Essentially isolated determinantal singularities
Their generic sections - Their characteristic classes

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Generic determinantal varieties

$M_{m,n}$ = set of $m \times n$ matrices with complex entries.

Definition

For each integer t such that $1 \leqslant t \leqslant \min\{m, n\}$,

$$M^t_{m,n} = \{A \in M_{m,n} | \operatorname{rank}(A) < t\}.$$

$M^t_{m,n}$ is a singular variety of codimension $(m - t + 1)(n - t + 1)$ in $M_{m,n}$, called generic determinantal variety.

The singular set of $M^t_{m,n}$ is $M^{t-1}_{m,n}$.
The partition of $M^t_{m,n}$ defined by

$$M^t_{m,n} = \cup_{i=1,\dots,t}(M^i_{m,n} \setminus M^{i-1}_{m,n})$$

is a Whitney stratification.

Determinantal varieties

Let $F : U \to M_{m,n}$ be a map defined by $F(x) = (f_{ij}(x))$, whose entries are complex analytic functions defined on an open domain U of $\mathbb{C}^N$.

Definition

The analytic variety $X = F^{-1}(M^t_{m,n})$ is called

determinantal variety of type (m, n, t),

if one has

$$\operatorname{codim}_{\mathbb{C}^N} X = \operatorname{codim}_{M_{m,n}} M^t_{m,n} = (m - t + 1)(n - t + 1).$$

Essentially nonsingular points

Definition (Ebeling and Gusein-Zade, 2009)

Let $F : U \to M_{m,n}$ be a map as above and t, such that $1 \leq t \leq \min(m, n)$. We say that $F : U \to M_{m,n}$ is transversal to the stratification of $M^t_{m,n}$ if for all $x \in U$, with $\operatorname{rank} F(x) = i - 1$, $0 < i \leq t$, then F is transversal to the stratum $\{M^i_{m,n} \backslash M^{i-1}_{m,n}\}$ at the point $F(x)$. In a precise way, if one has:

$$DF(x)(T_x(U)) \oplus T_{F(x)}(M^i_{m,n} \backslash M^{i-1}_{m,n}) = T_{F(x)}(M_{m,n}).$$

Definition (Ebeling and Gusein-Zade, 2009)

A point $x \in X = F^{-1}(M^t_{m,n})$ such that $\operatorname{rank} F(x) = i - 1$, $i \leq t$, is called essentially nonsingular if the map F is transversal to the stratum $\{M^i_{m,n} \backslash M^{i-1}_{m,n}\}$ at the point $F(x)$.

Notice that an essentially nonsingular point of a determinantal variety of type (m, n, t) might be singular.

Essentially Isolated Determinantal Singularity, EIDS

Definition (Ebeling and Gusein-Zade, 2009)

One says that a germ $(X, 0) \subset (\mathbb{C}^N, 0)$ of a determinantal variety of type (m, n, t) has an essentially isolated singular point at the origin (or is an essentially isolated determinantal singularity, EIDS) if it has only essentially nonsingular points in a punctured neighbourhood of the origin in X.

An EIDS $X \subset \mathbb{C}^N$, $X = F^{-1}(M^t_{m,n})$ has an isolated singularity at the origin if and only if $N \leqslant (m - t + 2)(n - t + 2)$.
This follows from the transversality away from the origin of F to the singular set $M^{t-1}_{m,n}$ of $M^t_{m,n}$.
In fact, the codimension of $M^{t-1}_{m,n}$ in $M_{m,n}$ is $(m - t + 2)(n - t + 2)$, hence the condition $N \leqslant (m - t + 2)(n - t + 2)$ implies that $F^{-1}(M^{t-1}_{m,n})$ is either empty or just the origin.
An EIDS with isolated singularity will be called isolated determinantal singularity, denoted by IDS.

Essential smoothing

We will consider deformations of an EIDS obtained by deformations of the matrix which defines the EIDS, so these deformations are themselves determinantal varieties of the same type.
$F : U \to M_{m,n}$ a representative of the germ F and $\widetilde{F} : U \times \mathbb{C} \to M_{m,n}$ an analytic deformation of F with $\widetilde{F}_s(x) = \widetilde{F}(x, s)$ and $\widetilde{F}_0(x) = F(x)$.
If $X = F^{-1}(M^t_{m,n})$ is an EIDS, then $\widetilde{X}_s = \widetilde{F}_s^{-1}(M^t_{m,n})$ also defines an EIDS of type (m, n, t) in U.

Definition

If $\widetilde{F}_s : U \to M_{m,n}$ is transversal to the stratification of $M^t_{m,n}$, we say that $\widetilde{F}_s(x) = \widetilde{F}(x, s)$ is an essential smoothing of $(X, 0)$.

We denote $\widetilde{X} = \widetilde{F}^{-1}(M^t_{m,n})$.

Smoothings

When $N \geq (m-t+2)(n-t+2)$, an essential smoothing is in general not smooth, but the following holds.

Theorem (Wahl, 1981)

Let $(X,0) \subset (\mathbb{C}^N, 0)$ be the germ of a determinantal variety with isolated singularity at the origin. Then,

X admits a smoothing if and only if $N < (m-t+2)(n-t+2)$.

The singular set of the essential smoothing $\widetilde{X}_s$ is $\widetilde{F}_s^{-1}(M_{m,n}^{t-1})$. Since $\widetilde{F}$ is transversal to the strata of the Whitney stratification $M_{m,n}^t$, the partition $\widetilde{X}_s = \cup_{1 \leq i \leq t} \widetilde{F}_s^{-1}(M_{m,n}^i \backslash M_{m,n}^{i-1})$ is a Whitney stratification of $\widetilde{X}_s$.

Example

Let

$$\begin{array}{rccl} F: & \mathbb{C}^4 & \to & M_{2,3} \\ & (x,y,z,w) & \mapsto & \begin{pmatrix} z & y+w & x \\ w & x & y \end{pmatrix} \end{array}$$

and $X = F^{-1}(M^2_{2,3})$.
The matrix

$$\begin{array}{rccl} \widetilde{F}: & \mathbb{C}^4 \times \mathbb{C} & \to & M_{2,3} \\ & (x,y,z,w,s) & \mapsto & \begin{pmatrix} z & y+w & x+s \\ w & x & y \end{pmatrix} \end{array}$$

defines an essential smoothing $\widetilde{X}_s = \widetilde{F}_s^{-1}(M^2_{2,3})$ *of* X.
In this case the essential smoothing $\widetilde{X}_s$ *is a smoothing.*

Local smoothing - Milnor fibre

Let $(X,0) \subset (\mathbb{C}^N,0)$ be the germ of a d-dimensional variety with isolated singularity at the origin. A local smoothing of $(X,0)$ is a flat morphism of local spaces $\pi : (\mathcal{X},0) \to (\mathbb{C},0)$ such that $\pi^{-1}(0) = X$ and $\pi^{-1}(s) = \mathcal{X}_s$ is smooth for all $s \neq 0$ small. Let $(X,0) \hookrightarrow (\mathcal{X},0) \subset (\mathbb{C}^N \times \mathbb{C},0)$, be a local embedding of the family in $\mathbb{C}^N \times \mathbb{C}$ such that $\pi : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}$ is the projection into the second factor. Then, we can choose sufficiently small ε-ball $\overline{B}_\varepsilon$ in $\mathbb{C}^N \times \mathbb{C}$, and a disk $D_\eta \subset \mathbb{C}$, $0 < \eta \ll \varepsilon$ such that $\mathcal{X}$ intersects $S_\varepsilon = \partial\overline{B}_\varepsilon$ transversely; then
$\pi|_{\pi^{-1}(D_\eta \setminus \{0\})} : \pi^{-1}(D_\eta \setminus \{0\}) \cap \overline{B}_\varepsilon \to D_\eta \setminus \{0\}$ is a locally trivial fibration.
The fiber $\overline{B}_\varepsilon \cap \mathcal{X}_s = X_s$ is the Milnor fibre of X at the origin.

Let $p : (X, 0) \to (\mathbb{C}, 0)$ be a complex analytic function defined in X with isolated singularity at the origin. Let us consider a function

$$\begin{array}{rccc} \widetilde{p}: & \mathbb{C}^N \times \mathbb{C} & \to & \mathbb{C} \\ & (x,s) & \mapsto & \widetilde{p}(x,s), \end{array}$$

such that $\widetilde{p}(x, 0) = p(x)$ and for all $s \neq 0$, $\widetilde{p}(\cdot, s) = p_s$ is a complex Morse function on X_s. That is its singularities are complex non-degenerate critical points. We have the following diagram

$$\begin{array}{ccc} X_s & \hookrightarrow & \overline{B}_\varepsilon \cap \mathcal{X} \subset \mathbb{C}^N \times \mathbb{C} \\ \Big\downarrow {\scriptstyle p_s} & & \Big\downarrow {\scriptstyle (\widetilde{p}, \pi)} \\ \mathbb{C} \times \{s\} & \hookrightarrow & \mathbb{C} \times \mathbb{C} \end{array}$$

Polar multiplicity

Let us denote by $\Sigma(\widetilde{p}, \pi)|_{\widetilde{X}_{reg}}$ the singular set of the map $(\widetilde{p}, \pi)$ restricted to $\widetilde{X}_{reg}$ and by $P_d((X,0), \pi, p) = \overline{\Sigma(\widetilde{p}, \pi)|_{\widetilde{X}_{reg}}}$ the relative polar variety of X related to π and p.

Definition (Gaffney, 1993; Pereira and Ruas, 2014)

The d-local polar multiplicity *of* $(X,0)$ *is defined by*

$$m_d((X,0), \pi, p) = m_0(P_d((X,0), \pi, p)),$$

where m_0 *is the multiplicity of* $P_d((X,0), \pi, p)$ *at the origin.*

In general $m_d((X,0), \pi, p)$ depends on the choices of $\widetilde{X}$ and $\widetilde{p}$ in the neighbourhood of the origin. When $(X,0)$ has a unique smoothing $(\widetilde{X}, 0)$, then $m_d((X,0), \pi, p)$ depends only on X and p in a neighbourhood of 0. If p is a generic linear embedding, $m_d((X,0), \pi, p)$ is an invariant of the EIDS X, denoted by $m_d(X,0)$, d-polar multiplicity of X at the origin.

Lê-Greuel formula type for IDS with smoothing

A determinantal variety is Cohen-Macaulay, then an IDS is normal and the smoothing X_s satisfies : $b_1(X_s) = 0$.

Definition (Pereira and Ruas, 2014)

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be a determinantal surface with isolated singularity at the origin. The local Milnor number of $(X, 0)$, denoted by $\mu(X, 0)$, is defined as

$$\mu(X, 0) = b_2(X_s).$$

The following result for determinantal surfaces in $(\mathbb{C}^4, 0)$ [Damon and Pike, 2014; Nuño Ballesteros, Oréfice-Okamoto and Tomazella, 2013; Pereira and Ruas, 2014], also holds for any determinantal surface with isolated singularity in $\mathbb{C}^N$ admitting smoothing.

Theorem

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be the germ of a determinantal surface with isolated singularity at the origin admitting smoothing. One has the Lê-Greuel formula : $\mu(X, 0) + \mu(X \cap p^{-1}(0), 0) = m_2(X, 0)$.

Vanishing local Euler characteristic

When $d = \dim X > 2$, then $b_i(X)$, $2 \leqslant i < d$ are not necessarily zero.

Definition

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an IDS such that $N < (m - t + 2)(n - t + 2)$. The vanishing local Euler characteristic is defined by

$$\nu(X, 0) = (-1)^d (\chi(X_s) - 1),$$

where X_s is a smoothing of X and $\chi(X_s)$ is the Euler characteristic of X_s .

Theorem (Nuño Ballesteros, Oréfice-Okamoto and Tomazella, 2013)

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an IDS with $N < (m - t + 2)(n - t + 2)$ and let $p : (\mathbb{C}^N, 0) \to (\mathbb{C}, 0)$ be a generic linear projection whose restriction to X has isolated singularity at the origin. Then,

$$\nu(X, 0) + \nu(X \cap p^{-1}(0), 0) = m_d(X, 0).$$

When $d = 2$, then $\nu(X, 0) = \mu(X, 0)$.

Example

Let $X = F^{-1}(M^2_{2,3}) \subset \mathbb{C}^4$ *be the variety defined by:*

$$\begin{array}{cccc} F: & \mathbb{C}^4 & \to & M_{2,3} \\ & (x,y,z,w) & \mapsto & \begin{pmatrix} z & y+w & x \\ w & x & y \end{pmatrix} \end{array}.$$

The following matrix defines an essential smoothing $\widetilde{X}_s = \widetilde{F}_s^{-1}(M^2_{2,3})$ *of* X

$$\begin{array}{cccc} \widetilde{F}: & \mathbb{C}^4 \times \mathbb{C} & \to & M_{2,3} \\ & (x,y,z,w,s) & \mapsto & \begin{pmatrix} z & y+w & x+s \\ w & x & y \end{pmatrix} \end{array}$$

Let $p : \mathbb{C}^4 \to \mathbb{C}$ *be given by* $p(x,y,z,w) = w$*, then it follows that* $m_2(X,0) = 3$ *and* $\mu(X \cap p^{-1}(0), 0) = 2$*, then* $\mu(X,0) = 1$.

General and strongly general hyperplanes

We define general and strongly general hyperplanes over an EIDS in order to extend previous results by J. Snoussi and Lê D. T.
The two definitions are equivalent when the variety has an isolated singularity.
The sections defined by the intersection of the IDS (resp. EIDS) and the general hyperplane (resp. strongly general hyperplane) determine another IDS (resp. EIDS).

Definition

Let $X \subset \mathbb{C}^N$ be a d-dimensional analytic complex variety, and let $\{V_\lambda\}_{\lambda \in \Lambda}$ be a stratification of X. The hyperplane $H \subset \mathbb{C}^N$, given by the kernel of the linear function $p : \mathbb{C}^N \to \mathbb{C}$ is called general for X at 0 if H is transversal to all T which are limits at 0 of tangent spaces to the regular part of X. We say that H is strongly general at the origin if it is general and there exists a neighbourhood U of 0 such that for all strata V_λ of X, with $0 \in \overline{V}_\lambda$, we have that H is transversal to V_λ at x, for all $x \in U \setminus \{0\}$.

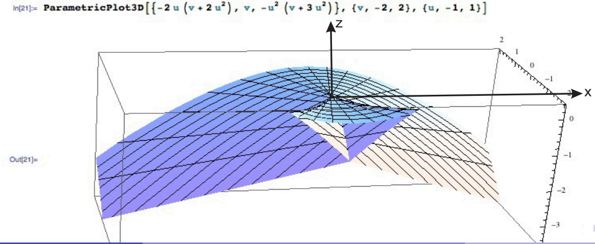
Example of a general hyperplane for X at 0 which is not strongly general.

Example

Let X be the swallowtail surface in $\mathbb{C}^3$ defined by the zeros of

$$256z^3 - 27x^4 - 128z^2y^2 + 144zx^2y + 16zy^4 - 4x^2y^3 = 0.$$

The set of the limits of tangent hyperplanes to X at zero is given by the hyperplane $z = 0$.
Any hyperplane different of the hyperplane $z = 0$ is general, in particular the plane $H = \{(x, y, z) : x = 0\}$ is general, but H is not transversal to the limits of lines tangent to the strata of dimension 1.



Theorem

Let $(X, 0) \subset (\mathbb{C}^N, 0)$ be an EIDS of type (m, n, t). If $H \subset \mathbb{C}^N$ is a strongly general hyperplane at the origin, then $(X \cap H, 0)$ is a $(d-1)$-dimensional EIDS in $(\mathbb{C}^{N-1}, 0)$ of the same type.

Dowód.

Let $F : \mathbb{C}^N \to M_{m,n}$ be a map defining $X = F^{-1}(M^t_{m,n})$. As X is an EIDS at the origin, then F is transversal to the strata of $M^t_{m,n}$ in $U \backslash \{0\}$, where U is a sufficiently small neighbourhood of the origin. By hypothesis H is transversal to the strata of X outside the origin. Let $i : \mathbb{C}^{N-1} \to \mathbb{C}^N$ be a linear embedding such $i(\mathbb{C}^{N-1}) = H$. Then $F \circ i : \mathbb{C}^{N-1} \to M_{m,n}$ is transversal to all the strata of $M^t_{m,n}$ outside the origin, so $X \cap H = (F \circ i)^{-1}(M^t_{m,n}) \subset \mathbb{C}^{N-1}$ is an EIDS of the same type of X and $\dim(X \cap H) = d - 1$ in $\mathbb{C}^{N-1}$. □

Minimality of the Milnor number

B. Teissier, J.-P. Henry and Lê D. T. studied the minimality of the Milnor number of generic sections of hypersurfaces with isolated singularities.
T. Gaffney proved the result for isolated complete intersection singularity (ICIS) and J. Snoussi considered the case of normal surfaces in $\mathbb{C}^N$.
We denote by $C_{X,x}$ the tangent cone of X at x with its reduced analytic structure.

Proposition (Snoussi, 2001)

Let $(X,0) \subset (\mathbb{C}^N, 0)$ be the germ of a normal analytic surface. A hyperplane H in $\mathbb{C}^N$ that does not contain any irreducible component of the tangent cone $C_{X,0}$ is general for X at 0 if and only if the section $X \cap H$ is reduced and the local Milnor number $\mu(X \cap H, 0)$ is minimum.

We extend the Snoussi Proposition to 3-determinantal varieties with isolated singularities.

Theorem

Let $(X,0) \subset (\mathbb{C}^N, 0)$ be the germ of a 3-dimensional determinantal variety with isolated singularity and H a hyperplane in $\mathbb{C}^N$. Suppose that $X \cap H$ has an isolated singular point at the origin, then the following conditions are equivalent.

(i) *the hyperplane H is general for X at the isolated singularity 0.*

(ii) *the Milnor number $\mu(X \cap H, 0)$ is minimum and the Milnor number $\mu(X \cap H \cap H', 0)$ is minimum for any hyperplane H' general for X and $X \cap H$ at 0.*

Example

Let $X \subset \mathbb{C}^5$ be a 3-determinantal variety with isolated singularity defined by

$$\begin{array}{rcl} F: \quad \mathbb{C}^5 & \to & M_{2,3} \\ (x,y,z,w,v) & \mapsto & \begin{pmatrix} x & y & z \\ w & v & x^2+y^2 \end{pmatrix} \end{array}$$

Let H and H' be hyperplanes given by the kernels of $p(x,y,z,w,v) = w - z$ and $p'(x,y,z,w,v) = x - v$. The surfaces $X \cap H$ and $X \cap H'$ are represented by the following matrices

$$\begin{pmatrix} x & y & z \\ z & v & x^2+y^2 \end{pmatrix} \text{ and } \begin{pmatrix} x & y & z \\ w & x & y^2 \end{pmatrix} \text{ respectively.}$$

It follows (from [Pereira and Ruas, 2014]) that $\mu(X \cap H, 0) = 4$ and $\mu(X \cap H', 0) = 2$, and that $\mu(X \cap H', 0) = 2$ is the minimum Milnor number.

Sections of EIDS

The following result is a generalization of a result of Lê D. T.

Theorem

Let $(X^d, 0) \subset (\mathbb{C}^N, 0)$ be an EIDS, $X^d = F^{-1}(M^t_{m,n})$. Let H, H' be strongly general hyperplanes for $(X, 0)$ at the origin.
Then H and H' contain planes P and P' respectively such that $\operatorname{codim} P = \operatorname{codim} P' = d - 2$ and for which the determinantal surfaces $X \cap P$ and $X \cap P'$ satisfy the following conditions :

a) *The varieties $X \cap P$ and $X \cap P'$ have isolated singularity.*
b) *The varieties $X \cap P$ and $X \cap P'$ admit smoothing.*
c) *The Milnor numbers $\mu(X \cap P, 0)$ and $\mu(X \cap P', 0)$ are the same.*

Example

Let $F : \mathbb{C}^N \to M_{2,3}$ *be an analytic map,* $N \geqslant 7$ *defined by*

$$F(x,y) = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 + g(y) \end{pmatrix},$$

where $x = (x_1, x_2, \dots, x_6)$*,* $y = (y_1, \dots, y_{N-6})$ *and* $g : \mathbb{C}^{N-6} \to \mathbb{C}$ *is an analytic function with* $g(0) = 0$*. Then* $X = F^{-1}(M^t_{m,n})$ *is a Cohen-Macaulay codimension 2 singularity in* $\mathbb{C}^N$*.*
Let $P = \{(x,y) |\ x_1 = x_5,\ x_2 = x_6,\ y = 0\}$*. Then* $X \cap P$ *is a determinantal surface in* $\mathbb{C}^4$ *defined by*

$$\begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_1 & x_2 \end{pmatrix}$$

and $\mu(X \cap P, 0) = 1$*.*

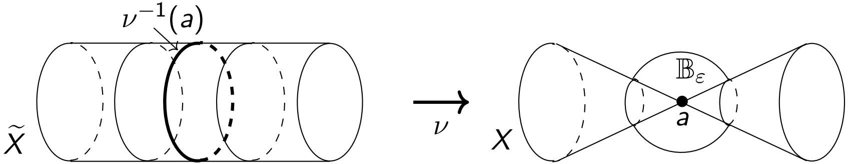
Nash transformation

Let M be a complex analytic manifold, of complex dimension N. Let X be an d-dimensional semi-analytic complex variety, $X \subset M$. We denote by X_{reg} the regular part of X.
The Grassmann bundle of d (complex) planes in TM is denoted by $G_d(TM)$. The fiber $G_d(T_x M)$ over $x \in M$ is the set of d-planes in $T_x(M)$ and is isomorphic to $G_d(\mathbb{C}^N)$. An element of $G_d(TM)$ is denoted by (x, P) where $x \in M$ and $P \in G_d(T_x M)$.
On the regular part of X, the Gauss map is defined by

$$\gamma : X_{\mathrm{reg}} \longrightarrow G_d(TM) \qquad \gamma(x) = (x, T_x(X_{\mathrm{reg}})).$$

The Nash transformation $\widetilde{X}$ is the closure of the image of γ in $G_d(TM)$.

$$\begin{array}{ccc} & & G_d(TM) \\ & \overset{\gamma}{\nearrow} & \downarrow \\ X_{\mathrm{reg}} & \hookrightarrow & M \end{array} \qquad\qquad \begin{array}{ccc} \widetilde{X} = \overline{\mathrm{Im}\gamma} & \hookrightarrow & G_n(TM) \\ \downarrow \nu & & \downarrow \\ X & \hookrightarrow & M \end{array}$$



Rysunek: The Nash transformation.

Mather classes

The fiber of the tautological bundle $\mathcal{T}$ over $G_d(TM)$, at point $P \in G_d(TM)$, is the set of vectors v in the d-plane P. The bundle $\widetilde{E}$, restriction of $\mathcal{T}$ to $\widetilde{X}$, is called the Nash bundle of X. It is equipped with the projection map π.

An element of $\widetilde{E}$ is written (x, P, v) where $x \in X$, P is a d-plane in $T_x(M)$ and v is a vector in P. We have the following diagram:

$$\begin{array}{ccc} \widetilde{E} & \hookrightarrow & \mathcal{T} \\ \pi \downarrow & & \downarrow \\ \widetilde{X} & \hookrightarrow & G_d(TM) \\ \nu \downarrow & & \downarrow \\ X & \hookrightarrow & M. \end{array}$$

MacPherson defined the Mather classes, by the formula

$$c_{\mathrm{Ma}}(X) = \nu_*(c(\widetilde{E}) \cap [\widetilde{X}]),$$

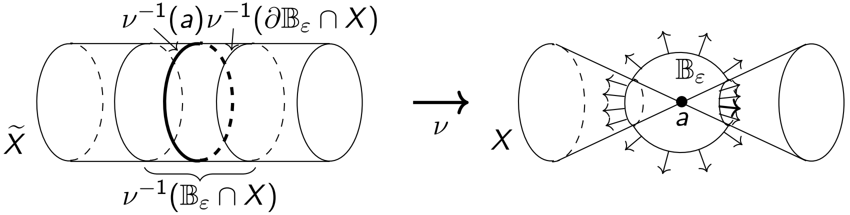
where $[\widetilde{X}]$ denotes the fundamental (orientation) homology class of $\widetilde{X}$.

Local Euler obstruction

Whitney condition (a) implies that a stratified vector field v defined on $A \subset X$ admits a canonical lifting $\tilde{v}$ on $\nu^{-1}(A)$ as a section of $\widetilde{E}$.

$$\begin{array}{ccc} \widetilde{E} & \xrightarrow{\nu} & TM|_X \\ {\scriptstyle \tilde{v}}\uparrow\downarrow & & {\scriptstyle v}\uparrow\downarrow \\ \widetilde{X} & \xrightarrow{\nu} & X \end{array} \qquad\qquad \tilde{v}(\tilde{x}) = (x, \tilde{x}, v(x))$$

Consider a radial stratified vector field v in a neighbourhood of the point $\{a\} \in X$ so that there exists $\varepsilon_0 > 0$ such that for all ε, $0 < \varepsilon < \varepsilon_0$, the vector $v(x)$ is pointing outwards the ball $\mathbb{B}_\varepsilon$ (centered at a) over the boundary $\mathbb{S}_\varepsilon = \partial\mathbb{B}_\varepsilon$. If ε is small enough, then $\mathbb{S}_\varepsilon$ is transverse to the strata V_α.



Rysunek: The local Euler obstruction.

Definition

Let v be a stratified vector field pointing outwards $\mathbb{B}_\varepsilon$ along $\mathbb{S}_\varepsilon$ and $\tilde{v}$ the lifting of v on $\nu^{-1}(X \cap \mathbb{S}_\varepsilon)$. The local Euler obstruction $\mathrm{Eu}_0(X)$ is the obstruction to extend $\tilde{v}$ as a nowhere zero section of $\tilde{E}$ over $\nu^{-1}(X \cap \mathbb{B}_\varepsilon)$, evaluated on the orientation class $\mathcal{O}_{\nu^{-1}(\mathbb{B}_\varepsilon),\nu^{-1}(\mathbb{S}_\varepsilon)}$:

$$Eu_a(X) = Obs(\tilde{v}, \widetilde{E}, \nu^{-1}(X \cap \mathbb{B}_\varepsilon)).$$

In (2018) Nancy Chachapoyas–Siesquén computes the local Euler obstruction of EIDS. The author obtains explicit formulae to calculate the local Euler obstruction for the determinantal varieties whose singular set is an isolated complete intersection singularity (ICIS).
In (2019) Terence Gaffney, Nivaldo G. Grulha Jr. and Maria A. S. Ruas compute the local Euler obstruction of generic determinantal varieties and apply this result to compute the Chern-Schwartz-MacPherson class of such varieties.

Theorem

Let $\Sigma^s \subset \mathrm{Hom}(\mathbb{C}^n, \mathbb{C}^{n+k})$ be a generic determinantal variety. The local Euler obstruction of Σ^s at 0 is

$$\mathrm{Eu}_0(\Sigma^s) = \binom{n}{s-1}, \qquad \text{for } 1 \leq s \leq n.$$

The authors provide explicit formulae, in different particular situations and, in particular, they recover the Chachapoyas formula.

In (2018) Xiping Zhang gives explicit formulae computing the Chern-Mather class and the Chern-Schwartz-MacPherson class of generic determinantal varieties. He also obtains formulae for the conormal cycles and the characteristic cycles of these varieties. For some small values of n, k and s, Zhang uses Macaulay2 to exhibit examples of the considered classes.
In (2021), Xiping Zhang finds explicit formulae for Chern-Schwartz-MacPherson classes and Chern-Mather classes of EIDS via Schubert calculus. As corollaries the author obtains formulae for their generic sectional Euler characteristics, characteristic cycles and polar classes.

Main Bibliography

J.-P. Brasselet, N. Chachapoyas, and M. Ruas. Generic sections of essentially isolated determinant singularities. International Journal of Mathematics. Vol. 28, No. 11, (2017)

S. M. Gusein-Zade and W. Ebeling. On the indices of 1-forms on determinantal singularities. Mosc. Math. J., 2003, Volume 3, Number 2, Pages 439–455

T. Gaffney, N. Grulha and M. A. Soares Ruas. Equisingularity and EIDS. Proc. Amer. Math. Soc. 149 (2021), 1593-1608

T. Gaffney and M. A. Soares Ruas. The local Euler obstruction and topology of the stabilization of associated determinantal varieties. April 2019 Mathematische Zeitschrift 291 (6), 2019.

D. T. Lê. Generic sections of singularities. In Singularity theory, pages 67782. World Sci. Publ., Hackensack, NJ, 2007.

J. Snoussi. Limites d'espaces tangents à une surface normale. Comment. Math. Helv., 76(1):618, 2001.

Thank you for your attention.

Dziękuję za uwagę