# Hahn-Banach type theorems for normed modules

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Abstract

Let $A$ be a normed algebra, $\mathcal{K}$ some class of left normed $A$-modules. A left normed $A$-module $Z$ is called *extremely $\mathcal{K}$-injective* if, for every $A$-module $Y$ and its submodule $X$, every bounded morphism $X \to Z$ can be extended to a morphism $Y \to Z$ of the same norm. (Thus, $Z$ plays the role of $\mathbb{C}$ in the classical Hahn-Banach theorem).

In the following theorem we consider, as $A$, the algebra $\mathcal{B}(L)$ of all bounded operators on an infinite-dimensional Hilbert space $L$, and, as $\mathcal{K}$, the class of left Ruan modules.

**Theorem 1**. *Let $H$ be an arbitrary Hilbert space, and $L \otimes H$ a Hilbert $A$-module with the outer multiplication $a \cdot (\xi \otimes \eta) := a(\xi) \otimes \eta$. Then such a module is extremely $\mathcal{K}$-injective.*

This theorem, combined with some general facts about Ruan modules, gives, as an easy corollary, Arveson-Wittstock Theorem.

Later Wittstock generalized and strengthened the formulated theorem in several directions. In particular, he proved that, with $A$ and $\mathcal{K}$ as above, every dual to a Ruan module is $\mathcal{K}$-injective.

Turn to the opposite class of commutative algebras. What about modules over one of the simplest, the algebra $c_0$ of vanishing sequences? The following theorem describes extremely $\mathcal{K}$-injective modules within a certain reasonable class of $c_0$-modules. Namely, we call a $c_0$-module $Z$ *homogeneous*, if, for $z', z'' \in Z$, the equalities $\|p^n \cdot z'\| = \|p^n \cdot z''\|; n = 1, 2, \ldots,$ where $p^n = (0, \ldots, 0, 1, 0, \ldots)$, imply $\|z'\| = \|z''\|$.

**Theorem 2**. *Let $\mathcal{K}$ be a class of homogeneous $c_0$-modules, and $Z$ is a non-degenerate homogeneous $c_0$-module. Then the module $Z^*$ is extremely $\mathcal{K}$-injective if, and only if, for every $n = 1, 2, ..,$ the normed space $\{p^n \cdot z; z \in Z\}$ is, up to an isometric isomorphism, a dense subspace of $L_1(\Omega_n)$ for some measure space $\Omega_n$.*

In particular, all $c_0$-modules $l_p; 1 \leq p \leq \infty$ are extremely $\mathcal{K}$-injective.

The condition of the non-degeneracy of $Z$ can not be omitted: $Z := l_\infty$ provides the relevant counter-example.

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