On the Uniformity of Zero-Dimensional Complete Intersections

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\( \mathbb{X} \) is called \( (i, j) \)-uniform iff \( \text{HF}_\mathbb{Y}(j) = \text{HF}_\mathbb{X}(j) \) for all \( \mathbb{Y} \subset \mathbb{X} \) consisting of \( s - i \) points
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1. $X$ is $(1, a_X)$-uniform iff $X$ has the **Cayley-Bacharach property** (i.e. every hypersurface of degree $a_X$ which passes through all points of $X$ but one passes also through the last point).
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2. $X$ is $(i, j)$-uniform for all $1 \leq i < s$ and $1 \leq j \leq a_X$ iff $X$ is in **uniform position** (e.g. if $X$ is the generic hyperplane section of an irreducible curve)
**Special Cases of (i, j)-Uniformity**

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3. If $X$ spans $\mathbb{P}^n$ then $X$ is $(s - n - 1, 1)$-uniform iff $X$ is in **linearly general position** (i.e. any $n + 1$ points of $X$ span $\mathbb{P}^n$)
The Region of Uniformity

If \( X \) is \((i, j)\)-uniform then \( X \) is also \((i - 1, j)\)-uniform and \((i, j - 1)\)-uniform.
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2. The General Cayley-Bacharach Conjecture

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**Conjecture CB12.** Let \( Y \subseteq \mathbb{P}^n \) be a subscheme of a zero-dimensional complete intersection of hypersurfaces of degrees \( d_1 \leq \cdots \leq d_n \). If \( Y \) fails to impose independent conditions on hypersurfaces of degree \( m \), then we have \( \text{deg}(Y) \geq e \cdot d_t \cdot d_{t+1} \cdots d_n \) where \( t \) and \( e \) are defined by the relations

\[
\sum_{i=t}^{n} (d_i - 1) \leq m + 1 < \sum_{i=t-1}^{n} (d_i - 1) \quad \text{and} \quad e = m + 2 - \sum_{i=t}^{n} (d_i - 1)
\]

(Notice that we have corrected the definition of \( e \) given in [EGH2], which was an obvious misprint.)
Hilbert Function Version of the GCB Conjecture
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Let $X \subseteq \mathbb{P}^n_K$ be a zero-dimensional complete intersection of type $(d_1, \ldots, d_n)$, where $d_1 \leq \cdots \leq d_n$. It is conjectured that:

- $(\mathcal{I}_1)$ Let $1 \leq e < d_{n-1}$ and $Y \subseteq X$ be such that
  \[ \Delta \text{HF}_Y(e + d_n - 2) \neq 0. \]
  Then we have $\deg(Y) \geq e \cdot d_n$.

- $(\mathcal{I}_2)$ Let $1 \leq e < d_{n-2}$ and $Y \subseteq X$ be such that
  \[ \Delta \text{HF}_Y(e + d_{n-1} + d_n - 3) \neq 0. \]
  Then we have $\deg(Y) \geq e \cdot d_{n-1} \cdot d_n$.

  :  

- $(\mathcal{I}_{n-1})$ Let $1 \leq e < d_1$ and $Y \subseteq X$ be such that
  \[ \Delta \text{HF}_Y(e + d_2 + \cdots + d_n - n) \neq 0. \]
  Then we have $\deg(Y) \geq e \cdot d_2 \cdots d_n$. 

The GCB Conjecture can be translated to a conjecture about the region of uniformity of a 0-dimensional complete intersection $X \subseteq \mathbb{P}^n$ of type $(d_1, \ldots, d_n)$, where $d_1 \leq \cdots \leq d_n$. It is conjectured that:
The GCB Conjecture can be translated to a conjecture about the region of uniformity of a 0-dimensional complete intersection \(X \subseteq \mathbb{P}^n\) of type \((d_1, \ldots, d_n)\), where \(d_1 \leq \cdots \leq d_n\). It is conjectured that:

- **(U\(_1\))** Let \(1 \leq e < d_{n-1}\). Then \(X\) is
  \((e \cdot d_n - 1, a_X - e - d_n + 3)\)-uniform.

- **(U\(_2\))** Let \(1 \leq e < d_{n-2}\). Then \(X\) is
  \((e \cdot d_{n-1} d_n - 1, a_X - e - d_{n-1} - d_n + 4)\)-uniform.

- **(U\(_{n-1}\))** Let \(1 \leq e < d_1\). Then \(X\) is
  \((e \cdot d_2 \cdots d_n - 1, a_X - e - d_2 - \cdots - d_n + n + 1)\)-uniform.

Here we have \(a_X = d_1 + \cdots + d_n - n - 1\), of course.
Examples for the Uniformity Conjecture

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The dotted region is the conjectured region of uniformity.
Example 2. Let $X \subset \mathbb{P}^3$ be a complete intersection of type $(2, 3, 5)$. 
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In the following we shall prove special cases of a slightly stronger conjecture. W.l.o.g. let $x_0 \in R$ be a non-zerodivisor. The ring $A = R/(x_0)$ is called the Artinian reduction of $R$. 
Let $P = K[x_1, \ldots, x_n]$ be standard graded, let $(f_1, \ldots, f_n) \in P^n$ be a homogeneous regular sequence, let $I = \langle f_1, \ldots, f_s \rangle$, and let $d_i = \deg(f_i)$ for $i = 1, \ldots, n$. Assume that $d_1 \leq \cdots \leq d_n$, and let $J \supseteq I$ be an ideal. It is conjectured that:

1. **(A_1)** Let $1 \leq e < d_{n-1}$, and suppose that $HF_{P/J}(e + d_n - 2) \neq 0$. Then we have $\dim_K(P/J) \geq e \cdot d_n$.

2. **(A_2)** Let $1 \leq e < d_{n-2}$, and suppose that $HF_{P/J}(e + d_{n-1} + d_n - 3) \neq 0$. Then $\dim_K(P/J) \geq e \cdot d_{n-1} \cdot d_n$.

3. **(A_{n-1})** Let $1 \leq e < d_1$, and suppose that $HF_{P/J}(e + d_2 + \cdots + d_n - n) \neq 0$. Then we have $\dim_K(P/J) \geq e \cdot d_2 \cdots d_n$. 

3. The First Interval \((A_1)\)

**Proposition.** Let \(A = P/I\) be a zero-dimensional complete intersection of type \((d_1, \ldots, d_n)\), where \(d_1 \leq \cdots \leq d_n\), let 
\[1 \leq e \leq d_{n-1},\] and let \(J \supseteq I\) be a homogeneous ideal in \(P\) such that 
\[HF_{P/J}(e + d_n - 2) \neq 0.\] Then we have 
\[\dim_K(P/J) \geq e d_n.\]

In other words, the AU Conjecture is true in the interval \(A_1\) and for 
\(e = 1\) in the interval \(A_2\).
Proposition. Let $A = P/I$ be a zero-dimensional complete intersection of type $(d_1, \ldots, d_n)$, where $d_1 \leq \cdots \leq d_n$, let $1 \leq e \leq d_{n-1}$, and let $J \supseteq I$ be a homogeneous ideal in $P$ such that $HF_{P/J}(e + d_n - 2) \neq 0$. Then we have $\dim_K(P/J) \geq e d_n$.

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Method of Proof. (1) Show that $HF_{P/J}(d_n - 1) \geq e$. 
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In other words, the \(AU\) Conjecture is true in the interval \(A_1\) and for \(e = 1\) in the interval \(A_2\).

**Method of Proof.** (1) Show that \(HF_{P/J}(d_n - 1) \geq e\).

Let \(i \geq d_n - 1\) and \(HF_{P/J}(i) < e\). Then **Macaulay’s Growth Theorem** shows \(HF_{P/J}(i + 1) \leq HF_{P/J}(i)\).

If there is an \(i \in \{d_n - 1, \ldots, e + d_n - 3\}\) for which equality holds, we have a case of maximal growth in Macaulay’s Growth Theorem.
Then Gotzmann’s Persistence Theorem implies that \( \tilde{J} = \langle J_{\leq i+1} \rangle \) satisfies \( \text{HF}_{P/\tilde{J}}(j) = \text{HF}_{P/\tilde{J}}(i) \) for all \( j \geq i \).

Therefore we get \( \dim(P/\tilde{J}) = 1 \) which contradicts \( I \subseteq \tilde{J} \) since all generators of \( I \) have degree \( \leq d_n \leq i + 1 \).
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Therefore we get \( \dim(P/\tilde{J}) = 1 \) which contradicts \( I \subseteq \tilde{J} \) since all generators of \( I \) have degree \( \leq d_n \leq i + 1 \).

(2) Next we show that \( \text{HF}_{P/J} \) is greater or equal to

\[ H(i) \]

\[ 1 \quad e-1 \quad d_n-1 \quad e+d_n-1 \]
This follows again from Macaulay’s Growth Theorem.
Finally, counting dimensions, we get $\dim_K P/J \geq ed_n$. 
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**Corollary.** The Artinian Uniformity Conjecture holds for \( n = 2 \).

*In particular, the General Cayley-Bacharach Conjecture holds in \( \mathbb{P}^2 \).*
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**Proposition.** Let \(A = P/I\) be a zero-dimensional complete intersection of type \((d_1, \ldots, d_n)\), where \(d_1 \leq \cdots \leq d_n\), let \(1 \leq e \leq d_1 - 1\), and let \(J \supseteq I\) be a homogeneous ideal in \(P\) such that \(\text{HF}_{P/J}(e + d_2 + \cdots + d_n - n) \neq 0\). Then we have

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\dim_K(P/J) \geq e d_2 \cdots d_n.
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In particular, the AU Conjecture holds true in the last interval.
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**Proposition.** Let \(A = P/I\) be a zero-dimensional complete intersection of type \((d_1, \ldots, d_n)\), where \(d_1 \leq \cdots \leq d_n\), let \(1 \leq e \leq d_1 - 1\), and let \(J \supseteq I\) be a homogeneous ideal in \(P\) such that \(\text{HF}_{P/J}(e + d_2 + \cdots + d_n - n) \neq 0\). Then we have \(\dim_K(P/J) \geq e d_2 \cdots d_n\).

In particular, the AU Conjecture holds true in the last interval.

**Method of Proof:** Consider the linked ideal

\[
J' = \{ f \in P \mid f \cdot J \subseteq I \}.
\]

Then \(\text{HF}_{P/J}(e + d_2 + \cdots + d_n - n) \neq 0\) implies \(J'_{d_1 - e} \neq 0\).
Thus the **depth sequence** of $J'$ is componentwise less or equal to $(d_1 - e, d_2, \ldots, d_n)$ and $J'$ contains a complete intersection of type $(d_1 - e, d_2, \ldots, d_n)$. 
Thus the depth sequence of $J'$ is componentwise less or equal to $(d_1 - e, d_2, \ldots, d_n)$ and $J'$ contains a complete intersection of type $(d_1 - e, d_2, \ldots, d_n)$.

**Corollary 1.** The Artinian Uniformity Conjecture holds for $n = 3$. In particular, the General Cayley-Bacharach Conjecture holds in $\mathbb{P}^3$. 
Thus the **depth sequence** of $J'$ is componentwise less or equal to $(d_1 - e, d_2, \ldots, d_n)$ and $J'$ contains a complete intersection of type $(d_1 - e, d_2, \ldots, d_n)$.

**Corollary 1.** The Artinian Uniformity Conjecture holds for $n = 3$. In particular, the General Cayley-Bacharach Conjecture holds in $\mathbb{P}^3$.

**Corollary 2.** The Artinian Uniformity Conjecture is true for $n = 4$ and $d_1 = d_2 = 2$. In particular, the General Cayley Bacharach Conjecture holds true for complete intersections of type $(2, 2, d_3, d_4)$ in $\mathbb{P}^4$, where $2 \leq d_3 \leq d_4$. 
Further Results
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Proposition. The Artinian Uniformity Conjecture is true in the interval \((A_{n-2})\) for all \(d_1 \leq e < d_2\).

The case \(e < d_1\) is still open.
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**Proposition.** The Artinian Uniformity Conjecture is true in the interval \((A_{n-2})\) for all \(d_1 \leq e < d_2\).

The case \(e < d_1\) is still open.

**Proposition.** The General Cayley-Bacharach Conjecture is true for complete intersections of type \((2, d_2, d_3, d_4)\) in \(\mathbb{P}^4\) if the quadric is reducible.
Let $K = \mathbb{F}_q$ be a finite field. The image of the map

$$\Phi_j : \mathbb{R}_j \rightarrow K^s$$

defined by $\Phi_j(f) = (f(p_1), \ldots, f(p_s))$ is called the $j^{th}$ generalized Reed-Muller code (or the $j^{th}$ evaluation code) $C_j(X)$ associated to $X$. 

5. An Application to Coding Theory
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Let $K = \mathbb{F}_q$ be a finite field. The image of the map

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**Proposition.** The minimal distance of $C_j(X)$ is

$$d = 1 + \max\{i \mid X \text{ is } (i, j)-\text{uniform}\}.$$
A zero-dimensional subscheme $X \subset \mathbb{P}^n$ is called \textbf{level} if the Artinian reduction $\overline{R} = R/(x_0)$ of its homogeneous coordinate ring $R$ satisfies $\text{socle}(\overline{R}) = \overline{R}_{a_X+1}$. 
A zero-dimensional subscheme $X \subset \mathbb{P}^n$ is called **level** if the Artinian reduction $\overline{R} = R/(x_0)$ of its homogeneous coordinate ring $R$ satisfies $\text{socle}(\overline{R}) = \overline{R}_{a_X+1}$.

**Proposition.** Let $X \subseteq \mathbb{P}^n_K$ be a level scheme, and let $i \in \{1, \ldots, a_X\}$. Then $X$ is $(i, a_X + 1 - i)$-uniform.
A zero-dimensional subscheme \( X \subset \mathbb{P}^n \) is called level if the Artinian reduction \( \overline{R} = R/(x_0) \) of its homogeneous coordinate ring \( R \) satisfies \( \text{socle}(\overline{R}) = \overline{R}_{a_X + 1} \).

**Proposition.** Let \( X \subseteq \mathbb{P}^n_K \) be a level scheme, and let \( i \in \{1, \ldots, a_X\} \). Then \( X \) is \((i, a_X + 1 - i)\)-uniform.

**Example 1.** Let \( X \subset \mathbb{P}^2 \) be a complete intersection of type \((5, 5)\).
The dotted region of uniformity is due to the level scheme property.
The asterisks mark the region of uniformity coming from GCB.
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**Example 2.** Let \( X \subset \mathbb{P}^3 \) be a complete intersection of type \((3, 3, 4)\).
Example 3. Let $X \subset \mathbb{P}^4$ be a complete intersection of type $(2, 2, 3, 3)$. 


References


Thank you for your attention!