Effective methods in algebraic and analytic geometry IV

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“Reduction method” in multivariate polynomial interpolation

Marcin Dumnicki

Harbourne–Hirschowitz Conjecture

Definition. Let $p_1, \ldots, p_r \in \mathbb{P}^2$ be points in general position, let $d, m_1, \ldots, m_r \in \mathbb{N}$. Define the linear system of *curves of degree* d *with multiplicities* $m_1, \ldots, m_r$ *in general position*

$$\mathcal{L}_d(m_1, \ldots, m_r) = |dH|(-\sum_{i=1}^{r} m_i p_i),$$

where H denotes a general line in $\mathbb{P}^2$.

Question. $\dim \mathcal{L}_d(m_1, \ldots, m_r) = ?$

We can guess that
$\dim \mathcal{L}_d(m_1, \dots, m_r) = \operatorname{edim} \mathcal{L}_d(m_1, \dots, m_r)$, where

$$\operatorname{edim} \mathcal{L}_d(m_1, \dots, m_r) = \max\left\{ \binom{d+2}{2} - \sum_{i=1}^{r} \binom{m_i+1}{2} - 1, -1 \right\}.$$

Example. Consider $L = \mathcal{L}_2(2,2)$. Then $\operatorname{edim} L = -1$. Let p_1, p_2 be points in general position, let H be the line passing through p_1 and p_2. Then $2H$ (or, looking at the equation, H^2) belongs to L.

Conjecture (Harbourne–Hirschowitz). Let $\Pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be a blow-up of $\mathbb{P}^2$ in r points in general position with exceptional divisors $E_1, \ldots, E_r$. Let $L = \mathcal{L}_d(m_1, \ldots, m_r)$. Then $\dim L > \operatorname{edim} L$ (such system will be called *special*) if and only if there exists an irreducible curve $C \subset \mathbb{P}^2$ such that its proper transform $\widetilde{C}$ by Π satisfies

(1) $\widetilde{C}^2 = -1$,
(2) $\widetilde{L}.\widetilde{C} < -1$, where $\widetilde{L} = |d\Pi^*(H) - \sum_{i=1}^r m_i E_i|$.

Conjecture (Nagata). Let $C \in \mathbb{P}^2$ be a curve, let $m_1, \ldots, m_r \in \mathbb{N}$, $r \geq 9$, let $p_1, \ldots, p_r \in \mathbb{P}^2$ be points in general position. If $\operatorname{mult}_{p_i} C \geq m_i$ for $i = 1, \ldots, r$, then

$$\deg C \geq \sqrt{\frac{1}{r}} \sum_{i=1}^{r} m_i.$$

Methods

- many “ad hoc” methods, both computer algebra and algebraic geometry (vanishing theorems) can be applied;
- blowing-up of the plane along a line (C. Ciliberto and R. Miranda);
- “reduction method” (M.D.).

Results

- a lot of partial results;
- multiplicities bounded by 7 (S. Yang, 2004);
- multiplicities bounded by 11 (M.D., W. Jarnicki, 2005);
- homogeneous multiplicities bounded by 20 (C. Ciliberto, R. Miranda, 2000);
- homogeneous multiplicities bounded by 42 (M.D., 2006).

Linear systems generated by monomials

Definition. Let B be a finite set of monomials, let $p_1, \dots, p_r \in \mathbb{K}^2$, let $m_1, \dots, m_r \in \mathbb{N}$. Define

$$\mathcal{L}_B(m_1 p_1, \dots, m_r p_r) = \{f = \sum_{\beta \in B} c_\beta X^\beta \mid \operatorname{mult}_{p_i} f \geq m_i, i = 1, \dots, r\}.$$

Define *the dimension* and *the expected dimension* of the *space of curves* $\mathcal{L}_B(m_1, \dots, m_r)$ as follows

$$\dim \mathcal{L}_B(m_1, \dots, m_r) = \min_{\{p_i\} \subset \mathbb{K}^2} \dim \mathcal{L}_B(m_1 p_1, \dots, m_r p_r) - 1,$$

$$\operatorname{edim} \mathcal{L}_B(m_1, \dots, m_r) = \max\{\#B - \sum_{i=1}^{r} \binom{m_i + 1}{2} - 1, -1\}$$

Definition. We say that system of curves $L = \mathcal{L}_B(m_1, \ldots, m_r)$ is *non-special* if $\dim L = \operatorname{edim} L$.

We can identify $\mathcal{L}_d(m_1, \ldots, m_r)$ with $\mathcal{L}_B(m_1, \ldots, m_r)$ where $B = \{X^\beta : |\beta| \leq d\}$.

Let

$$\varphi_{(j,\alpha)} : \mathbb{K}[X] \ni f \mapsto \frac{\partial^{|\alpha|} f}{\partial X^{\alpha}}(P_j) \in \mathbb{K}[P_j^X, P_j^Y].$$

Consider the following map

$$\varphi = (\varphi_{(j,\alpha)}|_{\mathrm{span}(B)})_{|\alpha|<m_j, j=1,\ldots,r}.$$

We have

$$\mathcal{L}_B(m_1, \ldots, m_r) \text{ “=” } \ker \varphi.$$

We can compute $\dim \mathcal{L}_B(m_1, \ldots, m_r)$ as

$$\#B - \operatorname{rank} \mathcal{M}_B(m_1, \ldots, m_r) - 1,$$

where

$$\mathcal{M}_B(m_1, \ldots, m_r) = \left[\varphi_{(j,\alpha)}(X^\beta)\right]^{\beta \in B}_{|\alpha| < m_j, j=1,\ldots,r}.$$

Reduction method

Let B be a finite set of monomials, $m_1, \ldots, m_r \in \mathbb{N}$, let $B = B_1 \cup B_2$, $B_1 \cap B_2 = \varnothing$.

Consider the matrix $\mathcal{M}_B(m_1, \ldots, m_r)$ in the following form:

$$\left[\begin{array}{c|c} \mathcal{M}_{B_1}(m_1, \ldots, m_{r-1}) & K_1 \\ \hline K_2 & \mathcal{M}_{B_2}(m_r) \end{array}\right].$$

Theorem. Let B be a finite set of monomials, $m_1, \ldots, m_r \in \mathbb{N}$, let $B = B_1 \cup B_2$, $B_1 \cap B_2 = \varnothing$, $\#B_2 = \binom{m_r+1}{2}$. Assume that the following hold:

(1) the system $\mathcal{L}_{B_2}(m_r)$ is non-special,
(2) there is no $P \subset B$ satisfying $\#P = \#B_2$, $P \neq B_2$, $\mathcal{L}_P(m_r)$ being non-special and

$$\sum_{\alpha \in P} \alpha = \sum_{\alpha \in B_2} \alpha.$$

Then $\dim \mathcal{L}_B(m_1, \ldots, m_r) \leq \dim \mathcal{L}_{B_1}(m_1, \ldots, m_{r-1})$.

Theorem. Let $\#B = \binom{m+1}{2}$. Then $\mathcal{L}_B(m)$ is non-special if and only if B does not lie on a curve of degree $m-1$.

“Reduction method”

Reducing the “tail”:

- complicated proof,
- algorithmic and easy to use,
- computation of dimension of $\mathcal{L}_d(m^{\times k})$ takes $\Theta((m^2)^3)$ instead of $\Theta((km^2)^3)$ steps,
- very often insufficient.

“Cutting”:

- easy to prove
- hard to find an algorithm,
- possible connection with symplectic packing,
- provides nice proofs,
- possible “rescaling”.