# ON LOCAL REDUCTION THEOREMS FOR SINGULAR SYMPLECTIC FORMS ON A 4-DIMENSIONAL MANIFOLD

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We study local invariants of singular symplectic forms with structurally smooth Martinet hypersurfaces on a 4-dimensional manifold $M$. We prove that the equivalence class of a germ at $p \in M$ of a singular symplectic form $\omega$ is determined by the Martinet hypersurface, the canonical orientation of it, the pullback of the singular symplectic form to it and the 2-dimensional kernel of $\omega$ at $p$. We also show which germs of closed 2-forms on a 3-dimensional submanifold can be realizable as pullbacks of singular symplectic forms to structurally smooth Martinet hypersurfaces.

*Keywords*: Symplectic forms; Singularities; Normal forms

## 1. Introduction

Let $\omega$ be a closed 2-form on a $2n$-dimensional manifold $M$. $\omega$ is a symplectic form on $M$ if for any $p \in M$

$$\omega^n|_p = \omega \wedge \cdots \wedge \omega|_p \neq 0. \tag{1}$$

By the Darboux Theorem there exists a system of local coordinates $(p_1, \cdots, p_n, q_1, \cdots, q_n)$ around any point $p \in M$ such that

$$\omega = \sum_{i=1}^{n} dp_i \wedge dq_i.$$

If the set of points $p \in M$, where $\omega$ does not satisfy (1), is nowhere dense we call $\omega$ a *singular symplectic form*.

In this paper we study local invariants of singular symplectic forms on a 4-dimensional manifold.

Because our consideration is local, we may assume that $\omega$ is a germ of a $\mathbb{K}$-analytic or smooth closed 2-form on $\mathbb{K}^4$ for $\mathbb{K} = \mathbb{R}$ or $\mathbb{K} = \mathbb{C}$ . Then

$\omega^2 = f\Omega$, where $f$ is a function-germ at 0 and $\Omega$ is a germ at 0 of a volume form on $\mathbb{K}^4$.

The *Martinet hypersurface* $\Sigma_2 = \Sigma_2(\omega)$ is the following set

$$\left\{p \in \mathbb{K}^4 : \omega^2|_p = 0\right\} = \{f = 0\}.$$

We assume that $f(0) = 0$ and $df_0 \neq 0$. Then $\Sigma_2$ is called *structurally smooth at* 0. In dimension 4 such situation is generic (see [12]).

Let $\omega$ be a germ of a singular symplectic form with a structurally smooth Martinet hypersurface at 0. It is obvious that $\Sigma_2$ is an invariant of $\omega$. It is also obvious that the pullback of $\omega$ to $\Sigma_2$ is an invariant of $\omega$. In this paper we consider the following problem.

*Do the Martinet hypersurface $\Sigma_2$ and the pullback of $\omega$ to $\Sigma_2$ form a complete set of invariants?*

The starting point of this paper is the articles [8,9] where an affirmative answer to the above question is given for all local *singular contact structures* excluding degenerations of infinite codimension. B. Jakubczyk and M. Zhitomirskii show that local $\mathbb{C}$-analytic singular contact structures on $\mathbb{C}^3$ with structurally smooth Martinet hypersurfaces are diffeomorphic if their Martinet hypersurfaces and restrictions of singular structures to them are diffeomorphic. In the $\mathbb{R}$-analytic category a complete set of invariants contains, in general, one more independent invariant. It is a canonical orientation on the Martinet hypersurface. The same is true for smooth local singular contact structures $P = (\alpha)$ on $\mathbb{R}^3$ provided $\alpha|_S$ is either not flat at 0 or $\alpha|_S = 0$. The authors also study local singular contact structures in higher dimensions. They find more subtle invariants of a singular contact structure $P = (\alpha)$ on $\mathbb{K}^{2n+1}$ : a line bundle $L$ over the Martinet hypersurface $S$, a canonical partial connection $\Delta_0$ on the line bundle $L$ at $0 \in \mathbb{K}^{2n+1}$ and a 2-dimensional kernel $ker(\alpha \wedge (d\alpha)^{n-1})|_0$. They also consider the more general case when $S$ has singularities.

For the first occurring singularities of singular symplectic forms on a 4-dimensional manifold the answer for the above question follows from Martinet's normal forms of types $\Sigma_{20}$ and $\Sigma_{220}$ (see [11,12,15]). In fact it is proved that the Martinet hypersurface $\Sigma_2$ and a characteristic line field on $\Sigma_2$ (i.e. $\{X \textit{ is a smooth vector field } : X\rfloor(\omega|_{T\Sigma_2}) = 0\}$) form a complete set of invariants. Since $(\omega|_{T\Sigma_2})|_0 \neq 0$ for $\Sigma_{20}$-singularity, then its characteristic line field is generated by a non-vanishing vector field. But for $\Sigma_{220}$-singularity both $\omega|_{T\Sigma_2}$ and the characteristic line vanish at 0 (see [11,15]).

In this paper we assume that $\omega|_{T\Sigma_2}$ vanishes at 0 (if $\omega|_{T\Sigma_2}$ does not vanish at 0 then $\omega$ is a symplectic singular form of type $\Sigma_{20}$ and these problems for this singularity are solved in [12]). We show that a complete set of invariants for local $\mathbb{C}$-analytic singular symplectic forms on $\mathbb{C}^4$ with structurally smooth Martinet hypersurfaces consists of the Martinet hypersurface, the pullback of the singular symplectic form to it and the 2-dimensional kernel of the singular symplectic form at 0 (Theorem 3.1). The same is true for local $\mathbb{R}$-analytic and smooth singular symplectic forms on $\mathbb{R}^4$ with structurally smooth Martinet hypersurfaces if we add to the invariants the canonical orientation of the Martinet hypersurface (Theorem 3.2). These results are obtained as corollaries of Theorem 2.1 on 'normal' forms of singular symplectic forms with a given pullback to the Martinet hypersurface. Another corollary of Theorem 2.1 is a realization theorem (Theorem 2.2), where we show which closed 2-forms on $\mathbb{K}^3$ vanishing at 0 can be obtained as a pullback of a singular symplectic form to its Martinet hypersuface.

In section 4 (see Theorems 4.1, 4.2) we also prove that an equivalence class of a $\mathbb{K}$-analytic singular symplectic form $\omega$ on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface is determined only by the Martinet hypersurface, its canonical orientation (only if $\mathbb{K} = \mathbb{R}$) and the pullback of the singular form to it if $\omega$ satisfies the following condition :

$$\forall X \ (X \ is \ a \ \mathbb{K} - analytic \ vector \ field \ and \ X\rfloor(\omega|_{T\Sigma_2}) = 0) \implies X|_0 = 0.$$

The same statement holds for local smooth singular symplectic forms $\omega$ on $\mathbb{R}^4$ with structurally smooth Martinet hypersurfaces if the two generators of the ideal generated by coefficients of $\omega|_{T\Sigma_2}$ form a regular sequence of length 2 (Theorem 4.3).

The local invariants of singular symplectic forms in higher dimensions and with singular Martinet hypersurfaces will be studied in [4].

## 2. The normal form and realization theorems

The main result of this section is Theorem 2.1. In this theorem a 'normal' form of $\omega$ with the given pullback to the Martinet hypersurface is presented and sufficient conditions for the equivalence of germs of singular symplectic forms with the same pullback to the common Martinet hypersurface are found. We also show which germs of closed 2-forms on $\mathbb{K}^3$ vanishing at 0 can be obtained as a pullback of a germ of a singular symplectic form on $\mathbb{K}^4$ to its structurally smooth Martinet hypersurface. All results of this section hold in $\mathbb{C}$-analytic, $\mathbb{R}$-analytic and ($C^\infty$) smooth categories.

Let $\Omega$ be a germ of a volume form on $\mathbb{K}^4$. Let $\omega_0$ and $\omega_1$ be two germs of singular symplectic forms on $\mathbb{K}^4$ with structurally smooth Martinet hypersurfaces at 0. It is obvious that if there exists a diffeomorphism-germ of $\mathbb{K}^4$ at 0 such that $\Phi^*\omega_1 = \omega_0$ then $\Phi(\Sigma_2(\omega_0)) = \Sigma_2(\omega_1)$. Therefore we assume that these singular symplectic forms have the same Martinet hypersurface.

If the singular symplectic forms are equal on their common Martinet hypersurface then we obtain the following result (see see [7]).

**Proposition 2.1.** *Let $\omega_0$ and $\omega_1$ be two germs at 0 of singular symplectic forms on $\mathbb{K}^4$ with the common structurally smooth Martinet hypersurface $\Sigma_2$.*

*If $\frac{\omega_1^2}{\omega_0^2}|_0 > 0$ for $\mathbb{K} = \mathbb{R}$ ($\Re e\left(\frac{\omega_1^2}{\omega_0^2}|_0\right) > 0$ or $\Im m\left(\frac{\omega_1^2}{\omega_0^2}|_0\right) \neq 0$ for $\mathbb{K} = \mathbb{C}$) and $\omega_0|_{T_{\Sigma_2}\mathbb{K}^4} = \omega_1|_{T_{\Sigma_2}\mathbb{K}^4}$ then there exists a diffeomorphism-germ $\Phi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ such that*

$$\Phi^*\omega_1 = \omega_0$$

*and $\Phi|_{\Sigma_2} = Id_{\Sigma_2}$.*

**Proof.** We present the proof in $\mathbb{R}$-analytic and smooth categories. The proof in the $\mathbb{C}$-analytic category is similar. Firstly we simplify the forms $\omega_0$ and $\omega_1$. We find a local coordinate system $(p_1, p_2, p_3, p_4)$ such that $\omega_0^2 = p_1\Omega$, $\omega_1^2 = p_1(A + g)\Omega$, where $\Omega = dp_1 \wedge dp_2 \wedge dp_3 \wedge dp_4$ , $g$ is a function-germ, $g(0) = 0$ and $A > 0$ (see [12]). In this coordinate system $\omega_i = \sum_{1\leq j<k\leq 4} f_{i,j,k} dp_j \wedge dp_k$, where $f_{i,j,k}$ is a function-germ on $\mathbb{K}^4$ for $i = 0, 1$ and $1 \leq j < k \leq 4$. We can decompose $f_{i,j,k}$ in the following way $f_{i,j,k}(p_1, p_2, p_3, p_4) = p_1 g_{i,j,k}(p_1, p_2, p_3, p_4) + h_{i,j,k}(p_2, p_3, p_4)$, where $g_{i,j,k}$ is a function-germ and $h_{i,j,k}$ is a function-germ that does not depend on $p_1$ for $i = 0, 1$ and $1 \leq j < k \leq 4$. Let $\alpha_i = \sum_{1\leq j<k\leq 4} g_{i,j,k} dp_j \wedge dp_k$ and $\tilde{\omega}_i = \sum_{1\leq j<k\leq 4} h_{i,j,k} dp_j \wedge dp_k$. Then we have $\omega_i = p_1\alpha_i + \tilde{\omega}_i$ for $i = 0, 1$. By assumptions we have $\tilde{\omega}_0|_{T_{\Sigma_2}\mathbb{K}^4} = \tilde{\omega}_1|_{T_{\Sigma_2}\mathbb{K}^4}$. It implies that $\tilde{\omega}_0 = \tilde{\omega}_1$, because $h_{i,j,k}$ does not depend on $p_1$. We denote $\tilde{\omega}_1 = \tilde{\omega}_0$ by $\tilde{\omega}$. Then $\omega_i = p_1\alpha_i + \tilde{\omega}$ for $i = 0, 1$.

Further on we use the Moser homotopy method (see [14]). Let $\omega_t = t\omega_1 + (1 - t)\omega_0$, for $t \in [0; 1]$.

We want to find a family of diffeomorphisms $\Phi_t$, $t \in [0; 1]$ such that $\Phi_t^*\omega_t = \omega_0$, for $t \in [0; 1]$, $\Phi_0 = Id$. Differentiating the above homotopy equation by $t$, we obtain

$$d(V_t \rfloor \omega_t) = \omega_0 - \omega_1 = p_1(\alpha_0 - \alpha_1),$$

where $V_t = \frac{d}{dt}\Phi_t$. We need to solve the above equation for $V_t$. Now we prove the following lemmas.

**Lemma 2.1 ( [2] ).** *Let $\gamma$ be a germ of a 2-form on $\mathbb{R}^4$ and $\theta$ be a germ of a 1-form on $\mathbb{R}^4$. If $p_1\gamma + dp_1 \wedge \theta$ is a germ of a closed 2-form on $\mathbb{R}^4$ then there exists a germ of a 1-form $\delta$ such that $p_1\gamma + dp_1 \wedge \theta = d(p_1\delta)$.*

**Proof.** $p_1\gamma + dp_1 \wedge \theta$ is closed, therefore there exists a 1-form $\xi$ such that $d\xi = p_1\gamma + dp_1 \wedge \theta$. There exist a germ of a 1-form $\xi_1$ on $\mathbb{R}^4$, a function-germ $g$ on $\mathbb{R}^4$ and a germ of 1-form $\xi_2$ on $\{p_1 = 0\}$ such that $\xi = p_1\xi_1 + g dp_1 + \pi^*\xi_2$, where $\pi : \mathbb{R}^4 \ni (p_1, p_2, p_3, p_4) \mapsto (p_2, p_3, p_4) \in \{p_1 = 0\}$. The pullback of $d\xi$ to $\{p_1 = 0\}$ vanishes. It implies that $d\xi_2 = 0$. Thus $d(p_1\xi_1 + g dp_1) = d(\xi - \pi^*\xi_2) = p_1\gamma + dp_1 \wedge \theta$. It implies that $d(p_1(\xi_1 - dg)) = p_1\gamma + dp_1 \wedge \theta$, which finishes the proof of Lemma 2.1. □

**Lemma 2.2.** *Let $\alpha$ be a germ of a 2-form on $\mathbb{R}^4$. If $p_1\alpha$ is a germ of a closed 2-form on $\mathbb{R}^4$ then there exists a germ of a 1-form $\beta$ such that $p_1\alpha = d(p_1^2\beta)$.*

**Proof.** By Lemma 2.1 there exists a germ of a 1-form $\gamma$ such that $p_1\alpha = d(p_1\gamma) = dp_1 \wedge \gamma + p_1 d\gamma$. It implies that $dp_1 \wedge \gamma|_{T_{\{p_1=0\}}\mathbb{R}^4} = 0$. Hence there exist a germ of a 1-form $\delta$ and a smooth function-germ $f$ such that $\gamma = p_1\delta + f dp_1$. If we take $\beta = \delta - \frac{df}{2}$ then

$$p_1\alpha = d(p_1\gamma - d(\frac{p_1^2 f}{2})) = d(p_1^2\beta),$$

which finishes the proof of Lemma 2.2. □

Let us notice that $p_1(\alpha_0 - \alpha_1) = \omega_1 - \omega_0$ is closed. By the above lemma it is enough to solve for $V_t$ the equation

$$V_t \rfloor \omega_t = p_1^2\beta. \tag{2}$$

Now we calculate $\Sigma_2(\omega_t)$. It is easy to see that

$$\omega_i^2 = (p_1\alpha_i + \tilde{\omega})^2 = \tilde{\omega}^2 + p_1(2\alpha_i \wedge \tilde{\omega} + p_1\alpha_i^2).$$

But $\omega_i^2|_{T_{\{p_1=0\}}\mathbb{R}^4} = 0$. This clearly forces $\tilde{\omega}^2|_{T_{\{p_1=0\}}\mathbb{R}^4} = 0$. It implies that $\tilde{\omega}^2 = 0$, because coefficients of $\tilde{\omega}$ do not depend on $p_1$. By the above formula we get

$$2\alpha_0 \wedge \tilde{\omega} = \Omega - p_1\alpha_0^2$$

and

$$2\alpha_1 \wedge \tilde{\omega} = (A + g)\Omega - p_1\alpha_1^2$$

The above formulas imply the following formula

$$\begin{aligned} \omega_t^2 &= (p_1(t\alpha_1 + (1-t)\alpha_0) + \tilde{\omega})^2 = \\ &= p_1(1 + t(A + g - 1))\Omega + \\ &\quad + p_1^2 \left((t\alpha_1 + (1-t)\alpha_0)^2 - t\alpha_1^2 - (1-t)\alpha_0^2\right). \end{aligned} \tag{3}$$

From (3) we obtain

$$\omega_t^2 = p_1(1 + t(A + g - 1) + p_1 h_t)\Omega, \tag{4}$$

where $h_t$ is a function-germ. Let us notice that $(1 + t(A + g(0) - 1)) \neq 0$ for $A > 0$ and for $t \in [0, 1]$. Since $V_t \rfloor \omega_t^2 = 2(V_t \rfloor \omega_t) \wedge \omega_t$ and $\Sigma_2(\omega_t) = \{p_1 = 0\}$ is nowhere dense, equation (2) is equivalent to the following equation

$$V_t \rfloor \omega_t^2 = 2p_1^2 \beta \wedge \omega_t. \tag{5}$$

Combining (5) with (4) we obtain

$$V_t \rfloor (1 + t(A + g - 1) + p_1 h_t)\Omega = 2p_1 \beta \wedge \omega_t \tag{6}$$

But if $A > 0$ then $(1 + t(A - 1)) \neq 0$ for $t \in [0; 1]$. Therefore we can find a germ of smooth (or $\mathbb{R}$-analytic) vector field $V_t$ that satisfies (6). $V_t|_{\Sigma_2} = 0$, because the right hand side of (6) vanishes on $\Sigma_2$. Hence there exists a diffeomorphism $\Phi_t$ such that $\Phi_t^* \omega_t = \omega_0$ for $t \in [0, 1]$ and $\Phi_t|_{\Sigma_2} = Id_{\Sigma_2}$. This completes the proof of Theorem 2.1.

Now we define

$$\iota : \Sigma_2 = \{p_1 = 0\} \ni (p_2, p_3, p_4) \mapsto (0, p_2, p_3, p_4) \in \mathbb{K}^4$$

and

$$\pi : \mathbb{K}^4 \ni (p_1, p_2, p_3, p_4) \mapsto (p_2, p_3, p_4) \in \Sigma_2 = \{p_1 = 0\}.$$

If $rank \iota^* \omega|_0$ is 2 then $\omega$ is equivalent to $\Sigma_{20}$ Martinet's singular form (see [12]). Therefore we study singular symplectic forms such that $rank \iota^* \omega|_0 = 0$.

In the next theorem we describe all germs of singular symplectic forms $\omega$ on $\mathbb{K}^4$ with structurally smooth Martinet hypersurfaces at 0 and $rank \iota^* \omega|_0 = 0$. We also find the sufficient conditions for equivalence of singular symplectic forms of this type.

**Theorem 2.1.** *Let $\omega$ be a germ of a singular symplectic form on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface at 0.*

*(a) If $rank \iota^* \omega|_0 = 0$ then there exists a germ of a diffeomorphism $\Phi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ such that*

$$\Phi^* \omega = d\left(p_1 \pi^* \alpha\right) + \pi^* \sigma,$$

*where* $\sigma = \iota^*\Phi^*\omega$ *is a germ of a closed* 2*-form on* $\{p_1 = 0\}$ *and* $\alpha$ *is a germ of a contact form on* $\{p_1 = 0\}$ *such that* $\alpha \wedge \sigma = 0$.

*(b)Moreover if* $\omega_0 = d\,(p_1\pi^*\alpha_0) + \pi^*\sigma$ *and* $\omega_1 = d\,(p_1\pi^*\alpha_1) + \pi^*\sigma$ *are two germs of singular symplectic forms satisfying the above conditions and*

*(1)* $\frac{\alpha_1\wedge d\alpha_1}{\alpha_0\wedge d\alpha_0}|_0 > 0$ *if* $\mathbb{K} = \mathbb{R}$,
*(2)* $\alpha_1|_0 \wedge \alpha_0|_0 = 0$,

*then there exists a germ of a diffeomorphism* $\Psi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ *such that*

$$\Psi^*\omega_1 = \omega_0.$$

**Remark 2.1.** Assumption (1) is only needed in $\mathbb{R}$-analytic and smooth categories. In the $\mathbb{C}$-analytic category we have

$$\Phi^*(d\,(p_1\pi^*\alpha) + \pi^*\sigma) = d\,(p_1\pi^* i\alpha) + \pi^*\sigma,$$

where $\Phi$ is the following diffeomorphism $\Phi(p_1, p_2, p_3, p_4) = (ip_1, p_2, p_3, p_4)$ and $i^2 = -1$. It is obvious that $\Phi|_{\Sigma_2} = Id_{\Sigma_2}$, where $\Sigma_2 = \{p_1 = 0\}$ and $i\alpha \wedge d(i\alpha) = -\alpha \wedge d\alpha$.

**Proof.** By Lemma 2.1 there exists a 1-form $\gamma$ such that $\omega = d(p_1\gamma) + \pi^*\sigma$. It is clear that we can write $\gamma$ in the following form $\gamma = \pi^*\alpha + p_1\delta + gdp_1$, where $\alpha$ is a germ of a 1-form on $\{p_1 = 0\}$, $g$ is a function-germ and $\delta$ is a germ of a 1-form. Then

$$d(p_1(p_1\delta + gdp_1)) = p_1(2dp_1 \wedge \delta + p_1 d\delta + dg \wedge dp_1).$$

By Lemma 2.2 we have $\omega = d(p_1\pi^*\alpha) + \pi^*\sigma + d(p_1^2\theta)$.

It is easy to see that

$$\begin{aligned}\omega^2 &= 2dp_1 \wedge \pi^*\alpha \wedge \pi^*\sigma + 4p_1dp_1 \wedge \theta \wedge \pi^*\sigma \\ &\quad + 2p_1dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha + p_1^2 v\Omega,\end{aligned}$$

where $v$ is a function-germ at 0. We have $\alpha \wedge \sigma = 0$, because $\omega^2|_{T_{\{p_1=0\}}\mathbb{K}^4} = 0$. From $\sigma|_0 = 0$, we have

$$\omega^2 = 2p_1dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha + p_1 g\Omega,$$

where $g$ is a function-germ vanishing at 0. From the above we obtain that

$$\alpha \wedge d\alpha|_0 \neq 0.$$

Let

$$\omega_0 = d\,(p_1\pi^*\alpha) + \pi^*\sigma.$$

Then

$$\omega_0^2 = 2p_1 dp_1 \wedge \pi^*\alpha \wedge d\pi^*\alpha + p_1 h\Omega,$$

where $h$ is a smooth function-germ at 0 such that $h(0) = 0$ . One can check that

$$\omega_0|_{T_{\{p_1=0\}}\mathbb{K}^4} = dp_1 \wedge \pi^*\alpha + \pi^*\sigma = \omega|_{T_{\{p_1=0\}}\mathbb{K}^4}.$$

Therefore by Proposition 2.1 there exists a germ of a diffeomorphism $\Theta : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ such that $\Theta^*\omega = \omega_0$ and $\Theta|_{\{p_1=0\}} = Id_{\{p_1=0\}}$.

This finishes the proof of part (a).

Now we prove part (b).

Assumption (2) implies that there exists $B \neq 0$ such that $\alpha_1|_0 = B\alpha_0|_0$. If $B \neq 1$ then $\Phi^*\omega_0 = d(p_1\pi^*(B\alpha_0)) + \pi^*\sigma$ where $\Phi$ is a diffeomorphism-germ of the form $\Phi(p) = (Bp_1, p_2, p_3, p_4))$. Thus we may assume that $B = 1$.

We use the Moser homotopy method. Let $\alpha_t = t\alpha_1 + (1-t)\alpha_0$ and $\omega_t = d\,(p_1\pi^*\alpha_t) + \pi^*\sigma$ for $t \in [0, 1]$. It is easy to check that $\alpha_t \wedge \sigma = 0$.

Now we look for germs of diffeomorphims $\Phi_t$ such that

$$\Phi_t^*\omega_t = \omega_0,\ \ for\ t \in [0; 1],\ \ \Phi_0 = Id. \tag{7}$$

Differentiating the above homotopy equation by $t$, we obtain

$$d(V_t \rfloor \omega_t) = d(p_1\pi^*(\alpha_0 - \alpha_1)),$$

where $V_t = \frac{d}{dt}\Phi_t$. Therefore we have to solve for $V_t$ the following equation

$$V_t \rfloor \omega_t = p_1\pi^*(\alpha_0 - \alpha_1). \tag{8}$$

We calculate the Martinet hypersurface of $\omega_t$. $\omega_t^2 = 2p_1 dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t)$, because $\sigma^2 = 0$, $d\alpha_t^2 = 0$ and $\alpha_t \wedge \sigma = 0$.

$\alpha_0|_0 = \alpha_1|_0$ and there exists $A > 0$ such that $(\alpha_1 \wedge d\alpha_1)|_0 = A(\alpha_0 \wedge d\alpha_0)|_0$. It implies that

$$\alpha_t \wedge d\alpha_t|_0 = (tA + (1-t))(\alpha_0 \wedge d\alpha_0)|_0.$$

Therefore

$$dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t)|_0 \neq 0 \tag{9}$$

for $t \in [0; 1]$. Thus $\Sigma_2(\omega_t) = \{p_1 = 0\}$.

Since $V_t \rfloor \omega_t^2 = 2(V_t \rfloor \omega_t) \wedge \omega_t$ and $\Sigma_2(\omega_t) = \{p_1 = 0\}$ is nowhere dense, equation (8) is equivalent to

$$V_t \rfloor \omega_t^2 = 2p_1\pi^*(\alpha_0 - \alpha_1) \wedge \omega_t.$$

Therefore we have to solve the following equation

$$V_t\rfloor\,(2dp_1 \wedge \pi^*(\alpha_t \wedge d\alpha_t)) = 2\pi^*(\alpha_0 - \alpha_1) \wedge \omega_t. \qquad (10)$$

Hence by (9) we can find a smooth solution $V_t$ of (10) and $V_t|_0 = 0$, because $\alpha_1|_0 = \alpha_0|_0$ Therefore there exist germs of diffeomorphisms $\Phi_t$, which satisfy (7). For $t = 1$ we have $\Phi_1^*\omega_1 = \omega_0$. □

We call a germ of a closed 2-form $\sigma$ on $\mathbb{K}^3$ *realizable with a structurally smooth Martinet hypersurface* if there exists a germ of a singular symplectic form $\omega$ on $\mathbb{K}^4$ such that $\Sigma_2(\omega) = \{0\} \times \mathbb{K}^3$ is structurally smooth and $\omega|_{T\Sigma_2(\omega)} = \sigma$.

From Martinet's normal form of type $\Sigma_{20}$ we know that all germs of closed 2-forms on $\mathbb{K}^3$ of the rank 2 are realizable with a structurally smooth Martinet hypersurface (see [12]). From part (a) of the Theorem 2.1 we obtain the following realization theorem of closed 2-forms on $\mathbb{K}^3$ of rank 0 at $0 \in \mathbb{K}^3$.

**Theorem 2.2.** *Let $\sigma$ be a germ of a closed 2-form on $\mathbb{K}^3$ and rank$\sigma|_0 = 0$. $\sigma$ is realizable with a structurally smooth Martinet hypersurface if and only if there exists a germ of a contact form $\alpha$ on $\mathbb{K}^3$ such that $\alpha \wedge \sigma = 0$.*

## 3. The canonical orientation and the 2-dimensional kernel of $\omega$ at 0

In $\mathbb{R}$-analytic and smooth categories assumption (1) of Theorem 2.1 means that $\omega_0$ and $\omega_1$ determine the same orientation. The orientation may be defined invariantly. Let $\omega$ be a germ of a singular symplectic structure on $\mathbb{R}^4$ with a structurally smooth Martinet hypersurface $\Sigma_2$ at 0. Then $\Sigma_2 = \{f = 0\}$ and $df|_0 \neq 0$. We define the volume form $\Omega_{\Sigma_2}$ on $\Sigma_2$ which determines the orientation of $\Sigma_2$ in the following way

$$\Omega_{\Sigma_2} \wedge df = \frac{\omega^2}{f}.$$

This definition is analogous to the definition in [8] proposed by V. I. Arnol'd. It is easy to see that this definition of the orientation does not depend on the choice of $f$ such that $\Sigma_2 = \{f = 0\}$ and $df|_0 \neq 0$. We call this orientation of $\Sigma_2$ the *canonical orientation of* $\Sigma_2$.

Assumption (2) of Theorem 2.1 can be also expressed invariantly. We call a subspace $\ker\omega|_0 = \{v \in T_0\mathbb{K}^4 : v\rfloor\omega|_0 = 0\}$ *the kernel of* $\omega$ *at* 0. It is easy to see that $\ker\omega|_0$ is 2-dimensional subspace of $T_0\Sigma_2$ if $\omega|_{T_0\Sigma_2} = 0$. $\ker\omega|_0$ can be also described as a kernel of a non-vanishing 1-form on $\Sigma_2$. Let $Y$ be a germ of a vector field on $\mathbb{K}^4$ that is transversal to $\Sigma_2$ at 0. Let $\iota : \Sigma_2 \hookrightarrow \mathbb{K}^4$ be an inclusion. Then the kernel of $\iota^*(Y\rfloor\omega)|_0$ is a 2-dimensional linear subspace of $T_0\Sigma_2$. By Theorem 2.1 it is easy to check that this definition does not depend on the choice of $Y$ and that the subspace $\ker \iota^*(Y\rfloor\omega)|_0$ is $\ker \omega|_0$. Assumption (2) of Theorem 2.1 means that $\ker \omega_0|_0 = \ker \omega_1|_0$, which is equivalent to $\ker \iota^*(Y\rfloor\omega_1)|_0 = \ker \iota^*(Y\rfloor\omega_0)|_0$. Now we formulate part (b) of Theorem 2.1 invariantly.

In the $\mathbb{C}$-analytic category $\omega$ is determined by the restriction to $T\Sigma_2$ and the 2-dimensional kernel of $\omega$ at 0.

**Theorem 3.1.** *Let $\omega_0$ and $\omega_1$ be germs of $\mathbb{C}$-analytic singular symplectic forms on $\mathbb{C}^4$ with a common structurally smooth Martinet hypersurface $\Sigma_2$ at 0 and $rank \iota^*\omega_0|_0 = rank \iota^*\omega_1|_0 = 0$.*

*If $\iota^*\omega_0 = \iota^*\omega_1$ and $\ker \omega_0|_0 = \ker \omega_1|_0$ then there exists a germ of a $\mathbb{C}$-analytic diffeomorphism $\Psi : (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0)$ such that*

$$\Psi^*\omega_1 = \omega_0.$$

In the $\mathbb{R}$-analytic and smooth categories $\omega$ is determined by the restriction to $T\Sigma_2$, the kernel of $\omega$ at 0 and the canonical orientation of $\Sigma_2$.

**Theorem 3.2.** *Let $\omega_0$ and $\omega_1$ be germs of smooth ($\mathbb{R}$-analytic) singular symplectic forms on $\mathbb{R}^4$ with a common structurally smooth Martinet hypersurface $\Sigma_2$ at 0 and $rank \iota^*\omega_0|_0 = rank \iota^*\omega_1|_0 = 0$.*

*If $\iota^*\omega_0 = \iota^*\omega_1$, $\ker \omega_0|_0 = \ker \omega_1|_0$ and $\omega_0$, $\omega_1$ define the same canonical orientation of $\Sigma_2$ then there exists a germ of a smooth ($\mathbb{R}$-analytic) diffeomorphism $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that*

$$\Psi^*\omega_1 = \omega_0.$$

## 4. Determination by the restriction of $\omega$ to $T\Sigma_2$ and the canonical orientation

In this section we find conditions in the $\mathbb{C}$-analytic category for the determination of the equivalence class of a singular symplectic form by its pullback to the Martinet hypersurface (Theorem 4.1). The same conditions are valid for the determination of the equivalence class of a singular symplectic form by its pullback to the Martinet hypersurface and the canonical orientation in the $\mathbb{R}$-analytic category (Theorem 4.2). In the smooth category we need a stronger condition to obtain an analogous result.

**Theorem 4.1.** *Let $\omega_0$ and $\omega_1$ be germs of $\mathbb{C}$-analytic singular symplectic* *forms on $\mathbb{C}^4$ with a common structurally smooth Martinet hypersurface $\Sigma_2$ at 0 and $rank\iota^*\omega_0|_0 = rank\iota^*\omega_1|_0 = 0$.*

*If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$ and there does not exist a germ of a $\mathbb{C}$-analytic vector field $X$ on $\Sigma_2$ at 0 such that $X\rfloor\sigma = 0$ and $X|_0 \neq 0$ then there exists a germ of a $\mathbb{C}$-analytic diffeomorphism $\Psi : (\mathbb{C}^4, 0) \to (\mathbb{C}^4, 0)$ such that*

$$\Psi^*\omega_1 = \omega_0.$$

**Theorem 4.2.** *Let $\omega_0$ and $\omega_1$ be germs of $\mathbb{R}$-analytic singular symplectic forms on $\mathbb{R}^4$ with a common structurally smooth Martinet hypersurface $\Sigma_2$ at 0 and $rank\iota^*\omega_0|_0 = rank\iota^*\omega_1|_0 = 0$.*

*If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$, $\omega_0$ and $\omega_1$ define the same canonical orientation of $\Sigma_2$ and there does not exist a germ of an $\mathbb{R}$-analytic vector field $X$ on $\Sigma_2$ at 0 such that $X\rfloor\sigma = 0$ and $X|_0 \neq 0$ then there exists a germ of an $\mathbb{R}$-analytic diffeomorphism $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that*

$$\Psi^*\omega_1 = \omega_0.$$

**Proof.** We present the proof of Theorem 4.2. The proof of Theorem 4.1 is similar.

By Theorem 2.1 we obtain $\omega_0 = d(p_1\pi^*\alpha_0) + \sigma$ and $\omega_1 = d(p_1\pi^*\alpha_1) + \sigma$, where $\alpha_0$, $\alpha_1$ are germs of analytic contact forms on $\Sigma_2 = \{p_1 = 0\}$ such that $\alpha_0 \wedge \sigma = \alpha_1 \wedge \sigma = 0$ and $\alpha_0 \wedge d\alpha_0$, $\alpha_1 \wedge d\alpha_1$ define the same orientation on $\Sigma_2$.

$\alpha_0$ is a contact form, therefore $\alpha_0|_0 \neq 0$. We can find a coordinate system $(x, y, z)$ on $\Sigma_2$ such that $\alpha_0 = f_0dx + g_0dy + h_0dz$, where $f_0$, $g_0$ and $h_0$ are function-germs on $\Sigma_2$ and $h_0(0) \neq 0$. Let $\sigma = ady\wedge dz + bdz\wedge dx + cdx\wedge dy$, where $a$, $b$, $c$ are function-germs on $\Sigma_2$ vanishing at 0. $\alpha_0 \wedge \sigma = 0$, thus we get $c = -\frac{f_0}{h_0}a - \frac{g_0}{h_0}b$.

Let $\alpha_1 = f_1dx + g_1dy + h_1dz$, where $f_1, g_1, h_1$ are functions-germs on $\Sigma_2$. From $\alpha_1 \wedge \sigma = 0$ we obtain the equation

$$a(f_1 - \frac{h_1}{h_0}f_0) + b(g_1 - \frac{h_1}{h_0}g_0) = 0 \qquad (11)$$

and $a(0) = b(0) = 0$.

Let $l$ be the greatest common divisor of $a$ and $b$ ($GCD(a, b)$). Then $a = la_1$ and $b = lb_1$, where $a_1$ and $b_1$ are germs of analytic functions on $\Sigma_2$ and $GCD(a_1, b_1) = 1$. Thus $\sigma = l(a_1dy \wedge dz + b_1dz \wedge dx - (\frac{f_0}{h_0}a_1 + \frac{g_0}{h_0}b_1)dx \wedge dy)$. If $a_1 \neq 0$ or $b_1 \neq 0$ then a germ of an analytic vector field $X = a_1\frac{\partial}{\partial x} + b_1\frac{\partial}{\partial y} - (\frac{f_0}{h_0}a_1 + \frac{g_0}{h_0}b_1)\frac{\partial}{\partial z}$ does not vanish at 0. It is easy to see that $X\rfloor\sigma = 0$. Therefore $a_1(0) = b_1(0) = 0$.

Thus the equation (11) has the following form

$$la_1(f_1 - \frac{h_1}{h_0} f_0) = -lb_1(g_1 - \frac{h_1}{h_0} g_0)$$

and $GCD(a_1, b_1) = 1$.

Therefore $f_1 - \frac{h_1}{h_0} f_0 = b_1 r$ and $g_1 - \frac{h_1}{h_0} g_0 = -a_1 r$, where $r$ is a function-germ on $\Sigma_2$ at 0.

Then $\alpha_1 = \frac{h_1}{h_0}(f_0 dx + g_0 dy + h_0 dz) + r(b_1 dx - a_1 dy)$. $\alpha_1|_0 \neq 0$ and $a_1(0) = b_1(0) = 0$ thus $h_1(0) \neq 0$.

Hence $\alpha_1|_0 = \frac{h_1(0)}{h_0(0)} \alpha_0|_0$.

It is easy to see that $\omega_i^2 = 2p_1 dp_1 \wedge \pi^*(\alpha_i \wedge d\alpha_i)$ for $i = 0, 1$. Therefore by assumptions of the theorem we have $\alpha_1 \wedge d\alpha_1 = A\alpha_0 \wedge d\alpha_0$, where $A > 0$.

Thus $\omega_0$ and $\omega_1$ satisfy the assumptions of Theorem 2.1. Then there exists a germ of an analytic diffeomorphism $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that

$$\Psi^* \omega_1 = \omega_0. \qquad \square$$

Now we find the normal form of a germ of a singular symplectic form on $\mathbb{K}^4$ at 0 which does not satisfy the assumptions of the above theorem. The following result is also true in the smooth category.

**Proposition 4.1.** *Let $\omega$ be a germ of a $\mathbb{K}$-analytic singular symplectic form on $\mathbb{K}^4$ with a structurally smooth Martinet hypersurface at 0 and $rank \iota^* \omega|_0 = 0$.*

*If there exists a germ of a $\mathbb{K}$-analytic vector field $X$ on $\Sigma_2$ at 0 such that $X \rfloor \sigma = 0$ and $X|_0 \neq 0$ then there exists a germ of a $\mathbb{K}$-analytic diffeomorphism $\Psi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0)$ such that*

$$\Psi^* \omega = d(p_1(dx + Cdy + zdy)) + g(x, y) dx \wedge dy$$

*or*

$$\Psi^* \omega = d(p_1(dy + Cdx + zdx)) + g(x, y) dx \wedge dy,$$

*where $C \in \mathbb{K}$ and $g$ is a $\mathbb{K}$-analytic function-germ on $\mathbb{K}^4$ at 0 that does not depend on $p_1$ and $z$.*

**Proof.** By Theorem 2.1 we may assume that $\omega = d(p_1 \pi^* \alpha) + \pi^* \sigma$, where $\sigma = \iota^* \omega$ and $\alpha$ is a germ of an analytic contact form on $\Sigma_2 = \{p_1 = 0\}$ such that $\alpha \wedge \sigma = 0$. Let $X$ be a germ of an analytic vector field on $\Sigma_2$ at 0 such that $X \rfloor \sigma = 0$ and $X|_0 \neq 0$. Then we may choose a coordinate system on $\Sigma_2$ such that $X = \frac{\partial}{\partial z}$. In this system the closed 2-form $\sigma$ has the following form $\sigma = h(x, y) dx \wedge dy$, where $h$ is an analytic function-germ on $\Sigma_2$ at 0 that does not depend on $z$. In this coordinate system $\alpha = a(x,y,z)dx + b(x,y,z)dy$, because $\alpha \wedge \sigma = 0$. Therefore $\omega$ has the following form

$$\omega = d(p_1(a(x,y,z)dx + b(x,y,z)dy)) + h(x,y)dx \wedge dy. \tag{12}$$

$a(0) \neq 0$ or $b(0) \neq 0$, because $\alpha_0 \neq 0$. Assume that $a(0) \neq 0$. Then by a diffeomorphism of the form

$$\Phi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0); (p_1, x, y, z) \mapsto (\frac{p_1}{a(x,y,z)}, x, y, z)$$

we obtain $\Phi^*\omega = d(p_1(dx + b_1(x,y,z)dy)) + h(x,y)dx \wedge dy$, where $b_1(x,y,z) = \frac{b(x,y,z)}{a(x,y,z)}$.

But $\alpha = dx + b_1(x,y,z)dy$ is a germ of a contact form on $\Sigma_2$. Therefore

$$\alpha \wedge d\alpha|_0 = \frac{\partial b_1}{\partial z}(0)dx \wedge dz \wedge dy \neq 0.$$

Thus $\frac{\partial b_1}{\partial z}(0) \neq 0$.

Then by a diffeomorphism of the form

$$\Phi : (\mathbb{K}^4, 0) \to (\mathbb{K}^4, 0); (p_1, x, y, z) \mapsto (p_1, x, y, b_1(x,y,z) - b_1(0))$$

we obtain $\Phi^*\omega = d(p_1(dx + Cdy + zdy)) + h(x,y)dx \wedge dy$, where $C = b_1(0)$.

If $a(0) = 0$ in (12) then $b(0) \neq 0$ and we obtain $\Psi^*\omega = d(p_1(dy + Cdx + zdx)) + g(x,y)dx \wedge dy$, by the analogous coordinate changes. □

Now we need some notions from commutative algebra (see Appendix 1 of [8], [3]) to formulate the result in the smooth category. We recall that a sequence of elements $a_1, \cdots, a_r$ of a proper ideal $I$ of a ring $R$ is called *regular* if $a_1$ is a nonzerodivisor of $R$ and $a_i$ is a nonzerodivisor of $R/ < a_1, \cdots, a_{i-1} >$ for $i = 2, \cdots, r$. Here $< a_1, \cdots, a_i >$ denotes the ideal generated by $a_1, \cdots, a_i$. The *length* of a regular sequence $a_1, \cdots, a_r$ is $r$.

The *depth* of the proper ideal $I$ of the ring $R$ is the supremum of lengths of regular sequences in $I$. We denote it by $depth(I)$. If $I = R$ then we define $depth(I) = \infty$.

Let $\sigma$ be a germ of a smooth ($\mathbb{K}$-analytic) closed 2-form on $\Sigma_2 = \mathbb{K}^3$ and $rank\sigma|_0 = 0$. In the local coordinate system $(x,y,z)$ on $\Sigma_2$ we have $\sigma = ady \wedge dz + bdz \wedge dx + cdx \wedge dy$, where $a, b, c$ are smooth ($\mathbb{K}$-analytic) function-germs on $\Sigma_2$. By $I(\sigma)$ we denote the ideal of the ring of smooth ($\mathbb{K}$-analytic) function-germs on $\Sigma_2$ generated by $a, b, c$ i.e. $I(\sigma) =< a, b, c >$. It is easy to see that $I(\sigma)$ does not depend on the local coordinate system on $\Sigma_2$. $\sigma$ satisfies the condition $\alpha \wedge \sigma = 0$, where $\alpha$ is a germ of a contact form on $\mathbb{K}^3$. It implies that $I(\sigma)$ is generated by two function-germs.

In the $\mathbb{K}$-analytic category if $depth I(\sigma) \geq 2$ then the two generators of $I(\sigma)$ form a regular sequence of length 2 (see [3]). One can easily check that it implies that there does not exist a germ of a $\mathbb{K}$-analytic vector field on $\Sigma_2$ such that $X \rfloor \sigma = 0$ and $X|_0 \neq 0$. The inverse implication is not true in general. Now we formulate the following result in the smooth category.

**Theorem 4.3.** *Let $\omega_0$ and $\omega_1$ be germs of smooth singular symplectic forms on $\mathbb{R}^4$ with a common structurally smooth Martinet hypersurface $\Sigma_2$ at 0 and $rank \iota^*\omega_0|_0 = rank \iota^*\omega_1|_0 = 0$.*

*If $\iota^*\omega_0 = \iota^*\omega_1 = \sigma$, $\omega_0$ and $\omega_1$ define the same canonical orientation of $\Sigma_2$ and the two generators of the ideal $I(\sigma)$ form a regular sequence of length 2 then there exists a germ of a smooth diffeomorphism $\Psi : (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ such that*

$$\Psi^*\omega_1 = \omega_0.$$

**Proof.** The proof is similar to the proof of Theorem 4.2. By Theorem 2.1 we obtain $\omega_0 = d(p_1\pi^*\alpha_0) + \sigma$ and $\omega_1 = d(p_1\pi^*\alpha_1) + \sigma$, where $\alpha_0,\ \alpha_1$ are germs of smooth contact forms on $\Sigma_2 = \{p_1 = 0\}$ such that $\alpha_0 \wedge \sigma = \alpha_1 \wedge \sigma = 0$ and $\alpha_0 \wedge d\alpha_0,\ \alpha_1 \wedge d\alpha_1$ define the same orientation on $\Sigma_2$.

$\alpha_0$ is a contact form therefore $\alpha_0|_0 \neq 0$. We can find a coordinate system $(x, y, z)$ on $\Sigma_2$ such that $\alpha_0 = f_0 dx + g_0 dy + h_0 dz$, where $f_0$, $g_0$ and $h_0$ are function-germs on $\Sigma_2$ and $h_0(0) \neq 0$. Let $\sigma = a dy \wedge dz + b dz \wedge dx + c dx \wedge dy$, where $a$, $b$, $c$ are function-germs on $\Sigma_2$ vanishing at 0. $\alpha_0 \wedge \sigma = 0$, thus we get $c = -\frac{f_0}{h_0} a - \frac{g_0}{h_0} b$. Thus $I(\sigma) = < a, b, c > = < a, b >$.

Let $\alpha_1 = f_1 dx + g_1 dy + h_1 dz$, where $f_1, g_1, h_1$ are functions-germs on $\Sigma_2$. From $\alpha_1 \wedge \sigma = 0$ we obtain the equation

$$a(f_1 - \frac{h_1}{h_0} f_0) + b(g_1 - \frac{h_1}{h_0} g_0) = 0 \tag{13}$$

and $a(0) = b(0) = 0$.

By assumptions $a, b$ is a regular sequence.

Therefore $f_1 - \frac{h_1}{h_0} f_0 = br$ and $g_1 - \frac{h_1}{h_0} g_0 = -ar$, where $r$ is a smooth function-germ on $\Sigma_2$ at 0.

Then proceeding in the same way as in the proof of Theorem 4.2 we get the result. □

**Acknowledgements**

The author wishes to express his thanks to S. Janeczko, B. Jakubczyk and M. Zhitomirskii for many helpful conversations and remarks during writing this paper. The author is also grateful to the organizers of the Singularity 5 weeks programme at CIRM Luminy for hospitality. The author thanks the referee of this paper for many useful comments, especially for a simpler description of the 2-dimensional kernel of $\omega$ at 0. .

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