Initial value problems for linear differential equations with a regular singular point

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Abstract

We consider linear differential equations of an arbitrary order $n \geq 1$ defined over an open interval $(0, X)$, $X \in \mathbb{R}$, with a regular singular point at $x = 0$ of order $m \leq n$. We also add initial conditions to that differential equation and consider initial value problems of the form:

$$\begin{cases} (x^m f_0 y)^{(n)} + (x^{m-1} f_1 y)^{(n-1)} + ... + (x f_{m-1} y)^{(n-m+1)} + (f_m y)^{(n-m)} ... + f_n y = h \\ \hfill \text{in } (0, X), \\ y(0) = y_0,\, y'(0) = y_1,\, ..., y^{(n-m-1)}(0) = y_{n-m-1}; \quad y \in \mathcal{C}[0, X], \end{cases}$$

with $y_0, ..., y_{n-m-1} \in \mathbb{R}$. The coefficient functions f_j and h satisfy certain regularity conditions and $f_0 > 0$ in $[0, X]$. When $n > m$ the condition $y \in \mathcal{C}[0, X]$ is superfluous and when $n = m$ there are not initial conditions. Using the convergence of the Liouville-Neumann expansion shown in a previous work for the particular case $n = 2$ and $m = 1$, we give existence and uniqueness theorems for the solution of this initial value problem. It is shown that this problem has a unique solution and the Liouville-Neumann expansion converges to that solution if $|f_j(0)| < (n - m + \delta_{n,m}) f_0(0)$, $j = 1, 2, 3, ..., m$. If we require an extra regularity condition for the solution of the form $y^{(n-m+k)} \in \mathcal{C}[0, X]$, $k = 0, 1, 2, ...$, this new problem has a unique solution and the Liouville-Neumann expansion converges to that solution if $|f_j(0)| < (n - m + k + 1) f_0(0)$, $j = 1, 2, 3, ..., m$.

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