone and F.V. Pacheco [3], application of a suitable variation of Bang's function yields the result for an arbitrary positive integer p.

Remark 2. The case $p = 2$ of Theorem 1 may be related to the following problem posed and investigated by Mandelbrojt [11] and [12] (Chap. VI):

Consider a smooth function $f(x)$ on the interval $[-1, 1]$ and suppose that $F(\xi) := f(\cos \xi)$ belongs to a class $\mathcal{E}(\mathbb{R}, M)$. To which class on the interval $[-1, 1]$ does f belong?

A complete solution for sequences M such that $\lim_{n\to\infty} \sqrt[n]{M_n} = \infty$ was given by Lalaguë [9], Chap. III. He proved that those functions f must belong to the class $\mathcal{E}(I, N^{(2)})$, $N_n^{(2)} = 1/n^n \cdot N_{2n}$, where $N := M^c$ is the log-convex regularization of the initial sequence M, and gave a formula for the smallest class containing all those functions f. Moreover, the smallest class coincides with $\mathcal{E}(I, N^{(2)})$, if

$$\liminf_{n\to\infty} \frac{1}{n} \sqrt[n]{M_n} > 0.$$

The last condition is equivalent to the existence of a constant $C > 0$ such that $M_n \geq n! C^n$ for all $n \in \mathbb{N}$, and thus implies that the Denjoy-Carleman class corresponding to the sequence M contains analytic functions.

Whenever the sequence M' (corresponding to M) is increasing, it is not difficult to draw Lalaguë's result from Theorem 1 for the case $p = 2$. Then those functions f must belong to the class $\mathcal{E}'(I, M'^{(2)})$. The proof relies on the observation that the class $\mathcal{E}'(I, N')$ is closed under analytic substitutions if a sequence N' is increasing. This, in turn, follows immediately from Cauchy's inequalities and the formula for the derivatives of a composite function, recalled below:

$$(f \circ g)^{(n)}(x) = \sum_{k=1}^{n} f^{(k)}(g(x)) \cdot \frac{1}{k!} \cdot \frac{d^n}{dX^n} \left(g(X) - g(x)\right)^k |_{X=x}.$$

Before establishing Theorem 1, we need two lemmas stated below.

Lemma 1. *Consider the Taylor expansions*

$$\sum_{i=1}^{\infty} \frac{x^i}{i} = -\log(1-x) \quad \textit{and} \quad \left(\sum_{i=1}^{\infty} \frac{x^i}{i}\right)^k = \sum_{n=1}^{\infty} c_n x^n.$$

Then we have the estimate

$$c_n \leq (2e)^n \cdot \frac{k!}{n^k} \quad \text{for all} \quad k, n \in \mathbb{N}.$$

Indeed, it is easy to verify the estimate:

$$|\log(1-z)| \leq \left|\log 2 + \frac{\pi}{6}\sqrt{-1}\right| \leq 1 \quad \text{for all} \quad z \in \mathbb{C},\ |z| \leq 1/2.$$

Hence and by Cauchy's inequalities, we get $|c_n| \leq 2^n$. Since $e^n > n^k/k!$ for all $n, k \in \mathbb{N}$, we have

$$c_n \leq 2^n < 2^n \cdot e^n \cdot \frac{k!}{n^k} = (2e)^n \cdot \frac{k!}{n^k},$$

as asserted.

As an immediate consequence, we obtain the

Corollary.

$$\sum_{i_1+\ldots+i_k=n} \frac{1}{i_1} \cdot \ldots \cdot \frac{1}{i_k} = c_n \leq (2e)^n \cdot \frac{k!}{n^k} \quad \textit{for all} \quad k, n \in \mathbb{N}.$$

Lemma 2. *Let $p, k \in \mathbb{N}$ with $p > 1$, $k \geq 1$, and*

$$\alpha_k(X, x) := \frac{1}{k!} \left(X^{1/p} - x^{1/p} \right) \quad \textit{for} \quad X, x > 0,$$

where α_k is regarded as a function in one variable X and parameter x. Then we have the estimate

$$\left| \alpha_k^{(n)}(x, x) \right| \leq (2e)^n \cdot n^{n-k} \cdot x^{-\frac{pn-k}{p}} \quad \textit{for all} \quad n, k \in \mathbb{N}, \quad x > 0.$$

Consider first the case $k = 1$, $\alpha_1(X, x) = X^{1/p} - x^{1/p}$. Then

$$\alpha_1^{(n)}(x, x) = \pm \frac{(p-1)(2p-1) \cdot \ldots \cdot ((n-1)p - 1)}{p^n} \cdot x^{-\frac{pn-1}{p}},$$

whence

$$\left| \alpha_1^{(n)}(x, x) \right| \leq (n-1)! \leq n^{n-1} \cdot x^{-\frac{pn-1}{p}},$$

as asserted. The Taylor expansion of $\alpha_1(X, x)$ at $X = x$ is

$$\alpha_1(X, x) = \sum_{i=1}^{\infty} a_i \cdot x^{-\frac{pi-1}{p}} (X - x)^i,$$

where

$$a_i := \pm \frac{1}{i!} \cdot \frac{(p-1)(2p-1) \cdot \ldots \cdot ((i-1)p - 1)}{p^i};$$

obviously, $|a_i| \le (i-1)!/i! = 1/i$. We get

$$\alpha_k(X,x) = \left(\sum_{i=1}^{\infty} a_i x^{-\frac{pi-1}{p}}(X-x)^i\right)^k = \sum_{j=1}^{\infty} b_j \cdot x^{-\frac{pj-k}{p}}(X-x)^j,$$

where

$$b_j := \frac{1}{k!}\sum_{i_1+\ldots+i_k=j} a_{i_1}\cdot\ldots\cdot a_{i_k}.$$

Then

$$|b_n| \le \frac{1}{k!}\sum_{i_1+\ldots+i_k=n} |a_{i_1}|\cdot\ldots\cdot|a_{i_k}| \le \frac{1}{k!}\sum_{i_1+\ldots+i_k=n} \frac{1}{i_1}\cdot\ldots\cdot\frac{1}{i_k} =$$

$$= \frac{c_n}{k!} \le \frac{(2e)^n}{n^k} \quad \text{for all} \quad n,k\in\mathbb{N};$$

the last inequality follows from the corollary to Lemma 1. Hence

$$\left|\alpha_k^{(n)}(x,x)\right| \le n!\cdot|b_n|\cdot x^{-\frac{pn-k}{p}} \le (2e)^n\cdot\frac{n!}{n^k}\cdot x^{-\frac{pn-k}{p}} \le (2e)^n\cdot n^{n-k}\cdot x^{-\frac{pn-k}{p}}$$

for all $n,k\in\mathbb{N}$, $x>0$, as asserted.

Now we can readily pass to the proof of Theorem 1. So suppose that

$$\left|F^{(n)}(\xi)\right| \le A^n M_n$$

for all $n\in\mathbb{N}$, $\xi\in I$ and some constant $A>0$. We are going to estimate the growth of the n-th derivative $f^{(n)}$. Fix $n\in\mathbb{N}$ and put:

$$p_n(x) := T_0^n(x) = \sum_{k=0}^{n-1} f^k(0)\frac{x^k}{k!}, \quad r_n(x) := f(x) - p_n(x),$$

$$P_n(\xi) := p_n(\xi^p) \quad \text{and} \quad R_n(\xi) := r_n(\xi^p).$$

Obviously,

$$P_n^{(pn)} \equiv 0 \quad \text{and} \quad R_n^{(pn)} \equiv F^{(pn)}.$$

From the Taylor formula, we therefore obtain the estimate

$$\left|R_n^{(pn)}(\xi)\right| \le A^{pn}M_{pn}\cdot\frac{\xi^{pn-k}}{(pn-k)!}$$

for all $k < pn$, $\xi \in I$. We still need an elementary inequality

$$\frac{1}{(pn-k)!} \leq \frac{e^{pn}}{n^{pn-k}} \quad \text{for all} \quad k < pn, \tag{$*$}$$

which can be proven via the following well-known Stirling formula

$$\sqrt{2\pi n} \cdot \frac{n^n}{e^n} < n! < e\sqrt{n} \cdot \frac{n^n}{e^n}.$$

Indeed, it suffices to show that

$$\frac{e^{pn-k}}{(pn-k)^{pn-k}} \leq \frac{e^{pn}}{n^{pn-k}}.$$

When $pn-k \geq n$ or, equivalently, $k \leq (p-1)n$, the last inequality is evident. Suppose thus that $pn-k < n$ or, equivalently, $k > (p-1)n$. This inequality is, of course, equivalent to

$$\left(\frac{n}{pn-k}\right)^{pn-k} \leq e^k,$$

which holds as shown below:

$$\left(\frac{n}{pn-k}\right)^{pn-k} = \left(1 + \frac{k-(p-1)n}{pn-k}\right)^{pn-k} =$$

$$= \left[\left(1 + \frac{k-(p-1)n}{pn-k}\right)^{\frac{pn-k}{k-(p-1)n}}\right]^{k-(p-1)n} < e^k.$$

Now, the foregoing estimate along with inequality (*) yield

$$\left|R_n^{(pn)}(\xi)\right| \leq A^{pn} M_{pn} \cdot \frac{e^{pn}}{n^{pn-k}} \cdot \xi^{pn-k}.$$

Applying the formula for the derivatives of a composite function, we obtain

$$r_n^{(n)}(x) = \sum_{k=1}^{n} R_n^{(k)}(\xi) \cdot \alpha_k^{(n)}(x,x).$$

Hence and by Lemma 2, we get

$$\left|f^{(n)}(x)\right| = \left|r_n^{(n)}(x)\right| \leq \sum_{k=1}^{n} A^{pn} M_{pn} \cdot \frac{e^{pn}}{n^{pn-k}} \cdot |\xi|^{pn-k} \cdot (2e)^n n^{n-k} \cdot |\xi|^{-(pn-k)} =$$

$$= n \cdot (2e)^n \cdot (eA)^{pn} \cdot \frac{M_{pn}}{n^{(p-1)n}},$$

which completes the proof of Theorem 1.

Finally, we show that, whenever the sequence M is log-convex, $\mathcal{E}(I, M^{(2)})$ is the smallest class containing all smooth functions $f(x)$ on the interval $I = [0,1]$ such that $F(\xi) = f(\xi^2) \in \mathcal{E}(I, M)$. We make use of a classical function constructed by Bang [1], applied in his proof that the classes determined by log-convex sequences contain functions with sufficiently large derivatives (the result due to Cartan [4] and Mandelbrojt [5]; see also [21], Section 1, Theorem 1).

The logarithmic convexity of the sequence M yields for every $j, k \in \mathbb{N}$ the inequality

$$\left(\frac{1}{m_k}\right)^{k-j} \leq \frac{M_j}{M_k} \quad \text{where} \quad m_k := \frac{M_{k+1}}{M_k}.$$

Consequently,

$$F(\xi) := \sum_{k=0}^{\infty} \frac{M_k}{(2m_k)^k} \cos(2m_k \xi)$$

is an even smooth function on $\mathbb{R}$ such that

$$F(\xi) \in \mathcal{E}(\mathbb{R}, M) \quad \text{and} \quad \left|F^{(2n)}(0)\right| \geq M_{2n}$$

for all $n \in \mathbb{N}$. Therefore $F(\xi) = f(\xi^2)$ for some smooth function f on $\mathbb{R}$, and we get

$$f^{(n)}(0) = \frac{n!}{(2n)!} F^{(2n)}(0) \quad \text{and} \quad \left|f^{(n)}(0)\right| \geq \frac{n! M_{2n}}{(2n)!},$$

which is the desired result.

It was communicated to us by E. Bierstone and F.V. Pacheco [3] that, in order to obtain this result for an arbitrary positive integer p, one can replace $\cos x$ by the function

$$C_p(x) := \sum_{j=0}^{\infty} \frac{x^{jp}}{(jp)!}$$

with the properties $C_p^{(pn)}(x) = C_p(x)$, $C_p^{(pn)}(0) = 1$ and $\left|C_p^{(n)}(x)\right| \leq e$ for all $n \in \mathbb{N}$, $x \in [-1,1]$. Then

$$F(\xi) := \sum_{k=0}^{\infty} \frac{M_k}{(2m_k)^k} C_p(2m_k \xi)$$

is a smooth function on $\mathbb{R}$ such that

$$F(\xi) \in \mathcal{E}([-1,1], M) \quad \text{and} \quad \left|F^{(pn)}(0)\right| \geq M_{pn}$$

for all $n \in \mathbb{N}$. As before, there exists a smooth function f on $\mathbb{R}$ such that

$$F(\xi) = f(\xi^p), \quad f^{(n)}(0) = \frac{n!}{(pn)!} F^{(pn)}(0) \quad \text{and} \quad \left|f^{(n)}(0)\right| \geq \frac{n! M_{pn}}{(pn)!}.$$

We conclude this section with some examples, one of which (namely, for $k = 2$) will be applied to the construction of our counter-example in the last section.

Example. Fix an integer $k \in \mathbb{N}$, $k \geq 1$, and put

$$\log^{(k)} := \underbrace{\log \circ \ldots \circ \log}_{k \text{ times}}, \quad \text{and} \quad e \uparrow\uparrow k := (\underbrace{\exp \circ \ldots \circ \exp}_{k \text{ times}})(1);$$

let n_k be the smallest integer greater than $e \uparrow\uparrow k$. Then the sequence

$$\left(\log^{(k)} n\right)^n \quad \text{for} \quad n \geq n_k$$

is log-convex. Further, the shifted sequence:

$$M' = (M'_n), \quad M'_n := \frac{1}{\left(\log^{(k)} n_k\right)^{n_k}} \cdot \left(\log^{(k)}(n_k + n)\right)^{(n_k+n)},$$

determines a quasianalytic class closed under derivatives; the former follows from Cauchy's condensation criterion. It is easy to check that the sequences $M'^{(p)}$, $p > 1$, are quasianalytic when $k > 1$, but are not quasianalytic when $k = 1$.

3. Non-extendability of quasianalytic germs. In this section we are concerned with a result by V. Thilliez [22] on the extension of quasianalytic function germs in one variable, recalled below. Consider two log-convex sequences M' and N' such that $\mathcal{E}'(M') \subset \mathcal{E}'(N')$. Denote by $\mathcal{E}'(M')^+$ the local ring of right-hand side germs at zero (i.e. germs of functions from $\mathcal{E}'([0,\varepsilon], M')$ for some $\varepsilon > 0$).

Theorem 2. *If* $\mathcal{E}'(N')$ *is a quasianalytic local ring, then*

$$\mathcal{O} \subsetneq \mathcal{E}'(M') \subset \mathcal{E}'(N') \quad \Longrightarrow \quad \mathcal{E}'(M')^{+} \setminus \mathcal{E}'(N') \neq \emptyset,$$

i.e. there exist right-hand side germs from $\mathcal{E}'(M')^{+}$ *which do not extend to germs from* $\mathcal{E}'(N')$*.*

Remark 3. Theorem 2 may be related to the research by M. Langenbruch [10] on the extension of ultradifferentiable functions in several variables, which is principally focused on the non-quasianalytic case, which seems to be more difficult in this context. His extension problem is, roughly speaking, as follows:

Given two compact convex subsets K, K_1 *of* $\mathbb{R}^m$ *such that* $\mathrm{int}\,(K) \neq \emptyset$ *or* $K = \{0\}$ *and* $K \subset \mathrm{int}\,(K_1)$*, characterize the sequences of positive numbers* M *and* N *such that every function from the class* $\mathcal{E}(K, M)$ *extends to a function from* $\mathcal{E}(K_1, N)$*.*

M. Langenbruch applies, however, different methods and techniques in comparison with V. Thilliez. In particular, his approach is based on the theory of Fourier transform and plurisubharmonic functions.

On the other hand, Thilliez's approach relies on Grothendieck's version of the open mapping theorem (cf. [7], Chap. 4, Part I, Theorem 2 or [13], Part IV, Chap. 24) and Runge approximation. It also enables the formulation of the non-extendability theorem for quasianalytic function germs on a compact convex subset $K \subset \mathbb{R}^m$ with $0 \in K$.

Nevertheless, in order to construct our counter-example in the next section, we need a refinement of the above non-extendability theorem, stated below. Thilliez's proof can be adapted mutatis mutandis. We shall outline it for the reader's convenience. Consider an increasing countable family $M'^{(p)}$, $p \in \mathbb{N}$, of log-convex sequences, i.e.

$$1 = M_0'^{(p)} \leq M_1'^{(p)} \leq M_2'^{(p)} \leq M_3'^{(p)} \leq \dots \quad \text{for all} \quad p \in \mathbb{N}.$$

Then we receive an ascending sequence of local rings

$$\mathcal{E}'(M'^{(1)}) \subset \mathcal{E}'(M'^{(2)}) \subset \mathcal{E}'(M'^{(3)}) \subset \dots$$

such that $\mathcal{E}'(M'^{(p)})$ is dominated by $\mathcal{E}'(M'^{(q)})$ for all $p, q \in \mathbb{N}$ with $p \leq q$.

Theorem 2*. *If every local ring $\mathcal{E}'(M'^{(p)})$ is quasianalytic, then*

$$\mathcal{O} \subsetneq \mathcal{E}'(M') \subset \bigcup_{p\in\mathbb{N}} \mathcal{E}'(M'^{(p)}) \implies \mathcal{E}'(M')^+ \setminus \bigcup_{p\in\mathbb{N}} \mathcal{E}'(M'^{(p)}) \neq \emptyset.$$

Before proving Theorem 2*, we adopt the following notation. For a smooth function f on an interval $I \subset \mathbb{R}$ and $r > 0$, put

$$\|f\|_{M',I,r} := \left\{ \sup \frac{|f^{(n)}(x)|}{r^n n! M_n} : \ n \in \mathbb{N}, \ x \in I \right\}.$$

For $k \in \mathbb{N}$, $k > 0$, let $B_k(M')$ or $B_k(M')^+$, respectively, denote the Banach space with norm

$$\| \cdot \|_{M',[-1/k,1/k],k} \quad \text{or} \quad \| \cdot \|_{M',[0,1/k],k},$$

of those smooth functions on the interval $[-1/k, 1/k]$ or $[0, 1/k]$) such that

$$\|f\|_{M',[-1/k,1/k],k} < \infty \quad \text{or} \quad \|f\|_{M',[0,1/k],k} < \infty, \quad \text{respectively.}$$

Because the canonical embeddings

$$B_k(M') \hookrightarrow B_l(M') \quad \text{and} \quad B_k(M')^+ \hookrightarrow B_l(M')^+, \quad k, l \in \mathbb{N}, \ k \leq l,$$

are compact linear operators, one can endow the local rings $\mathcal{E}'(M')$ and $\mathcal{E}'(M')^+$ with the inductive topologies. Similarly, the countable union of local rings $\bigcup_{p\in\mathbb{N}} \mathcal{E}'(M'^{(p)})$ is the inductive limit of the sequence $B_k(M'^{(k)})$, $k \in \mathbb{N}$, $k > 0$, of Banach algebras. Further, the local ring

$$\bigcup_{p\in\mathbb{N}} \mathcal{E}'(M'^{(p)}) \cap \mathcal{E}'(M')^+$$

is the inductive limit of the sequence

$$B_k(M'^{(k)}) \cap B_k(M')^+, \quad k \in \mathbb{N}, \ k > 0,$$

of Banach algebras with norms

$$\|f\|_k := \max \{ \|f\|_{M'^{(k)},[-1/k,1/k],k} \, , \ \|f\|_{M',[0,1/k],k} \}.$$

Clearly, the restriction operator

$$R : \bigcup_{p\in\mathbb{N}} \mathcal{E}'(M'^{(p)}) \cap \mathcal{E}'(M')^+ \longrightarrow \mathcal{E}'(M')^+$$

is continuous and injective by quasianalyticity.

To proceed with *reduction ad absurdum*, suppose that R is surjective. By Grothendieck's version of the open mapping theorem (cf. [7], Chap. 4, Part I or [13], Part IV, Chap. 24), R is a homeomorphism. Next, by Grothendieck's factorization theorem (*loc.cit.*), for each $k \in \mathbb{N}$ there is an $l \in \mathbb{N}$ and a constant $C > 0$ such that

$$R\left(B_l(M'^{(l)})\right) \supset R\left(B_l(M'^{(l)}) \cap B_l(M')^+\right) \supset B_k(M')^+,$$

and

$$\|f\|_{M'^{(l)},[-1/l,1/l],l} \;\leq\; \|R^{-1}f\|_l \;\leq\; C\|f\|_{M',[0,1/k],k}$$

for all $f \in B_k(M')^+$. In particular, there is an $l \in \mathbb{N}$ and a constant $A > 0$ such that

$$R\left(B_l(M'^{(l)})\right) \supset B_1(M')^+,$$

and

$$\|f\|_{M'^{(l)},[-1/l,1/l],l} \;\leq\; A\|f\|_{M',[0,1],1}$$

for all $f \in B_1(M')^+$. In particular,

$$\left|P\left(-\frac{1}{l}\right)\right| \leq A\|P\|_{M',[0,1],1}$$

for every polynomial $P \in \mathbb{C}[x]$. Put

$$W := \left\{z \in \mathbb{C} :\; \mathrm{dist}\,(z, [0,1]) \leq \frac{1}{2l}\right\} \quad \text{and} \quad B := \sup\left\{\frac{(2l)^n}{M'_n} \,:\; n \in \mathbb{N}\right\} < \infty;$$

the last inequality holds because $\mathcal{O} \subsetneq \mathcal{E}'(M')$ whence

$$\sup\left\{\sqrt[n]{M'_n} :\; n \in \mathbb{N}\right\} = \infty.$$

It follows from Cauchy's inequalities that

$$\sup\left\{\left|P^{(n)}(x)\right| : x \in [0,1]\right\} \leq n!(2l)^n \sup\left\{|P(x)| : x \in W\right\},$$

and hence

$$\|P\|_{M',[0,1],1} \leq B \sup \{|P(x)| : x \in W\}.$$

Consequently,

$$\left| P\left(-\frac{1}{l}\right)\right| \leq AB \sup \{|P(x)| : x \in W\}$$

for every polynomial $P \in \mathbb{C}[x]$. But, by virtue of Runge approximation, there exists a sequence of polynomials $P_\nu \in \mathbb{C}[x]$ which converges uniformly to 0 on W, and to 1 for $x = -1/l$. This contradicts the above estimate, and thus the theorem follows.

4. Construction of a counter-example. Now, we can readily give a counter-example indicating that quasianalytic structures, unlike the classical structure $\mathbb{R}_{an}$, may not admit quantifier elimination in the language augmented merely by the reciprocal function $1/x$. The example we construct is a plane curve through $0 \in \mathbb{R}^2$ which is definable in the quasianalytic structure corresponding to the log-convex sequence

$$M' = (M'_n), \quad M'_n := \frac{1}{(\log\log 3)^3} \cdot (\log\log(n+3))^{(n+3)};$$

this sequence determines a quasianalytic class closed under derivatives (cf. the example from the end of Section 2). By Theorem 2*, we can take a function germ

$$f \in \mathcal{E}'(M')^+ \setminus \bigcup_{p \text{ odd}} \mathcal{E}'(M'^{(p)}).$$

Let $V \subset \mathbb{R}^2$ be the graph of a representative of this germ in a right-hand side neighbourhood $[0, \varepsilon]$.

To proceed with *reductio ad absurdum*, suppose V is given by a term in the language augmented by the reciprocal function $1/x$. Then there would exist a rectilinearization of this term via a finite sequence of blowings-up at points (see [16] and [17], Section 4, Remark 5). Consequently, the germ of V at zero would be contained in the image $\varphi([-\delta, \delta])$, where

$$\varphi = (\varphi_1, \varphi_2) : [-\delta, \delta] \longrightarrow \mathbb{R}^2, \quad \varphi(0) = 0,$$

is a Q-analytic homeomorphism. But then the order of φ_1 at zero must be odd, and thus the set V would have a parametrization near zero of the form

$(\xi^p, g(\xi))$, where p is odd and g is a Q-analytic function at zero. Hence and by Theorem 1,

$$f(x) = g(x^{1/p}) \in \mathcal{E}'(I, M'^{(p)}),$$

which is a contradiction.

Remark. In view of Puiseux's theorem for definable functions (cf. [18], Section 2), every smooth definable function germ in one variable belongs to $\mathcal{E}'(M'^{(p)})$ for some positive integer p. Therefore the structure under study will not admit quantifier elimination, even considered with the richer language of restricted definable quasianalytic functions augmented by the reciprocal function $1/x$.

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