Regular separation with parameter

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I Introduction

By a result of Łojasiewicz any two complex, locally analytic sets $X, Y \subset \mathbb{C}^m$ are regularly separated at any point $a \in X \cap Y$, i.e. there is a neighbourhood $U \ni a$ and constants $\alpha, c > 0$ such that

$$\mathrm{dist}(z, X) + \mathrm{dist}(z, Y) \geqslant c \cdot \mathrm{dist}(z, X \cap Y)^\alpha, \; z \in U.$$

In subanalytic geometry there is also the following theorem of Łojasiewicz and Wachta (1982):

Theorem. (Regular separation with parameter)

If $X, Y \subset \mathbb{R}^m_z \times \mathbb{R}^n_w$ are subanalytic bounded sets, then there is an exponent $\alpha > 0$ such that for any $z \in \pi_m(X \cap Y)$,

$$\mathrm{dist}(w, Y^z) \geqslant c(z) \cdot \mathrm{dist}(w, X^z \cap Y^z)^\alpha \text{ for } w \in X^z,$$

with some $c(z) > 0$ depending in general on z.

It obviously implies a *complex counterpart*. One would like, however, to prove it using only complex analytic geometry tools and obtain also some *bound* on the uniform exponent.

II Main theorem

Let $X, Y \subset \mathbb{C}^m_z \times \mathbb{C}^n_w$ be locally analytic sets such that $0 \in X \cap Y$. Put as earlier $X^z := \{w \in \mathbb{C}^n \mid (z, w) \in X\}$. Suppose that $X \cap Y$ has pure dimension $k \geqslant 1$ and let $\pi(z, w) = z$ be proper on it, with multiplicity $\mu := \limsup_{z \to 0} \#(\pi|_{X \cap Y})^{-1}(z)$ at zero.

(1) Under these assumptions there is a neighbourhood $U \times V$ of zero and an exponent $s \geqslant 1$ such that for all $z \in \pi(X \cap Y) \cap U$

$$\mathrm{dist}(w, Y^z) \geqslant c(z)\mathrm{dist}(w, X^z \cap Y^z)^s, \ w \in X^z \cap V,$$

with some $c(z) > 0$, which may be chosen independent of z, whenever $0 \in \mathrm{Reg}\pi(X \cap Y)$.

(2) Moreover, $s \leqslant \mu \cdot \deg_0 \pi(X \cap Y) \cdot \mathcal{L}_0(X, Y)$, where $\mathcal{L}_0(X, Y)$ is the Łojasiewicz separation exponent of X, Y at zero.

By a result of Cygan we know that $\mathcal{L}_0(X, Y) \leqslant \deg_0(X \bullet Y)$, where $X \bullet Y$ denotes the cycle of intersection of X, Y in the sense of Tworzewski.

III C-holomorphic functions

Let $A \subset \mathbb{C}^m$ be a locally analytic set. The following definition is due to Remmert:

Definition. A mapping $f\colon A \to \mathbb{C}^n$ is called c-holomorphic if it is continuous and its restriction to $\operatorname{Reg} A$ is holomorphic.

The most important feature of c-holomorphic mappings, basic for all geometric considerations involving them, is given in the following theorem:

Theorem. A mapping $f\colon A \to \mathbb{C}^n$ is c-holomorphic iff it is continuous and its graph $\Gamma_f := \{(x, f(x)) \mid x \in A\}$ is a locally analytic subset of $\mathbb{C}^m \times \mathbb{C}^n$.

Moreover, each c-holomorphic mapping satisfies the Łojasiewicz inequality: for any $a \in f^{-1}(0)$ there is a neighbourhood $U \ni a$ and constants $\alpha, c > 0$ such that $|f(x)| \geqslant c \cdot \operatorname{dist}(x, f^{-1}(0))^\alpha$, when $x \in A \cap U$.

IV Nullstellensatz for c-holomorphic functions

Also an effective Nullstellensatz of the Płoski-Tworzewski type is valid in the c-holomorphic setting:

Nullstellensatz. Suppose A has pure dimension $k \geqslant 1$, $f = (f_1, \ldots, f_n)\colon A \to \mathbb{C}^n$ and $g\colon A \to \mathbb{C}$ are c-holomorphic, and $g^{-1}(0) \supset f^{-1}(0) \ni 0$.

If either $n \geqslant k$ and $f^{-1}(0) = \{0\}$, or $n \leqslant k$ and $f^{-1}(0)$ has pure dimension $k - n$, then in a neighbourhood of zero in A there is

$$g^{\deg_0 Z_f} = \sum_{j=1}^{n} h_j f_j$$

with some c-holomorphic functions $h_j\colon A \to \mathbb{C}$, $j = 1, \ldots, n$.

Here $Z_f := \Gamma_f \bullet (\mathbb{C}^m \times \{0\}^n)$ is the cycle of zeroes of f.

A more general but ineffective (i.e. we have *some* exponent to do with) c-holomorphic Nullstellensatz is valid on *locally irreducible* sets (with no assumption on their dimension).

V Łojasiewicz inequality with parameter

Let A be a pure k-dimensional analytic subset of some open set $U \times V \subset \mathbb{C}^m_z \times \mathbb{C}^n_w$ ($k \geqslant 1$). Let $f\colon A \to \mathbb{C}^p$ be c-holomorphic non-constant and such that $f^{-1}(0)$ has pure dimension $r \leqslant m$ and $\pi(z,w) = z$ is proper on it. We denote by μ_a its multiplicity at a point $a \in f^{-1}(0)$.

(1) Under these assumptions, for all $a \in f^{-1}(0)$ there is a neighbourhood $G \times H \ni a$ and an exponent $\alpha \geqslant 1$ such that for any $z \in G$,

$$|f(z,w)| \geqslant c(z) \cdot \mathrm{dist}(w, (f^{-1}(0))^z)^\alpha,\ w \in H \cap A^z,$$

with some $c(z) > 0$, which may be chosen independent of z, whenever $\pi(a) \in \mathrm{Reg}\pi(f^{-1}(0))$.

(2) There is $\alpha \leqslant \mu_a \cdot \deg_{\pi(a)}\pi(f^{-1}(0)) \cdot \mathcal{L}_a(\Gamma_f, A \times \{0\}^n)$.

If $k = m+n$, then $\mathcal{L}_a(\Gamma_f, A \times \{0\}^n)$ is just the Łojasiewicz exponent of f at a.

(3) If, moreover, $m = r = k - p$, then $\alpha \leqslant \mu_a \cdot \deg_a Z_f$.