Generalizations of Newton non-degeneracy for hypersurface singularities

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(GKŁW, 2022.11.18.)

Newton non-degenerate (Nnd) hypersurface germs, $V(f) \subset (\mathbb{C}^n, o)$, are (often) simple to deal with. Their topological type is determined by the Newton diagram. (Hence various topological invariants can be computed combinatorially.) But being Nnd is a highly restrictive condition, even for plane curve germs.

In arXiv:0807.5135 I have introduced a generalization of Nnd-hypersurface singularities. An isolated hypersurface germ is called "directionally Newton-non-degenerate" (dNnd) if the non-degeneracy holds "in each particular direction". Equivalently, such a singularity is resolvable by a "poly-toric blowup". For such singularities various invariants (e.g. the Milnor number, the zeta function) are determined by the collection of Newton diagrams.

The class of dNnd singularities is still restricted, even for plane curve germs. The broadest generalization of Newton-non-degeneracy is obtained by considering all the Newton diagrams (in all the possible coordinate systems).

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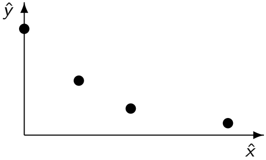

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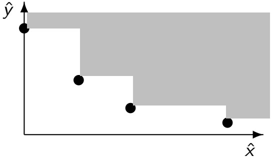

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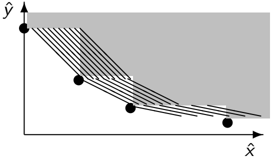

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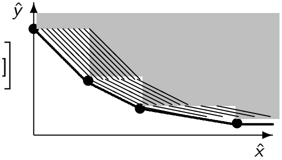

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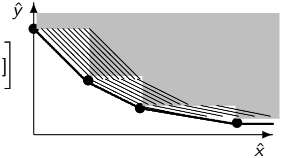

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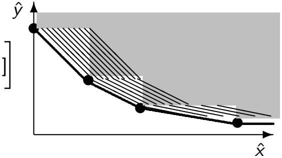

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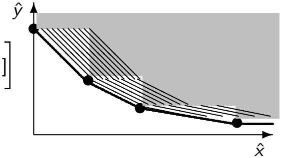

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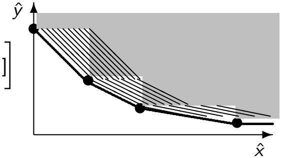

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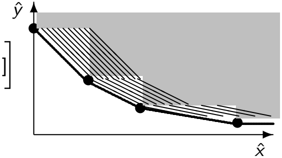

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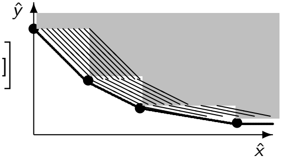

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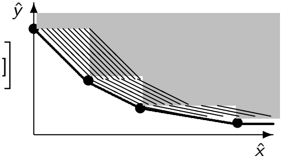

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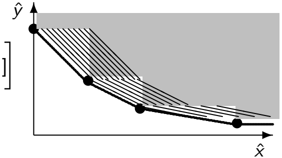

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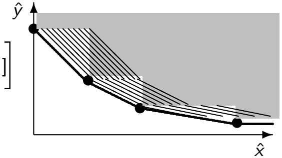

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Corollary. All the topological invariants of $V(f)$ are determined by Γ_f.

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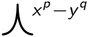
Explicit cases: 

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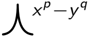
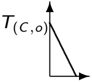
Explicit cases: 



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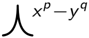
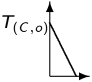
Explicit cases: 



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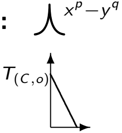

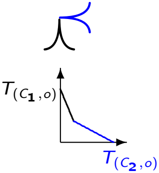

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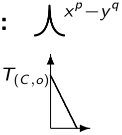

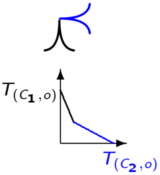

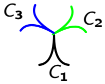

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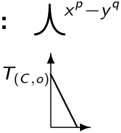

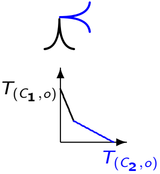

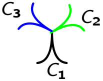

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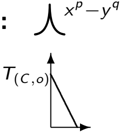

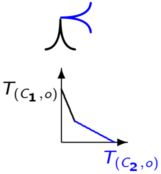

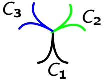

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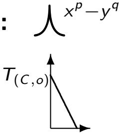

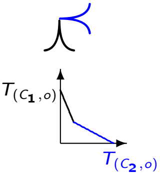

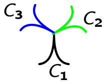

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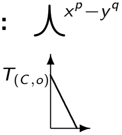

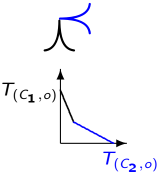

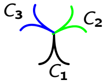

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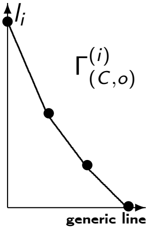

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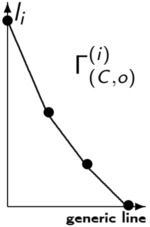

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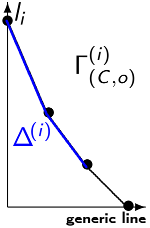

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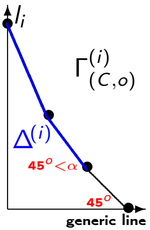

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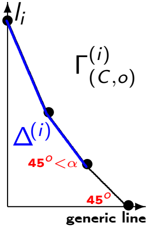

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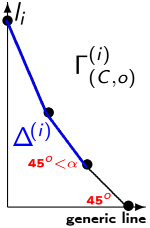

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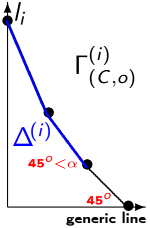

Example. Let $(C,o)=\cup(C_i,o)\subset(\mathbb{C}^2,o)$ pairwise non-tangent branches, each with one Puiseux pair. Then (C,o) is dNnd.
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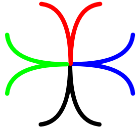

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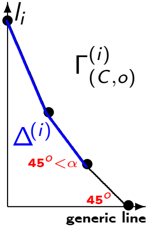

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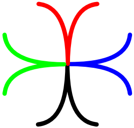

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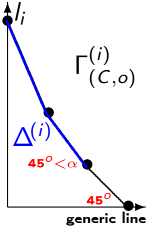

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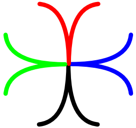

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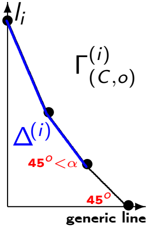

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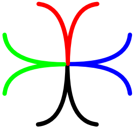

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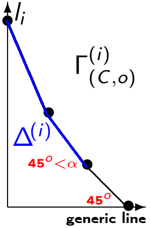

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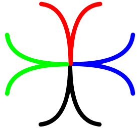

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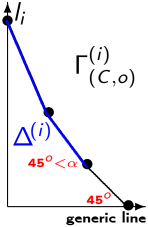

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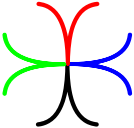

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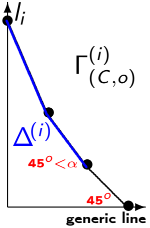

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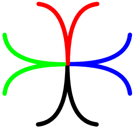

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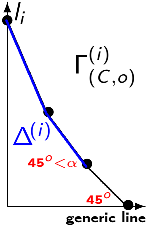

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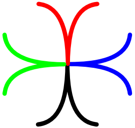

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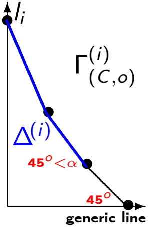

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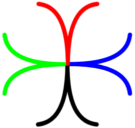

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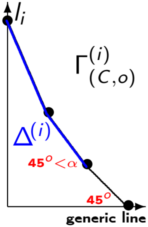

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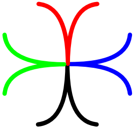

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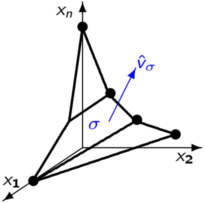

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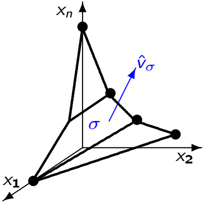

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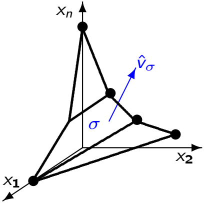

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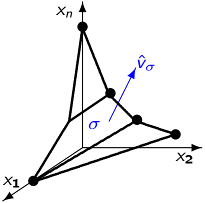

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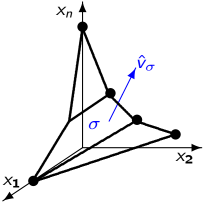

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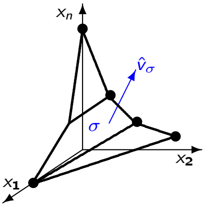

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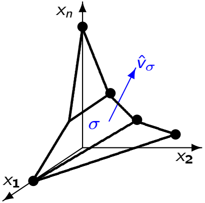

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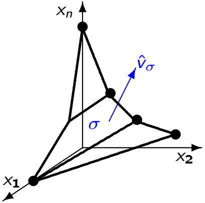

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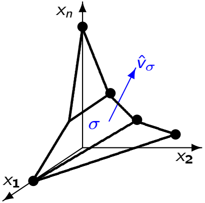

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Topological invariants for dNnd hypersurface germs

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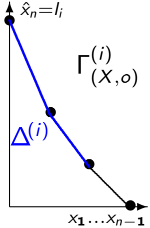

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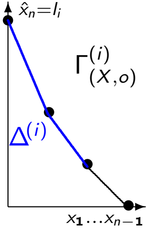

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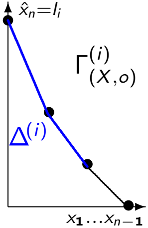

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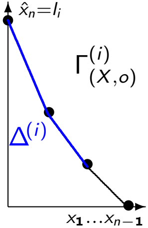

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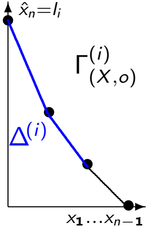

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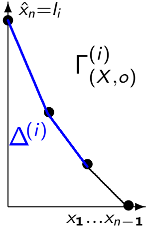

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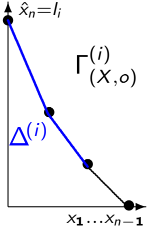

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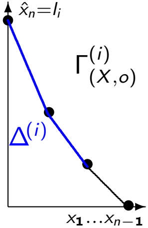

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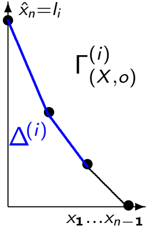

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The properties of wNnd germs are studied in arXiv:0807.5135.

Thanks for your attention!