Effective Bertini theorem and formulas for multiplicity and the local Łojasiewicz exponent

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Theorem

Let X be a nonsingular closed subvariety of $\mathbb{P}^n_k$, where k is an algebraically closed field. Then there exists a hyperplane $H \subset \mathbb{P}^n_k$ not containing X and such that the intersection $H \cap X$ is regular at every point. Furthermore, the set of hyperplanes with this property forms an open dense subset of the complete linear system $|H|$ considered as a projective space.

Algebraic cone C_0, Total degree $\delta(V)$

- $f_1, \dots, f_r$ - homogeneous polynomials,

$$C_0 = V(f_1, \dots, f_r) \text{ - algebraic cone.}$$

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- $V \subset \mathbb{C}^m$ - algebraic set,
  $V = V_1 \cup \cdots \cup V_s$ - decomposition of V into irreducible components,

  $$\delta(V) = \deg V_1 + \cdots + \deg V_s \text{ - total degree of } V.$$

Bertini's weak theorem

- For any $a \in \mathbb{C}$, denote

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- For any $a \in \mathbb{C}$, denote

$$N_a(x_1, \ldots, x_m) = x_1 + ax_2 + \cdots + a^{m-1}x_m.$$

Theorem

$C_0 \subset \mathbb{C}^m$ - algebraic cone of pure dimension $q \geq 1$, $\delta(C_0) \leq d$. Then the set

$$A = \{a \in \mathbb{C} : \ C_0 \cap V(N_a) \text{ - improper } \}$$

is finite. Moreover,

$$\#A \leq d(m - q).$$

The properness of the intersection $C_0 \cap V(N_a)$ in the above theorem cannot be replaced by transversality, as shown in the following example.

Example

Let

$$C = \{(x, y, z) \in \mathbb{C}^3 : y^2 - 4xz = 0\}.$$

Obviously $C \setminus \{(0,0,0)\}$ *is a smooth cone of dimension 2. Let* $a \in \mathbb{C}$

$$N_a(x, y, z) = x + ay + a^2 z.$$

Then $C \cap V(N_a) \neq \emptyset$ *for any* $a \in \mathbb{C}$. *Moreover, if* $(0,0,0) \neq (x_0, y_0, z_0) \in C \cap V(N_a)$, *then* $z_0 \neq 0$, $a = \frac{-y_0}{2z_0}$ *and* $x_0 = \frac{y_0^2}{4z_0}$. *Therefore*

$$N_a(x, y, z) = x + \frac{-y_0}{2z_0} y + \frac{y_0^2}{4z_0^2} z = \frac{-1}{4z_0}(-4z_0 x + 2y_0 y - 4x_0 z)$$

and $T_{(x_0,y_0,z_0)} C = V(N_a)$. *This shows that the intersection* $C \cap V(N_a)$ *is not transversal at* (x_0, y_0, z_0).

System of independent linear functions

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- $\mathbb{L}(m,1) = \{N\colon \mathbb{C}^m \to \mathbb{C},\ N \text{ - linear map}\}$.
- A system of functions $N_1, \ldots, N_s \in \mathbb{L}(m,1)$, $s \geq m$, we call *independent* if for any sequence $1 \leq i_1 < \cdots < i_m \leq s$ the system $N_{i_1}, \ldots, N_{i_m}$ is linearly independent over $\mathbb{C}$.

Main lemma

A crucial role in the proof of Bertini theorem plays the following

Lemma

Let m, q, d be positive integers, $m \geq 2$, $m \geq q$. Let $\ell = d(m-q)+q$, and let $N_j : \mathbb{C}^m \to \mathbb{C}$, $1 \leq j \leq \ell$, be a system of independent linear functions. Then for any algebraic cone $C_0 \subset \mathbb{C}^m$ with $\dim C_0 \leq q$ and $\delta(C_0) \leq d$, there exist $1 \leq i_1 < \cdots < i_q \leq \ell$ such that

$$C_0 \cap V(N_{i_1}, \ldots, N_{i_q}) = \{0\}.$$

Theorem

Let d, m, q be positive integers such that $m \geq q$, let $\ell = 2d^{m-q}[(m-q)(d-1)+1]^{q-1} + m - 1$, and let $N_j \in \mathbb{L}(m,1)$, $1 \leq j \leq \ell$, be a system of independent linear functions. Then

$$E_{j_1,\dots,j_{m-1}} =$$
$$= \{x \in \mathbb{C}^m : \exists_{a \in \mathbb{C}^m \setminus \{0\}}\ N_{j_1}(a) = \cdots = N_{j_{m-1}}(a) = 0,\ ax^T = 0\},$$

for $1 \leq j_1 < \cdots < j_{m-1} \leq \ell$ is a system of hyperplanes such that for any irreducible algebraic cone $V \subset \mathbb{C}^m$, $\dim V = q$, $\deg V \leq d$ such that $V \setminus \{0\}$ is smooth, there are $1 \leq j_1 < \cdots < j_{m-1} \leq \ell$ such that the intersection $X = V \cap E_{j_1,\dots,j_{m-1}}$ is transversal at any point $x \in X \setminus \{0\}$ and the set $X \setminus \{0\}$ is smooth.

Remark

The assertion of the above theorem holds for the system of linear functions

$$N_j(x_0, \ldots, x_n) = x_0 + a_j x_1 + \cdots + a_j^n x_m,$$

where $a_j \in \mathbb{C}$, $1 \leq j \leq \ell$, are pairwise different numbers.

Effective Bertini theorem (II)

Theorem

Let d, m, q, s be positive integers such that $m \geq q$ and $q-1 \geq s$, let $\ell = d^{m-q}[(m-q)(d-1)+q-1]+m(q-1)-1$, and let $N_j \in \mathbb{L}(m(q-1),1)$, $1 \leq j \leq \ell$, be a system of independent linear functions such that

$$K_{s,j_1,\dots,j_{m(q-1)-1}} = \{x \in \mathbb{C}^m : \exists_{a=(a_1,\dots,a_{q-1})\in(\mathbb{C}^m)^{q-1}\setminus\{0\}}$$
$$N_{j_1}(a) = \cdots = N_{j_{m(q-1)-1}}(a) = 0,\ a_1x^T = \cdots = a_sx^T = 0\},$$

for $1 \leq j_1 < \cdots < j_{m-1} \leq \ell$ is a system of linear subspaces of dimension $m-s$. Then for any irreducible algebraic cone $V \subset \mathbb{C}^m$ with $\dim V = q$ and $\deg V \leq d$ such that $V \setminus \{0\}$ is smooth, there are $1 \leq j_1 < \cdots < j_{m(q-1)-1} \leq \ell$ such that the intersection $X = V \cap K_{s,j_1,\dots,j_{m(q-1)-1}}$ is transversal at any point $x \in X \setminus \{0\}$ and the set $X \setminus \{0\}$ is smooth.

Remark

To get the effectivity of selection of the subspaces $K_{s,j_1,\dots,j_{m(q-1)-1}}$ and keep the formulation of Bertini theorem (II) simple, we assumed that these subspaces are orthogonal to $a_1,\dots,a_{q-1}$. Of course, it is enough to assume that the subspaces $K_{s,j_1,\dots,j_{m(q-1)-1}}$ are transverse to the subspaces spanning on vectors $a_1,\dots,a_{q-1}$.

Remark

For $d > 1$ and $q < m$ the estimation of the number of linear functions ℓ in the Bertini theorem (II) is better than the one obtained by repeated use of the estimation of ℓ in the Bertini theorem (I). Indeed, using the Bertini theorem (I) s-times we need at least

$$s - 1 + 2d^{m-q}[(m-q)(d-1)+1]^{q-1} + m - 1$$

linear functions N_j.

Remark

Since (using Bernoulli's inequality),

$$
\begin{aligned}
2d^{m-q}[(m-q)(d-1)+1]^{q-1}+m &\geq d^{m-q}[(m-q)(d-1)(q-1)+1] \\
&\quad + m + d^{m-q}[(m-q)(d-1)+1]^{q-1} \\
&\geq d^{m-q}[(m-q)(d-1)+q-1] + m + 2^{m-q}[m-q+1]^{q-1},
\end{aligned}
$$

it suffices to prove that $2^{m-q}[m-q+1]^{q-1} \geq m(q-2)$ for $2 < q < m$ or equivalently that

$$(m-q)+(q-1)\log_2(m-q+1) \geq \log_2 m + \log_2(q-2) \text{ for } 2 < q < m.$$

Obviously for $q = 2$ the assertion holds. We check directly the above inequality for $m \geq q+1$ and obtain the assertion.

Effective formula for local degree $\deg_0 V$

- U - neighborhood of $0 \in \mathbb{C}^m$,
  $V \subset \mathbb{C}^m$ - algebraic set of pure dimension, $0 \in V$,
  $H \subset \mathbb{C}^m$ - generic affine subspace, sufficiently close to 0,
  $\dim H = m - \dim V$.

  $\deg_0 V = \#(V \cap H \cap U)$ - local degree of V at 0.

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Corollary

For any algebraic set $V = V(f_1, \ldots, f_r) \subset \mathbb{C}^m$ of pure dimension q, where $f_j \in \mathbb{C}[x_1, \ldots, x_n]$, $\deg f_j \le d$ for $1 \le j \le r$, we have

$$\deg_0 V = \max_{1 \le j_1 < \cdots < j_{m(q-1)-1} \le \ell} \deg_0(V \cap K_{s, j_1, \ldots, j_{m(q-1)-1}})$$

for any $1 \le s \le q$.

Effective formula for multiplicity $i_0(f)$

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, $m \geq n$ - polynomial map, finite at 0.

  $i_0(f) := i(\text{graph}\, f \cdot (\mathbb{C}^n \times \{0\}); (0,0))$ - multiplicity of f at 0
  $=$ improper intresection multiplicity of graph f and $\mathbb{C}^n \times \{0\}$ at $(0,0)$ (Achilles, Tworzewski, Winiarski, 1990).

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- If $m = n$, then
  $i_0(f) = \dim_{\mathbb{C}} \mathcal{O}_0^n/(f)$,
  $\mathcal{O}_0^n$ - the ring of germs of holomorphic functions at $0 \in \mathbb{C}^n$.

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- (S. Spodzieja, 2000)

  If $m > n$, then:

  1) $i_0(f) \leq \dim_{\mathbb{C}} \mathcal{O}_0^n/(L \circ f)$,
  2) $i_0(f) = \dim_{\mathbb{C}} \mathcal{O}_0^n/(L \circ f)$ for the generic $L \in \mathbb{L}(m,n)$,
  3) $i_0(f) = \dim_{\mathbb{C}} \mathcal{O}_0^n/(L \circ f) \quad \Leftrightarrow \quad V(L) \cap C_0(f(U)) = \{0\}$,

  $U \subset \mathbb{C}^n$ - sufficiently small neighbourhood of 0,
  $C_0(f(U))$ - tangent cone to the set $f(U)$.

Effective formula for multiplicity $i_0(f)$

Theorem

Let $\ell = d(m-n)+n$. For any independent system

$$L_j \in \mathbb{L}(m,1), \quad 1 \le j \le \ell,$$

and any polynomial mapping $f:(\mathbb{C}^n,0)\to(\mathbb{C}^m,0)$, finite at 0, with $\deg_0 C_0(f(U)) = d$ we have

$$i_0(f) = \min_{1\le i_1<\cdots<i_n\le \ell} \dim_{\mathbb{C}} \mathcal{O}_0^n/((L_{i_1},\ldots,L_{i_n})\circ f).$$

Effective formula for multiplicity $i_0(f)$

Example

Let $f\colon (\mathbb{C}^2,0) \to (\mathbb{C}^3,0)$ *be a polynomial mapping defined by*

$$f(x,y) = (x^2, xy, y^3).$$

Obviously f *is finite at* 0*. Put* $P(X,Y,Z) = X^3Z^2 - Y^6$*. Then* $P(x^2,xy,y^3) = 0$ *and*

$$C_0(f(U)) = \{(X,Y,Z) \in \mathbb{C}^3 \colon XZ = 0\}.$$

Hence $d = \deg_0 C_0(f(U)) = 2$ *and* $\ell = 2(3-2)+2 = 4$*. Let*

$$\begin{aligned} L_1(X,Y,Z) &= X, \\ L_2(X,Y,Z) &= Y, \\ L_3(X,Y,Z) &= Z, \\ L_4(X,Y,Z) &= X+Y+Z. \end{aligned}$$

Effective formula for multiplicity $i_0(f)$

Example

The system $\{L_1, L_2, L_3, L_4\}$ *is independent and we have* $\binom{4}{2} = 6$ *possibilities. Namely*

$$\begin{aligned}
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_1, L_2) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(x^2, xy) = +\infty \\
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_1, L_3) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(x^2, y^3) = 6 \\
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_1, L_4) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(x^2, xy + y^3) = 6 \\
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_2, L_3) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(xy, y^3) = +\infty \\
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_2, L_4) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(xy, x^2 + y^3) = 5 \\
\dim_{\mathbb{C}} \mathcal{O}_0^2 /((L_3, L_4) \circ f) &= \dim_{\mathbb{C}} \mathcal{O}_0^2 /(y^3, x^2 + xy) = 6
\end{aligned}$$

Therefore

$$i_0(f) = 5.$$

Effective formula for local Łojasiewicz exponent $\mathcal{L}_0(f)$

- $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^m, 0)$, $m \geq n$ - polynomial map, finite at 0.

  $$\mathcal{L}_0(f) = \inf\{\nu \in \mathbb{R}_+ : \exists_{C>0} \quad |f(z)| \geq C|z|^\nu, \ |z| \to 0\}$$

  - local Łojasiewicz exponent of f at 0.

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$$\mathcal{L}_0(f) = \min_{1\leq i_1<\cdots<i_n\leq\ell} \mathcal{L}_0((L_{i_1},\ldots,L_{i_n})\circ f).$$

Thank you for your attention!