Germs of maps, group actions and large modules inside group orbits.

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Abstract

Consider map-germs $(\mathbb{k}^n, o) \to (\mathbb{k}^p, o)$ up to the groups of right/left-right/ contact equivalence. The group orbits are complicated and are traditionally studied via their tangent space. This transition is classically done by vector fields integration, thus binding the theory to the real/complex case.

I will present the new approach to this subject. One studies the maps of germs of Noetherian schemes, in any characteristic. The corresponding groups of equivalence admit 'good' tangent spaces. The submodules of the tangent spaces lead to submodules of the group orbits. This extends (and sometimes strengthens) classical results on 'determinacy vs infinitesimal determinacy'.

Based on arXiv:1212.6894 (jointly with G. Belitski), arXiv:1808.06185 (jointly with A.-F. Boix, G.-M. Greuel) and arXiv:2111.02715.

Prologue: $f \in Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)$ for $\mathbb{k}\in\mathbb{R},\mathbb{C}$.

1. Suppose $\mathbf{rank[f'|_o] = min(n,p)}$. (i.e. f is an immersion or a submersion)
   Then $\mathbf{f(\underline{x})}{=}(\mathbf{x_1},\dots,\mathbf{x_{min(n,p)}},\mathbf{0},..,\mathbf{0})$ in some coordinates on $(\mathbb{k}^n,o)$ and $(\mathbb{k}^p,o)$.
   (At a non-critical point "there is no local geometry/topology/algebra".)

2. Let $(\mathbb{k}^1,o)\stackrel{f}{\to}(\mathbb{k}^1,o)$, with $f'|_o=0$. Then (in some coordinates) $\mathbf{f(x)=x^d}$.

   For a general critical point $(\mathbb{k}^n,o)\stackrel{f}{\to}(\mathbb{k}^p,o)$ no nice canonical form is possible.

   Goal: (a weaker statement) **"The higher order terms are not important."**
   (i.e. can be eliminated by coordinate changes)

3. (Morse lemma) Take $(\mathbb{k}^n,o)\stackrel{f}{\to}(\mathbb{k}^1,o)$, with $f'|_o=(0,\dots,0)$. Assume $f''|_o\in Mat_{n\times n}(\mathbb{k})$ is non-degenerate. Then (in some local coordiates) $f(x)$=homogeneous polynomial of degree 2. And then diagonalize, $\mathbf{f}\rightsquigarrow\sum(\pm)\mathbf{x_i^2}$.

Prologue: $f \in Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ for $\Bbbk \in \mathbb{R}, \mathbb{C}$.

Consider maps up to coordinate changes in $(\Bbbk^n, o)$ and $(\Bbbk^p, o)$:

- (The right equivalence) $\mathscr{R} \circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ by $f \rightsquigarrow f \circ \Phi_X^{-1}$, here $\Phi_X \circlearrowright (\Bbbk^n,o)$.
- (The left equivalence) $\mathscr{L} \circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ by $f \rightsquigarrow \Phi_Y \circ f$, here $\Phi_Y \circlearrowright (\Bbbk^p,o)$.
- (The $\mathscr{A}$-equivalence) $\mathscr{A} := \mathscr{L} \times \mathscr{R} \circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ by $f \rightsquigarrow \Phi_Y \circ f \circ \Phi_X^{-1}$.
- (The $\mathscr{K}$-equivalence) $\mathscr{K} \circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ by $f \rightsquigarrow GL(p, \mathcal{O}_{(\Bbbk^n,o)}) \cdot f \circ \Phi_X^{-1}$.

Q. How large are the group orbits, $\mathscr{R}f$, $\mathscr{A}f$, $\mathscr{K}f$?
Can we eliminate higher order terms of f? Maybe bring f to a "nice" form?

These questions have been extensively studied. Numerous good criteria exist. But the whole theory was chained to the $\mathbb{R}, \mathbb{C}$ case.(Because of vector field integration.)

My goal: purely algebraic version (in any characteristic). There are some results. Surprisingly, they are often stronger than the classical ones, even for $\Bbbk \in \mathbb{R}, \mathbb{C}$.

$f \in Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)$ $\mathbb{k}$-any field, e.g. $\mathbb{Q},\mathbb{R},\mathbb{C},\mathbb{Q}_p$, a finite field, etc.

$(\mathbb{k}^n,o)$ is the (formal/analytic/...) germ. Namely, $\mathcal{O}_{(\mathbb{k}^n,o)}=\mathbb{k}[[x]],\mathbb{k}\{x\}$ ($\mathbb{k}$-normed, complete) or $\mathbb{k}\langle x\rangle$ (algebraic power series).

Fix coordinates $(y_1\dots,y_p)$ on $(\mathbb{k}^p,o)$. Then $Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)\ni f=(f_1\dots,f_p)$. Thus $Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)\cong\mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.

- $\mathscr{R}:=Aut(\mathbb{k}^n,o)=Aut_{\mathbb{k}}(\mathcal{O}_{(\mathbb{k}^n,o)})\circlearrowright Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)$ by $f\rightsquigarrow f\circ\Phi_X^{-1}$.
- $\mathscr{L}:=Aut(\mathbb{k}^p,o)=Aut_{\mathbb{k}}(\mathcal{O}_{(\mathbb{k}^p,o)})\circlearrowright Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)$ by $f\rightsquigarrow\Phi_Y\circ f$.
- $\mathscr{A}=\mathscr{L}\times\mathscr{R}\circlearrowright Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big)$ by $f\rightsquigarrow\Phi_Y\circ f\circ\Phi_X^{-1}$. And $\mathscr{K}=GL(p,\mathcal{O}_{(\mathbb{k}^p,o)})\rtimes\mathscr{R}$.

Q. Let $\mathscr{G}\in\mathscr{R},\mathscr{A},\mathscr{K}$. How large is the group orbit $\mathscr{G}f$?
Find the largest ideal $J\subset\mathcal{O}_{(\mathbb{k}^n,o)}$ satisfying: $\mathscr{G}f\supseteq\{f\}+J\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.
(Thus if $f-\tilde{f}\in J\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ then $f\overset{\mathscr{G}}{\sim}\tilde{f}$. E.g. for $J\supseteq\mathfrak{m}^d$ get finite determinacy.)

The $\mathscr{R}$-orbits (characteristic free, $p=1$) $f\in\mathfrak{m}\subset\mathcal{O}_{(\mathbb{k}^n,o)}$, $f\overset{\mathscr{R}}{\rightsquigarrow} f\circ\Phi_X^{-1}$.

The standard approach: study the orbit $\mathscr{R}f$ via its "tangent space" $T_{\mathscr{R}}f$. Note: the group $\mathscr{R}$ is not Lie/(pro-)algebraic/pro-finite. Thus we *define* $T_{\mathscr{R}}$, and then establish the relation $T_{\mathscr{R}}f \rightleftarrows \mathscr{R}f$.

The tangent space to the group: $T_{\mathscr{R}}$ =(germs of vector fields at o)= $\{\sum a_i(x)\frac{\partial}{\partial x_i}\} = Der_{\mathbb{k}}(\mathcal{O}_{(\mathbb{k}^n,o)})$ = ($\mathbb{k}$-linear derivations of the ring).
The "tangent space to the orbit": $(T_{\mathscr{R}f} \underset{char(\mathbb{k})=0}{=})T_{\mathscr{R}}f = Jac(f)\subseteq\mathcal{O}_{(\mathbb{k}^n,o)}$.

Proposition. *Let $f\in\mathfrak{m}^3$. Suppose an ideal $J\subset\mathcal{O}_{(\mathbb{k}^n,o)}$ satisfies: $J^2\subseteq J\cdot Jac(f)$. Then $\mathscr{R}f\supseteq\{f\}+J^2$.*
Example. $\mathscr{R}f\supseteq\{f\}+Jac(f)^2$. (Here f can have a non-isolated critical point.)

Theorem (Mather, $\mathbb{k}=\mathbb{R},\mathbb{C}$). *If $\mathfrak{m}^d\subseteq\mathfrak{m}^2\cdot Jac(f)$ then $\mathscr{R}f\supseteq\{f\}+\mathfrak{m}^d$.*

Let us compare these bounds. Assume $r\geq 4$ and $char(\mathbb{k})\nmid r$.

$f(x_1,x_2)=x_1^r+x_2^r$	$f(x_1,\dots,x_n)=\sum_{i=1}^n x_i^r$	The proposition is known for $\mathbb{k}=\mathbb{R},\mathbb{C}$.
$\mathfrak{m}^{2r-3}\subset\mathfrak{m}^2\cdot Jac(f)$	$\mathfrak{m}^{n(r-1)-1}\subset\mathfrak{m}^2\cdot Jac(f)$	[Ebeling]
Old: $f+\mathfrak{m}^{2r-3}\subseteq\mathscr{R}f$	Old: $f+\mathfrak{m}^{n(r-1)-1}\subseteq\mathscr{R}f$	(via vector fields
New: $f+(x^{r-1},y^{r-1})^2\subseteq\mathscr{R}f$	New: $f+(\{x_i^{r-1}\})^2\subseteq\mathscr{R}f$	integration)

Proposition. *Let $f \in \mathfrak{m}^3$. Suppose an ideal $J \subset \mathcal{O}_{(\mathbb{k}^n,o)}$ satisfies: $J^2 \subseteq J \cdot Jac(f)$. Then $\mathscr{R}f \supseteq \{f\} + J^2$.*

Remarks.

1. Given f we can eliminate various "higher order terms". (This is useful!)

2. The orbit $\mathscr{R}f \subset Maps((\mathbb{k}^n,o),(\mathbb{k}^p,o))$ is highly non-linear. But it contains a huge linear subspace.

3. This statement holds for more general germs/rings. e.g. for $\mathcal{O}_{(\mathbb{k}^n,o)}/_I$ (or local henselian Noetherian rings), with a technical assumption when $char(\mathbb{k}) > 0$. Geometricaly: we have functions on singular (germs of) spaces, and their $\mathscr{R}$-orbits.

4. Similar statements hold for $\mathscr{K}$ and $\mathscr{A}$-equivalences. The $\mathscr{A}$-case is essentially more complicated.

The action $\mathscr{A}=\mathscr{L}\times\mathscr{R}\circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ by $f\rightsquigarrow \Phi_Y\circ f\circ\Phi_X^{-1}$.

- $\Phi_X\circlearrowright(\Bbbk^n,o)$, i.e. $\Phi_X\in Aut_{\Bbbk}(\mathcal{O}_{(\Bbbk^n,o)})$ (local coordinate changes).
- $\Phi_Y\circlearrowright(\Bbbk^p,o)$, i.e. $\Phi_Y\in Aut_{\Bbbk}(\mathcal{O}_{(\Bbbk^p,o)})$.

The action $\mathscr{R}\circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ is $\Bbbk$-linear. $(f+\tilde{f})\circ\Phi_X^{-1}=f\circ\Phi_X^{-1}+\tilde{f}\circ\Phi_X^{-1}$.
The action $\mathscr{L}\circlearrowright Maps\big((\Bbbk^n,o),(\Bbbk^p,o)\big)$ is not $\Bbbk$-linear (neither additive, nor multiplicative).

The tangent space $T_{\mathscr{A}}:=T_{\mathscr{L}}\oplus T_{\mathscr{R}}=Der_{\Bbbk}(\mathcal{O}_{(\Bbbk^p,o)})\oplus Der_{\Bbbk}(\mathcal{O}_{(\Bbbk^n,o)})$.
The "tangent space to the orbit": $(T_{\mathscr{A}f}\underset{char(\Bbbk)=0}{=})T_{\mathscr{A}}f=T_{\mathscr{L}}f+T_{\mathscr{R}}f$.

Example. Let $(\Bbbk^n,o)\stackrel{f}{\to}(\Bbbk^1,o)$. Then $T_{\mathscr{R}}f=Jac(f)$, $T_{\mathscr{K}}f=Jac(f)+(f)\subset\mathcal{O}_{(\Bbbk^n,o)}$.

But $T_{\mathscr{A}}f=Jac(f)+Span_{\Bbbk}(1,f,f^2,\dots)\subset\mathcal{O}_{(\Bbbk^n,o)}$. Not an ideal, only a vector space!

The action $\mathscr{A} \circlearrowright Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big) \cong \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ and the orbits $\mathscr{A}f$ vs $T_{\mathscr{A}}f$

Goal: The largest ideal $J \subset \mathcal{O}_{(\mathbb{k}^n,o)}$ satisfying: $\mathscr{A}f \supseteq \{f\} + J\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.

Let $f \in \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ and assume: the field $\mathbb{k}$ is infinite.

Theorem (K.2021) Suppose an ideal $J \subset \mathcal{O}_{(\mathbb{k}^n,o)}$ satisfies:
$J^2\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)} \subseteq \mathfrak{m}\cdot J\cdot T_{\mathscr{R}}f + f^{-1}(y^2)\cdot T_{\mathscr{L}}f$. Then $\mathscr{A}f \supseteq \{f\} + J^2\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)} + f^{-1}(y^2)\cdot T_{\mathscr{L}}f$.

Example. $(p=1,\ J=\mathfrak{m}^d)$ If $\mathfrak{m}^{2d} \subseteq \mathfrak{m}^{d+1}\cdot Jac(f) + Span_{\mathbb{k}}(f^2, f^3, \dots)$,
then $\mathscr{A}f \supseteq \{f\} + \mathfrak{m}^{2d} + Span_{\mathbb{k}}(f^2, f^3, \dots)$.

Remarks. • This is a linearization result. It translates the study of the orbit $\mathscr{A}f$ to the tangent space $T_{\mathscr{A}}f$. It remains to verify that $T_{\mathscr{A}}f$ is "large enough" .

• For $\mathscr{R}$-case we had the condition $J^2 \subseteq J\cdot Jac(f)$. This is easy to ensure, e.g. one can take $J = Jac(f)$. For $\mathscr{A}$-case the condition is
$J^2\cdot R_X^{\oplus p} \subseteq \mathfrak{m}\cdot J\cdot T_{\mathscr{R}}f + f^{-1}(y^2)\cdot T_{\mathscr{L}}f$. More delicate.
To get effective bounds one needs "The structure of $T_{\mathscr{A}}f$". (A separate talk.)

• There are stronger/more general (but more technical) versions.

• The proof does not use unfoldings/vector field integration. It uses the implicit function theorem (Tougeron's style), the Weierstraß division and some commutative algebra.

• Guess: the assumption "$\mathbb{k}$ is infinite" is not necessary. (But I need it in the proof.)

Another approach to determinacy, via filtrations

Take a map $f \in Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big) \cong \mathfrak{m} \cdot \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$, we get the orbit $\mathscr{G} f \subset \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.
If $\mathscr{G} f \supseteq \{f\}+$(linear space) then $T_{\mathscr{G}} f \supseteq$(linear space).
Thus we want "the largest part of $T_{\mathscr{G}} f$" that lies inside $\mathscr{G} f$.

Theorem (K.2021). Fix some integers $1 \leq j < d$ and an ideal $I \subset \mathcal{O}_{(\mathbb{k}^n,o)}$.

- ($\mathscr{R}$-equivalence, $p=1$) Suppose $char(\mathbb{k})=0$ or $char(\mathbb{k}) > \lceil \frac{d-ord(f)}{j} \rceil$. Then: $\mathscr{R}^{(j)} f \supseteq \{f\} + I^d$ if and only if $T_{\mathscr{R}^{(j)}} f \supseteq I^d$.
- ($\mathscr{A}$-equivalence) Suppose $char(\mathbb{k})=0$ or $char(\mathbb{k}) > \lceil \frac{2d-1-ord(f)}{j} \rceil$. Then: $\mathscr{A}^{(j)} f \supseteq \{f\} + I^d \cdot \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ if and only if $T_{\mathscr{A}^{(j)}} f \supseteq I^d \cdot \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.

Now we define the subgroups $\mathscr{G}^{(j)} \leq \mathscr{G}$ and the subspaces $T_{\mathscr{G}^{(j)}} \subseteq T_{\mathscr{G}}$.

Filtrations $I^j \subset R$, $\mathscr{G}^{(j)} \leq \mathscr{G}$ and $T_{\mathscr{G}^{(j)}} \subseteq T_{\mathscr{G}}$. Fix an ideal $I \subseteq \mathfrak{m}$. Take the filtration on the space of maps, $Maps((\mathbb{k}^n,o),(\mathbb{k}^p,o)) \cong \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)} \supseteq I\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)} \supset I^2\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)} \supset \cdots$.

- For an(y) action $\mathscr{G} \circlearrowright \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ take those elements that preserve the filtration: $\mathscr{G} \geq \mathscr{G}^{(0)} := \{g \in \mathscr{G} |\ g(I^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}) = I^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)},\ \forall\ d\}$. For $j \geq 1$ define $\mathscr{G}^{(0)} \triangleright \mathscr{G}^{(j)} := \{g \in \mathscr{G}^{(0)} |\ g|_{I^d/I^{d+j}} = Id|_{I^d/I^{d+j}},\ \forall\ d\}$, i.e.,$g$ acts as identity modulo "higher" order terms. Hence the group-filtration: $\mathscr{G} \geq \mathscr{G}^{(0)} \triangleright \mathscr{G}^{(1)} \triangleright \mathscr{G}^{(2)} \triangleright \cdots$

- Similarly, given a tangent space $T_{\mathscr{G}} \circlearrowright \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$, take those derivations that preserve the filtration: $T_{\mathscr{G}} \supset T_{\mathscr{G}^{(0)}} = \{\xi \in T_{\mathscr{G}} |\ \xi(I^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}) \subseteq I^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}\}$. For $j \geq 1$ take the "filt.nilpotent" derivations $T_{\mathscr{G}} \supset T_{\mathscr{G}^{(j)}} = \{\xi \in T_{\mathscr{G}} |\ \xi(I^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}) \subseteq I^{d+j}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}\}$. Hence the tangent space filtration: $T_{\mathscr{G}} \supset T_{\mathscr{G}^{(0)}} \supseteq T_{\mathscr{G}^{(1)}} \supseteq \cdots$

Example. $\mathscr{G} = \mathscr{R}$, $I = \mathfrak{m}$. The filtration $\mathcal{O}_{(\mathbb{k}^n,o)} \supset \mathfrak{m} \supset \mathfrak{m}^2 \supset \cdots$. Then:

- $\mathscr{R} = \mathscr{R}^{(0)}$, as local coordinate changes preserve the origin. And $\mathscr{R}^{(j)} = \{x \to x + h(x) |\ h(x) \in (x)^{j+1}\}$.
- $T_{\mathscr{R}} = Der_{\mathbb{k}}(\mathcal{O}_{(\mathbb{k}^n,o)}) \supset T_{\mathscr{R}^{(0)}} = \{\xi |\ \xi(\mathfrak{m}) \subseteq \mathfrak{m}\} \supset T_{\mathscr{R}^{(j)}} = \{\xi |\ \xi(\mathfrak{m}) \subseteq \mathfrak{m}^{j+1}\}$. (vector fields that vanish at o up to order $j+1$.)

Theorem (K.2021). Fix some integers $1 \le j < d$ and an ideal $I \subset \mathcal{O}_{(\mathbb{k}^n,o)}$.

- ($\mathscr{R}$, $p = 1$) Suppose $char(\mathbb{k}) = 0$ or $char(\mathbb{k}) > \lceil \frac{d - ord(f)}{j} \rceil$. Then:
  $\mathscr{R}^{(j)} f \supseteq \{f\} + I^d$ if and only if $T_{\mathscr{R}^{(j)}} f \supseteq I^d$.

- ($\mathscr{A}$) Suppose $char(\mathbb{k}) = 0$ or $char(\mathbb{k}) > \lceil \frac{2d-1-ord(f)}{j} \rceil$. Then:
  $\mathscr{A}^{(j)} f \supseteq \{f\} + I^d \cdot \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$ if and only if $T_{\mathscr{A}^{(j)}} f \supseteq I^d \cdot \mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$.

Example. ($\mathscr{R}$, $p=1$, $I=\mathfrak{m}$ and $j = 1$) $\mathscr{R}^{(1)} f \supseteq \{f\} + \mathfrak{m}^d$ iff $\mathfrak{m}^2 \cdot Jac(f) \supseteq \mathfrak{m}^d$.
(Assuming $char(\mathbb{k}) = 0$ or $char(\mathbb{k}) > d - ord(f)$.)

Remarks.

- This theorem holds over local Noetherian rings, with technical assumptions.
- This result is new, but “the ideology" is well known, [Gaffney], [du Plessis], [Bruce - du Plessis - Wall], [Belitskii-K.], [Boix-Greuel-K.].
- This theorem generalizes the classical criteria over $\mathbb{R}, \mathbb{C}$. The proof does not use vector field integration/unfoldings/finite jets. Yet, this result is restricted by $char(\mathbb{k})$ assumption.
- What happens in low characteristic? The situation is more delicate. This was studied in [Boix-Greuel-K.] for $\mathscr{G} \in \mathscr{R}, \mathscr{K}$, with weaker bounds. (For low characteristic these weaker bounds are sharp!)

An application: relative algebraization of maps

Let $f \in Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big) \cong \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$. Suppose f is finitely $\mathscr{G}$-determined, i.e., $\mathscr{G}f \supseteq \{f\} + \mathfrak{m}^d\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$, for $d \gg 1$. Then $f \overset{\mathscr{G}}{\sim}$(a polynomial map).

What happens if f is not finitely $\mathscr{G}$-determined? (i.e. its critical/singular/ instability locus is of positive dimension)

Example (Whitney): $f(x,y,z) = xy(x+y)(x-zy)(x-e^z y) \in \mathbb{C}\{x,y,z\}$ is not $\mathscr{K}$-equivalent to a polynomial. Here $Sing(V(f)) = V(x,y) \subset \mathbb{C}^3$.

Note: at least f is a polynomial "in the direction transverse to the $\hat{z}$-axis". We get the natural generalization.

Proposition. Let $f \in Maps\big((\mathbb{k}^n,o),(\mathbb{k}^p,o)\big) \cong \mathfrak{m}\cdot\mathcal{O}^{\oplus p}_{(\mathbb{k}^n,o)}$, with $\mathbb{k}=\bar{\mathbb{k}}$ (any characteristic). Let $\mathscr{G} \in \mathscr{R},\mathscr{K},\mathscr{A}$. Suppose the critical/singular/instability locus is of codimension c in $(\mathbb{k}^n,o)$. Then f is $\mathscr{G}$-equivalent to an element of $\mathcal{O}_{n-c}[x_1,\dots,x_c]^{\oplus p}$.

Here $\mathcal{O}_{n-c} = \mathbb{k}[[x_{c+1},\dots,x_n]]$ or $\mathbb{k}\{x_{c+1},\dots,x_n\}$, thus $\mathcal{O}_{n-c}[x_1,\dots,x_c]$=polynomials in variables $x_1,\dots,x_c$, with coefficients - power series in the rest of the variables.

Thanks for your attention!