Approximate roots of quasi-ordinary polynomials

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Theorem ([1, Theorem 4.4], [4, Theorem 1])

Let A be a commutative ring that contains $\mathbb{Q}$ as a subring. If $f \in A[Y]$ is monic and $k | \deg f$, then there exists a unique monic polynomial $w \in A[Y]$ of degree $\frac{\deg f}{k}$ such that

$$\deg(f - w^k) < \deg f - \deg w.$$

The polynomial w from the above theorem is said to be the kth approximate root of f and is denoted by $\sqrt[k]{f}$.

Example

- If $f = Y^d + a_{d-1}Y^{d-1} + \cdots + a_0 \in A[Y]$, then $\sqrt[d]{f} = Y + \frac{a_{d-1}}{d}$.
- If $f = Y^4 + a_3Y^3 + a_2Y^2 + a_1Y + a_0 \in A[Y]$, then $\sqrt[2]{f} = Y^2 + \frac{1}{2}a_3Y + \frac{1}{2}a_2 - \frac{1}{8}a_3^2$.

Let $\mathbb{K}$ be an algebraically closed field of characteristic zero.
We say that a polynomial
$f = Y^m + a_{m-1}Y^{m-1} + \cdots + a_0 \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ is

- Weierstrass if $a_i(0) = 0$ for every $i \in \{0, \ldots, m-1\}$;
- quasi-ordinary if its discriminant is of the form $uX_1^{k_1} \cdots X_n^{k_n}$, where u is a unit in the ring $\mathbb{K}[[X_1, \ldots, X_n]]$ and $k_1, \ldots, k_n \in \mathbb{Z}_{\geqslant 0}$.

Remark

(i) Every monic factor of a quasi-ordinary polynomial is quasi-ordinary.

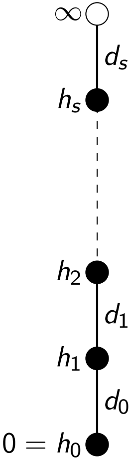
(ii) Every monic factor of a Weierstrass polynomial is Weierstrass.

(iii) In the one-variable case every Weierstrass polynomial is quasi-ordinary.

- f is an irreducible quasi-ordinary Weierstrass polynomial,
- $\mathrm{Zer}\, f = \{\alpha_1, \ldots, \alpha_m\} \subset \mathbb{K}[[X_1^{1/m}, \ldots, X_n^{1/m}]]$
- for $q = (q_1, \ldots, q_n) \in \mathbb{Q}^n_{\geqslant 0}$ we define $\underline{X}^q := X_1^{q_1} \cdots X_n^{q_n}$.
- for $i \neq j$ we have $\alpha_i - \alpha_j = u_{ij}\underline{X}^{\lambda_{ij}}$, where $u_{ij}(0) \neq 0$ and $\lambda_{ij} \in \mathbb{Q}^n_{\geqslant 0}$
- $q = (q_1, \ldots, q_n), q' = (q'_1, \ldots, q'_n) \in \mathbb{Q}^n_{\geqslant 0}$:

$$q \leqslant q' \Leftrightarrow \forall_{i \in \{1,\ldots,n\}}\ q_i \leqslant q'_i$$

- $\{\lambda_{ij} : i \neq j\} \cup \{\infty\} = \{h_1, \ldots, h_s, h_{s+1} = \infty\}$, $h_1 < \cdots < h_s < h_{s+1}$ (the characteristic of f)
- $d_0 := 1$, $d_i := [\mathbb{Z}^n + \mathbb{Z}h_1 + \cdots + \mathbb{Z}h_i : \mathbb{Z}^n]$, $i = 1, \ldots, s$,
- $c_i := \sum_{j=1}^{i} \left(\frac{1}{d_{j-1}} - \frac{1}{d_j}\right) h_j + \frac{1}{d_i} h_i$, $i = 1, \ldots, s$.



The Wall-Eggers tree of an irreducible quasi-ordinary Weierstrass polynomial of characteristic $h_1 < \cdots < h_s < h_{s+1} = \infty$.

Let $f = Y^4 - 2X_1^3X_2^2Y^2 - 4X_1^5X_2^4Y - X_1^7X_2^6 + X_1^6X_2^4$. The polynomial f is quasi-ordinary and irreducible in $\mathbb{C}[[X_1, X_2]][Y]$ with roots

$$\begin{array}{rcl} \alpha_1 & = & X_1^{3/2}X_2 + X_1^{7/4}X_2^{3/2} \\ \alpha_2 & = & X_1^{3/2}X_2 - X_1^{7/4}X_2^{3/2} \\ \alpha_3 & = & -X_1^{3/2}X_2 + \sqrt{-1}X_1^{7/4}X_2^{3/2} \\ \alpha_4 & = & -X_1^{3/2}X_2 - \sqrt{-1}X_1^{7/4}X_2^{3/2} \end{array}$$

and characteristic $h_1 = (\frac{3}{2}, 1)$, $h_2 = (\frac{7}{4}, \frac{3}{2})$. Since

$$\begin{array}{rcl} M_0 & = & \mathbb{Z}^2, \\ M_1 & = & \mathbb{Z}^2 + \mathbb{Z}(\frac{3}{2}, 1), \\ M_2 & = & \mathbb{Z}^2 + \mathbb{Z}(\frac{3}{2}, 1) + \mathbb{Z}(\frac{7}{4}, \frac{3}{2}), \end{array}$$

we have

$$d_0 = 1, \quad d_1 = 2, \quad d_2 = 4.$$

The Wall-Eggers tree of $f = Y^4 - 2X_1^3X_2^2Y^2 - 4X_1^5X_2^4Y - X_1^7X_2^6 + X_1^6X_2^4$.

Let f, g be irreducible quasi-ordinary Weierstrass polynomials, $\mathrm{Zer}\, f = \{\alpha_1, \ldots, \alpha_m\}$, $\mathrm{Zer}\, g = \{\beta_1, \ldots, \beta_l\}$. Assume that for every $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, l\}$ we have

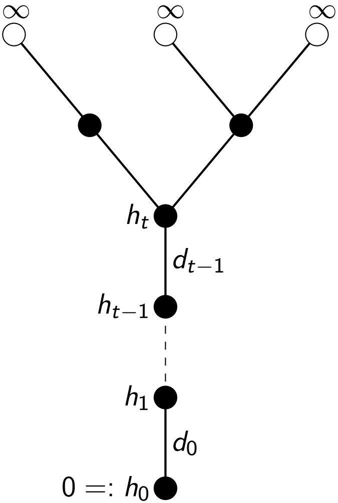
$$\alpha_i - \beta_j = u_{ij} \underline{X}^{\lambda_{ij}},$$

where $u_{ij}(0) \neq 0$ and $\lambda_{ij} \in \mathbb{Q}^n_{\geqslant 0}$. Then

$$\mathrm{cont}(f, g) := \max\{\lambda_{ij} : i \in \{1, \ldots, m\}, j \in \{1, \ldots, l\}\}$$

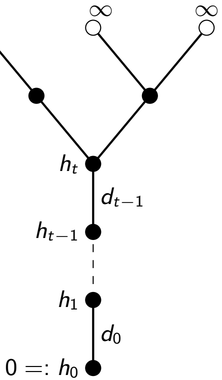
is well defined and we call it the contact between polynomials f and g.

If f is quasi-ordinary Weierstrass polynomial and $f = \varphi_1 \cdots \varphi_r$, where $\varphi_i \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ are irreducible, then we construct the Wall-Eggers tree of f by adding points $\mathrm{cont}(\varphi_i, \varphi_j)$ on the Wall-Eggers trees of polynomials φ_i, φ_j and gluing these trees together from h_0 up to the level $\mathrm{cont}(\varphi_i, \varphi_j)$.



The Wall-Eggers tree of some reducible quasi-ordinary Weierstrass polynomial.

The segment of the Wall-Eggers tree of f from $h_0 = 0$ up to the first ramification point, which correspond to $\mathrm{cont}(\varphi_i, \varphi_j)$ for some $i, j \in \{1, \ldots, r\}$ will be called the stock of the Wall-Eggers tree of f.



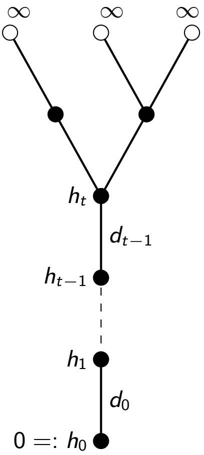
With every reducible quasi-ordinary Weierstrass polynomial f we can associate the sequence of vectors

$$h_0 < \cdots < h_t$$

from $\mathbb{Q}^n_{\geqslant 0}$, which corresponds to the sequence of all points on the stock of the Wall-Eggers tree of f. Moreover, for such a polynomial f we can also consider the sequence

$$d_0 < \cdots < d_{t-1}$$

of positive integers that correspond to all indices on the stock of the Wall-Eggers tree and that comes from an arbitrary irreducible factor of f, since this sequence is common for every irreducible factor of f.



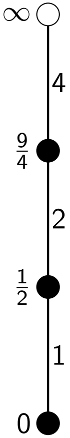
$f = Y^4 - 4X^2Y^3 + (-2X + 6X^4)Y^2 + (4X^3 - 4X^5 - 4X^6)Y + X^2 - 2X^5 + 4X^7 + X^8 - X^9$,
$g = Y^4 + 4X^2Y^3 + (-2X + 6X^4)Y^2 - 4X^3Y + X^2 - 2X^5 - 3X^8 - X^{11}$.
Then all roots of f are

$$\begin{aligned}\alpha_1 &= X^{1/2} + X^2 + X^{9/4},\\ \alpha_2 &= X^{1/2} + X^2 - X^{9/4},\\ \alpha_3 &= -X^{1/2} + X^2 + \sqrt{-1}X^{9/4},\\ \alpha_4 &= -X^{1/2} + X^2 - \sqrt{-1}X^{9/4}\end{aligned}$$

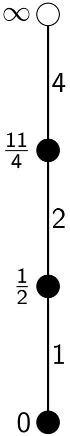
and all roots of polynomial g are

$$\begin{aligned}\beta_1 &= X^{1/2} - X^2 + X^{11/4},\\ \beta_2 &= X^{1/2} - X^2 - X^{11/4},\\ \beta_3 &= -X^{1/2} - X^2 - \sqrt{-1}X^{11/4},\\ \beta_4 &= -X^{1/2} - X^2 + \sqrt{-1}X^{11/4}.\end{aligned}$$

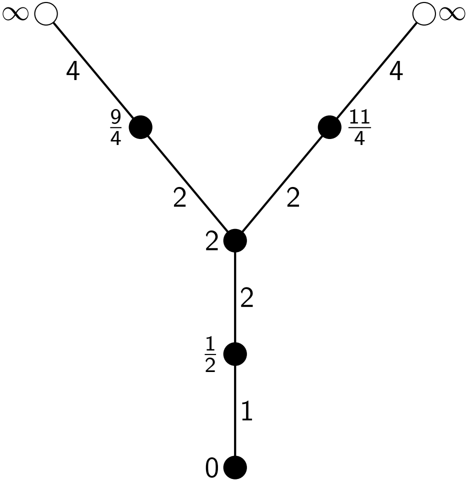
It is easy to see that $f, g \in \mathbb{C}[[X]][Y]$ are irreducible, quasi-ordinary and Weierstrass.



The Wall-Eggers tree of f.



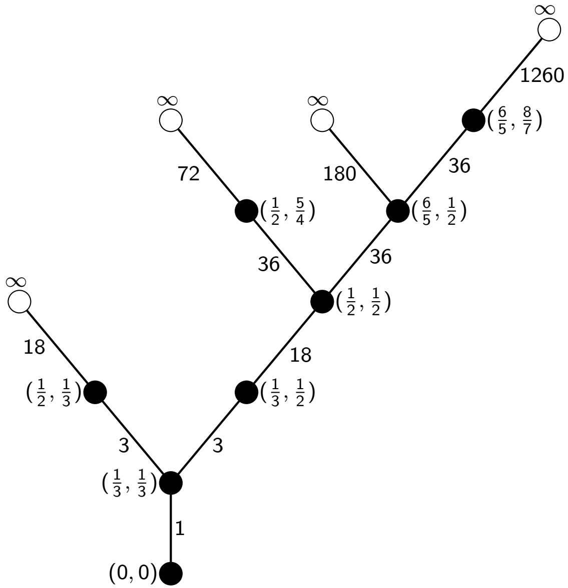
The Wall-Eggers tree of g.



The Wall-Eggers tree of $F = fg$.

Let $f = f_1 f_2 f_3 f_4$ be a quasi-ordinary Weierstrass polynomial such that f_i is the minimal polynomial of α_i for $i = 1, 2, 3, 4$, where

$$\begin{aligned}
\alpha_1 :=& X_1^{1/3} X_2^{1/3} + X_1^{1/2} X_2^{1/3}, \\
\alpha_2 :=& X_1^{1/3} X_2^{1/3} + X_1^{1/3} X_2^{1/2} + X_1^{1/2} X_2^{1/2} + X_1^{1/2} X_2^{5/4}, \\
\alpha_3 :=& X_1^{1/3} X_2^{1/3} + X_1^{1/3} X_2^{1/2} + X_1^{1/2} X_2^{1/2} + X_1^{6/5} X_2^{1/2}, \\
\alpha_4 :=& X_1^{1/3} X_2^{1/3} + X_1^{1/3} X_2^{1/2} + X_1^{1/2} X_2^{1/2} + X_1^{6/5} X_2^{8/7}.
\end{aligned}$$


∞
1260
∞
∞
$(\frac{6}{5}, \frac{8}{7})$
72
180
36
$(\frac{1}{2}, \frac{5}{4})$
$(\frac{6}{5}, \frac{1}{2})$
36
36
∞
$(\frac{1}{2}, \frac{1}{2})$
18
18
$(\frac{1}{2}, \frac{1}{3})$
$(\frac{1}{3}, \frac{1}{2})$
3
3
$(\frac{1}{3}, \frac{1}{3})$
1
$(0, 0)$

Let $\alpha \in \mathrm{Zer}\, f$ and $h \in \mathbb{Q}^n_{\geqslant 0}$. We define the h–truncation of α as the power series $\mathrm{trunc}_h(\alpha) \in \mathbb{K}[[\underline{X}^{1/m}]]$, which is obtained from α by omitting all terms of order $\geqslant h$. Consider also the minimal polynomial f_h of $\mathrm{trunc}_h(\alpha)$ over the field $\mathbb{K}((\underline{X}))$.

Lemma ([6],[7])

If $h_{r-1} < h \leqslant h_r$ for some $r \in \{1, \ldots, s\}$, then the polynomial f_h has the following properties:

(i) $\mathrm{Zer}\, f_h = \{\, \mathrm{trunc}_h(\alpha') : \alpha' \in \mathrm{Zer}\, f \}$,

(ii) $f_h \in \mathbb{K}[[\underline{X}]][Y]$ *is an irreducible quasi-ordinary Weierstrass polynomial,*

(iii) $\deg f_h = d_{r-1}$,

(iv) $\mathrm{Char}(f_h) = \{h_1, \ldots, h_{r-1}\}$.

For a power series

$$g = \sum_{a \in \mathbb{Z}^n_{\geqslant 0}} c_a \underline{X}^a \in \mathbb{K}[[X_1, \dots, X_n]]$$

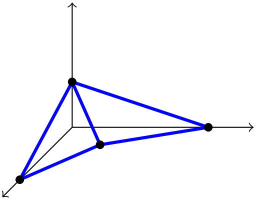
we define its Newton polyhedron $\Delta(g)$ as the convex hull of the set

$$\bigcup_{c_a \neq 0} (a + \mathbb{R}^n_{\geqslant 0}).$$

The Newton polyhedron of a polynomial

$$f = Z + X^3 + Y^3 + XY$$

is a subset of $\mathbb{R}^3_{\geqslant 0}$ that is bounded from below by two triangles, which are presented in the following picture. Note that points $(0,0,1), (0,3,0), (3,0,0)$ and $(1,1,0)$ are vertices of $\Delta(f)$.



The lower bound of the Newton polyhedron of f.

The logarithmic distance of irreducible Weierstrass polynomials $f, g \in \mathbb{K}[[\underline{X}]][Y]$ is the set

$$\mathrm{cont}_A(f, g) := \frac{1}{(\deg f)(\deg g)} \Delta(\mathrm{Res}(f, g)),$$

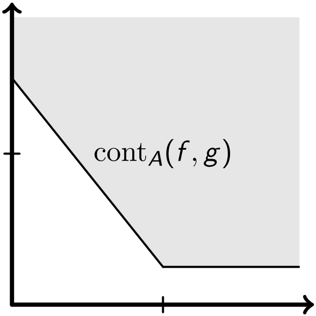
where $\mathrm{Res}(f, g)$ denotes the resultant of polynomials f, g.

Consider polynomials

$$f = Y^2 + 2X_1X_2Y + X_1^2X_2 \in \mathbb{C}[[X_1, X_2]][Y]$$

$$g = Y^2 + X_1Y + X_2^3 \in \mathbb{C}[[X_1, X_2]][Y].$$

$$\mathrm{Res}(f, g) = -X_1^4X_2^2 + X_1^4X_2 + 4X_1^2X_2^5 - 4X_1^2X_2^4 + X_2^6.$$



We introduce the partial order in the set of Newton polyhedra: $\Delta_1 \geqslant \Delta_2$ if and only if $\Delta_1 \subset \Delta_2$.
For irreducible quasi-ordinary Weierstrass polynomials f, g, h we have the so called triangle inequality. This means that

$$\mathrm{cont}_A(f,g) \geqslant \inf\{\mathrm{cont}_A(f,h), \mathrm{cont}_A(h,g)\}$$

holds, where $\inf\{A, B\}$ denotes the convex hull of the union of A and B (see [5],[7]). In this case the logarithmic distance is called the logarithmic contact.

Theorem (Abhyankar-Moh, 1973)

If $f \in \mathbb{K}[[X]][Y]$ is an irreducible Weierstrass polynomial of characteristic

$$h_1 < \cdots < h_s < h_{s+1} = \infty$$

and $k = \frac{\deg f}{d_t}$ for some $t \in \{1, \ldots, s\}$, then

(i) $\sqrt[k]{f}$ *is an irreducible Weierstrass polynomial of characteristic* $h_1 < \cdots < h_t < \infty$,

(ii) $\mathrm{cont}_A \left(f, \sqrt[k]{f}\right) = \Delta(X^{c_{t+1}})$ *in case $t < s$.*

Theorem (Abhyankar-Moh, 1975)

Let $f, g \in \mathbb{K}[T]$. If $\mathbb{K}[f,g] = \mathbb{K}[T]$, then

(i) $\deg f | \deg g$ *or* $\deg g | \deg f$*;*

(ii) $(f,g) = (F(T,0), G(T,0))$ *for some polynomial automorphism $(F,G) \in \mathbb{K}[X,Y]^2$.*

Theorem (Jung-van der Kulk, [11])

The group $\mathrm{GA}_2(\mathbb{K})$ *is generated by two types of polynomial automorphisms:*

- *elementary automorphisms:*

$$(X + h_1, Y) \ \text{ or } \ (X, Y + h_2),$$

  where $h_1 \in \mathbb{K}[Y]$ *and* $h_2 \in \mathbb{K}[X]$,

- *affine automorphisms:*

$$(a_1 X + b_1 Y + c_1, a_2 X + b_2 Y + c_2),$$

  where $a_1, b_1, c_1, a_2, b_2, c_2 \in \mathbb{K}$ *and* $a_1 b_2 - b_1 a_2 \neq 0$.

Conjecture (Tame Generators Conjecture)

The group $\mathrm{GA}_n(\mathbb{K})$ is generated by two types of polynomial automorphisms:

- *elementary automorphisms*:

$$F = (X_1, \ldots, X_{i-1}, X_i + f, X_{i+1}, \ldots, X_n) \in \mathrm{GA}_n(\mathbb{K}),$$

  where

$$f \in \mathbb{K}[X_1, \ldots, X_{i-1}, X_{i+1}, \ldots, X_n].$$

- *affine automorphisms*:

$$F = (F_1, \ldots, F_n) \in \mathrm{GA}_n(\mathbb{K}),$$

  where

$$\max_{i=1,\ldots,n} \deg F_i = 1.$$

Theorem (González Pérez, 2003)

If $f \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ *is an irreducible quasi-ordinary Weierstrass polynomial of characteristic*

$$h_1 < \cdots < h_s < h_{s+1} = \infty$$

and $k = \frac{\deg f}{d_t}$ *for some* $t \in \{1, \ldots, s\}$, *then*

(i) $\sqrt[k]{f}$ *is an irreducible quasi-ordinary Weierstrass polynomial of characteristic* $h_1 < \cdots < h_t < \infty$,

(ii) $\mathrm{cont}_A\left(f, \sqrt[k]{f}\right) = \Delta(\underline{X}^{c_{t+1}})$ *for* $t < s$.

Theorem (Brzostowski, 2011)

Let $f \in \mathbb{K}[[X]][Y]$ be an irreducible Weierstrass polynomial of characteristic

$$h_1 < \cdots < h_s < h_{s+1} = \infty$$

and $k | \deg f$. Assume that $t \in \{0, \ldots, s\}$ is the greatest index such that $k | \frac{\deg f}{d_t}$. If $t < s$ and a Weierstrass polynomial g is an irreducible factor of $\sqrt[k]{f}$, then

$$\mathrm{cont}_A(f, g) = \Delta(X^{c_{t+1}}).$$

Theorem 1

Let $f \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ be a quasi-ordinary Weierstrass polynomial and $h_1 < \cdots < h_t < h_{t+1}$ the sequence of initial points on the stock of the Wall-Eggers tree of f. Assume that $k = \frac{\deg f}{d_t}$ and h_{t+1} is not the last point on the stock of the Wall-Eggers tree of f. Then

(i) $\sqrt[k]{f}$ *is an irreducible quasi-ordinary Weierstrass polynomial of characteristic $h_1 < \cdots < h_t < \infty$,*

(ii) $\mathrm{cont}_A\left(f^*, \sqrt[k]{f}\right) = \Delta(\underline{X}^{c_{t+1}})$ *for every irreducible factor f^* of polynomial f.*

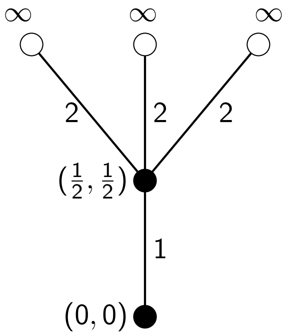
Theorem 2

Let $f \in \mathbb{K}[[X_1, \ldots, X_n]][Y]$ *be a quasi-ordinary Weierstrass polynomial,* $h_1 < \cdots < h_t < h_{t+1}$ *the sequence of initial points on the stock of the Wall-Eggers tree of* f *and* $k | \deg f$*. Assume that* t *is the greatest index such that* $k | \frac{\deg f}{d_t}$ *and* h_{t+1} *is not the last point on the stock of the Wall-Eggers tree of* f*. If* p *is an irreducible Weierstrass polynomial dividing* $\sqrt[k]{f}$*, then*

$$\mathrm{cont}_A(f^*, p) = \Delta(\underline{X}^{c_{t+1}})$$

for every irreducible factor f^* *of polynomial* f*.*

$$f := (Y^2+X_1X_2)(Y^2+(1+i)X_1X_2)(Y^2+(1-i)X_1X_2) \in \mathbb{C}[[X_1, X_2]][Y].$$



The Wall-Eggers tree of polynomial f.

$\sqrt[3]{f} = Y^2 + X_1X_2,$
$c_1 = (\frac{1}{2}, \frac{1}{2})$
$\mathrm{cont}_A(Y^2 + X_1X_2, \sqrt[3]{f}) = \infty > \Delta(\underline{X}^{c_1}),$

Let f be the minimal polynomial of

$$\beta := X_1^{1/2}(1+X_2^{1/3})^{1/2} + aX_1^{1/2}X_2^{1/2},$$

where a is a properly chosen non-zero element of $\mathbb{K}$.
Consider

$$w := (Y^2-X_1)(Y-X_1^{1/2}-X_1^3X_2^3(X_1+X_2))(Y+X_1^{1/2}-X_1^3X_2^3(X_1+X_2)),$$

$$g := \sum_{k=0}^{7} \mathrm{coeff}(f, Y^k)Y^k + \sum_{k=8}^{12} \mathrm{coeff}(w^3, Y^k)Y^k.$$

Then f, g are irreducible quasi-ordinary Weierstrass polynomials of characteristic

$$(\tfrac{1}{2}, 0) < (\tfrac{1}{2}, \tfrac{1}{3}) < (\tfrac{1}{2}, \tfrac{1}{2}) < \infty.$$

$$d_1 = 2, d_2 = 6, d_3 = 12.$$

Then $\sqrt[3]{g} = w$ is non-characteristic and is not quasi-ordinary.

Example (González Pérez)

Consider

$$h = Z^2 + X^5 Y^7 Z + X^5 Y^7 (X + Y) \in \mathbb{C}[[X, Y]][Z]$$

$$f = h^3 + X^5 Y^7 + X^5 Y^7 h \in \mathbb{C}[[X, Y]][Z].$$

Then f is an irreducible quasi-ordinary polynomial of characteristic

$$(0, 0) < (\tfrac{5}{6}, \tfrac{7}{6}) < \infty.$$

We have that $\sqrt[3]{f} = h$ is non-characteristic and not quasi-ordinary.

Take any Weierstrass polynomial $f \in \mathbb{K}[[\underline{X}]][Y]$, positive integer N, a power series $\gamma \in \mathbb{K}[[\underline{X}^{1/N}]]$, a new variable Z and $h \in \frac{1}{N}\mathbb{Q}^n$. By definition $f(\gamma + Z\underline{X}^h)$ is called the (γ, h)–deformation of f. If

$$f(\gamma + Z\underline{X}^h) = aF\underline{X}^q + \text{ higher order terms},$$

where $a \in \mathbb{K}^*$ and $F \in \mathbb{K}[Z]$ is monic, then we denote

$$\mathrm{Def}(f, \gamma, h) := F.$$

Remark

For every quasi-ordinary Weierstrass polynomial f the polynomial $\mathrm{Def}(f, \gamma, h)$ is well-defined.

Theorem 3

Let $f \in \mathbb{K}[[X_1,\ldots,X_n]][Y]$ be a quasi-ordinary Weierstrass polynomial, $h_1 < \cdots < h_t < h_{t+1}$ the sequence of initial points on the stock of the Wall-Eggers tree of f and $k|\deg f$. Assume that $t \in \{0,\ldots,s\}$ is the greatest index such that $k|\frac{\deg f}{d_t}$ and h_{t+1} is not the last point on the stock of the Wall-Eggers tree of f. Let γ be the h_{t+1}–truncation of some root of f and

$$F := \mathrm{Def}(f,\gamma,h_{t+1}).$$

Then

$$\sqrt[k]{F} = Z^s(Z^{n_{t+1}} - a_1^{n_{t+1}})\cdots(Z^{n_{t+1}} - a_r^{n_{t+1}}),$$

where $a_1,\ldots,a_r \in \mathbb{K}^*$ are such that $a_1^{n_{t+1}},\ldots,a_r^{n_{t+1}}$ are pairwise different, $s = n_{t+1}\{\frac{\deg f}{kd_{t+1}}\}$, $r = \lfloor\frac{\deg f}{kd_{t+1}}\rfloor$ and

$$\sqrt[k]{f} = \varphi^*\varphi_1\cdots\varphi_r,$$

Theorem 3

where $\varphi^* \in \mathbb{K}[[\underline{X}]][Y]$ *is such that*

$$\mathrm{Def}(\varphi^*, \gamma, h_{t+1}) = Z^s$$

and $\varphi_i \in \mathbb{K}[[\underline{X}]][Y]$ *are irreducible and quasi-ordinary of characteristic*

$$h_1 < \cdots < h_t < h_{t+1} < \infty$$

with

$$\mathrm{Def}(\varphi_i, \gamma, h_{t+1}) = (Z^{n_{t+1}} - a_i^{n_{t+1}})$$

for $i = 1, \ldots, r$. *Moreover, if* $s = 1$ *then* φ^* *is irreducible and quasi-ordinary of characteristic*

$$h_1 < \cdots < h_t < \infty.$$

Let C be a non-trivial subgroup of $(\mathbb{R}, +)$. Consider the field $\mathbb{K}_C((T))$ of formal power series $\sum_{i=1}^{\infty} a_i T^{\alpha_i}$, where $(\alpha_i)_{i=1}^{\infty}$ is an increasing sequence of elements from C that tends to infinity and $a_i \in \mathbb{K}$ for all $i \in \mathbb{Z}_{>0}$. Such power series will be called generalized Puiseux series.

We denote

$$\mathbb{K}_C((T^{1/m})) := \mathbb{K}_{C/m}((T)).$$

Theorem

If $f \in \mathbb{K}_C((T))[Y]$ is a polynomial of degree $m > 1$, then $\operatorname{Zer} f \subset \mathbb{K}_C((T^{1/N}))$ for some $N \in \mathbb{Z}_{>0}$.

Corollary

The field

$$\bigcup_{m \in \mathbb{Z}_{>0}} \mathbb{K}_C((T^{1/m}))$$

is the algebraic closure of $\mathbb{K}_C((T))$.

Theorem

Let $N \in \mathbb{Z}_{>0}$ and $C = r_1\mathbb{Z} + \cdots + r_n\mathbb{Z}$ for some linearly independend over $\mathbb{Q}$ positive real numbers $r_1, \ldots, r_n$. Then:

(i) $\mathbb{K}_C((T^{1/N})) = \mathbb{K}_C((T))(T^{r_1/N}, \ldots, T^{r_n/N})$,

(ii) *The Galois group $\mathrm{Gal}(\mathbb{K}_C((T^{1/N}))/\mathbb{K}_C((T)))$ consists of all automorphisms of the form*

$$\sigma\left(\sum_{i=1}^{\infty} a_i T^{(r_1 k_{i,1}+\cdots+r_n k_{i,n})/N}\right) = \sum_{i=1}^{\infty} a_i \varepsilon_1^{k_{i,1}} \cdots \varepsilon_n^{k_{i,n}} T^{(r_1 k_{i,1}+\cdots+r_n k_{i,n})/N}$$

for some $\underline{\varepsilon} = (\varepsilon_1, \ldots, \varepsilon_n) \in \mathbb{K}^n$ such that $\varepsilon_i^N = 1$ for $i = 1, \ldots, n$.

Theorem

Let $N \in \mathbb{Z}_{>0}$ *and* $C = r_1\mathbb{Z} + \cdots + r_n\mathbb{Z}$ *for some linearly independend over* $\mathbb{Q}$ *positive real numbers* $r_1, \ldots, r_n$*. If* $f \in \mathbb{K}_C((T))[Y]$ *has degree* > 1 *and* $\operatorname{Zer} f \subset \mathbb{K}_C((T^{1/N}))$*, then the Galois group*

$$\operatorname{Gal}(f) = \{\sigma|_{L_f} : \sigma \in \operatorname{Gal}(\mathbb{K}_C((T^{1/N}))/\mathbb{K}_C((T)))\},$$

where L_f *is the splitting field of* f*.*

Let $0 \neq C < (\mathbb{R}, +)$. We say that $f \in \mathbb{K}_C((T))[Y]$ is a Weierstrass polynomial if f is monic and all of its coefficients, except the leading coefficient, have positive orders.

Remark

For an irreducible Weierstrass polynomial $f \in \mathbb{K}_C((T))[Y]$ we can define:

- the *characteristic* of f: $h_1 < \cdots < h_s$,
- indices: $d_0 < d_1 < \cdots < d_s$,
- numbers: $c_0, c_1, \ldots, c_s$,
- the Wall-Eggers tree of f.

Let $N \in \mathbb{Z}_{>0}$, $\gamma \in \mathbb{K}[[\underline{X}^{1/N}]]$ and $C := r_1\mathbb{Z} + \cdots + r_n\mathbb{Z}$, where $r_1, \ldots, r_n \in \mathbb{R}_+$ are linearly independent over $\mathbb{Q}$. We define the monomial substitution, i.e. the monomorphism of $\mathbb{K}$–algebras

$$\mathbb{K}[[\underline{X}^{1/N}]] \ni \gamma \longmapsto \overline{\gamma} \in \mathbb{K}_C((T^{1/N}))$$

defined by

$$X_i^{1/N} \longmapsto T^{r_i/N}.$$

For a polynomial

$$f = a_m Y^m + \cdots + a_0 \in \mathbb{K}[[\underline{X}]][Y]$$

we consider

$$\overline{f} := \overline{a_m} Y^m + \cdots + \overline{a_0} \in \mathbb{K}_C((T))[Y].$$

For every $a \in \mathbb{Q}^n$ and $r := (r_1, \ldots, r_n)$ we define

$$\overline{a} := \langle a, r \rangle.$$

Remark

Let $f \in \mathbb{K}[[\underline{X}]][Y]$ be an irreducible quasi-ordinary Weierstrass polynomial of characteristic $h_1 < \cdots < h_s < h_{s+1} = \infty$. Then for an arbitrary monomial substitution we have

(i) $\overline{f} \in \mathbb{K}_C((T))[Y]$ is an irreducible Weierstrass polynomial,

(ii) the characteristic of $\overline{f}$ is equal to the sequence $\overline{h}_1 < \cdots < \overline{h_s} < \infty$,

(iii) if c_i is the ith vector of f, then $\overline{c_i}$ is the ith number associated with $\overline{f}$.

Thank you for your attention!

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