Equisingular families of surface singularities in $\mathbb{C}^3$

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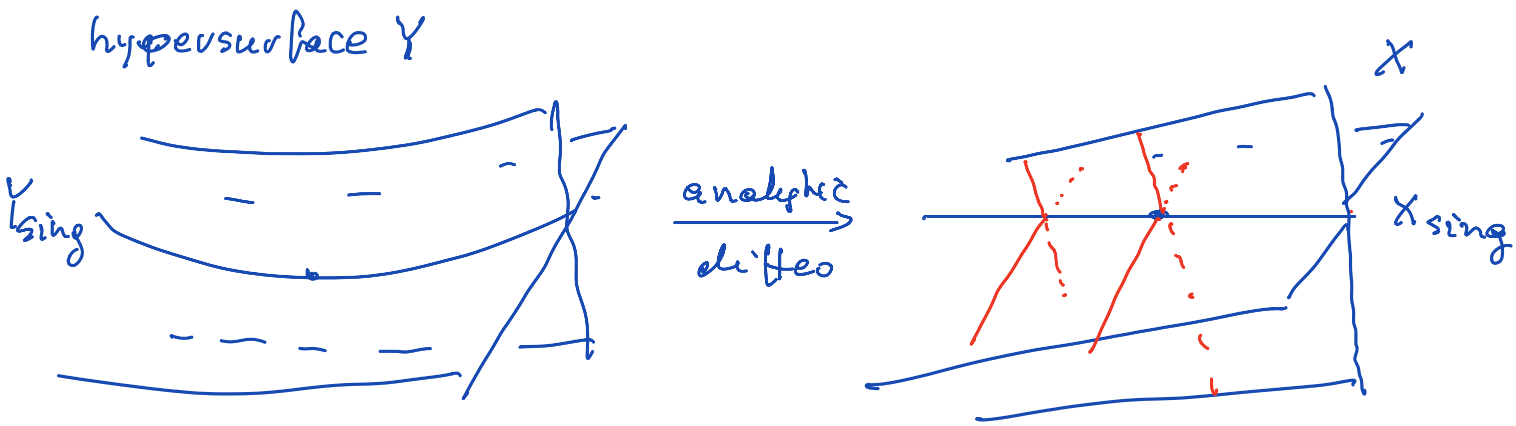
"Lipschitz stratification of complex hypersurfaces in codimension 2"
A.P & Laurentiu Paunescu, math Arxiv.

"Algebro-geometric equisingularity of Zariski" A.P. Math Arxiv.
to appear in Handbook of Geometry and Topology of Singularities II.

GKLW online seminar January 22, 2021.



equisingularity in codimension 1 $\longleftrightarrow$ equisingular families of plane curve singularities

I Equisingular families of complex plane curve singularities.

$$f_t(x,y) = f(x,y,t) = y^d + \sum_{j=1}^{d} a_j(x,t)\, y^{d-j},$$

$((x,y) \in (\mathbb{C}^2,0),\ f$ reduced $, t \in T = \{x=y=0\}$ parameter, $a_j(0,t) \equiv 0)$

Definition (Zariski, Studies in equisingularity I. 1965 Equivalent Singularities of Plane Algebroid Curves)

Family $X_t = \{f_t(x,y) = 0\}$ is **equisingular** if family of discriminants is equimultiple

$$D_f(x,t) = x^m \,\mathrm{unit}(x,t)$$

II Parameterized Puiseux Theorem

Let $y_i(x,t)$, $i = 1, \ldots, d$, be the roots of f_t. Then

$\exists n \in \mathbb{N} \setminus \{0\}$ s.t. $\quad y_i(u^n, t)$ are analytic

and $\quad (y_i - y_j) = u^{k_{ij}}\, \mathrm{unit}(u,t)$

(if f_t irreducible then $n = d$ and

$$y_i(x,t) = \sum_{k=1}^{\infty} b_{i,k}(t)\, x^{k/n}$$
)

III Theorem TFAE

(i) X_t is (Zariski) equisingular

(ii) X_t admits parameterized Puiseux

(iii) $(\mathbb{C}^2, X_t)$ is (ambient) topologically trivial.

$\exists$ homeomorphism $\Phi : (\mathbb{C}^2,0)\times(\mathbb{C}^\ell,0) \longrightarrow (\mathbb{C}^{2+\ell},0)$

$\Phi(x,y,t) = (\Psi_t(x,y), t)$ and $\Psi_t(X_0) = X_t$.

(iv) $X := f^{-1}(0)$, $T := \{x=y=0\}$. Then $T = X_{sing}$ and

$(X \setminus T, T)$ is a Whitney stratification of X.

(v) constancy of numerical invariants.

(Puiseux pairs and intersection numbers of branches)

IV Comments

- In (i) and (ii) one should add $\exists$ system of coordinates x,y.
- A syst. of coord. x,y is transverse $\pitchfork$ if

$$\mathrm{mult}_0 f = \mathrm{mult}_0 f_t = d$$

- We have (iii) $\Longleftrightarrow$ (iv) $\Longleftrightarrow$ (v) $\Longleftarrow$ (i) $\Longleftrightarrow$ (ii) for a coord syst

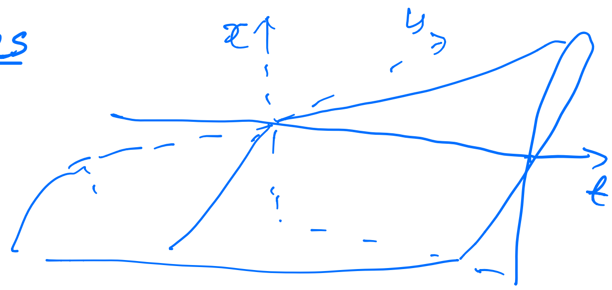
$\Longleftrightarrow$ (i) for any $\pitchfork$ coord syst.

$\Longleftrightarrow$ (ii) — " —

- Zariski equisingularity is a stratifying condition.

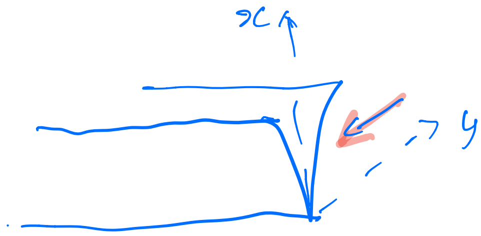
$\{t;\ f_t$ is not equisingular at t is Zariski closed nowhere dense$\}$

V Pictures

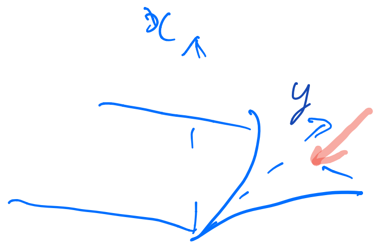


$$y^2 - t^3 x^2 + x^3 = 0$$

$$\Delta(x,t) = 4\, x^4\, \mathrm{unit}(x,t)$$



transverse projection



not transverse projection

VI Addendum

TFAE to (i) – (v)

(vi) (Pham–Teissier) Φ of (iii) can be choosen bi-Lipschitz

(vii) $(X \setminus T, T)$ is a Lipschitz stratification in the sense of Mostowski

- Lift of Lipschitz vector field from $\mathbb{C}_x \times \mathbb{C}^l_t$ to X is Lipschitz.

Example

$$y_i = \sum b_{ik}(t)\, x^{k/n} \implies \frac{\partial y_i}{\partial t} = \sum b'_{ik}(t)\, x^{i/k} \implies \left| \frac{\partial^2 y_i}{\partial t\, \partial x} = \sum \tfrac{i}{k}\, b'_{ik}(t)\, x^{i/k - 1} \right| \le C$$

bounded

$$\left| \frac{\partial}{\partial t}(y_i - y_j) \right| \le C\, |y_i - y_j| \quad \text{because} \quad y_i - y_j = x^{k_{ij}} \cdot \mathrm{unit}(x,t)$$

$\implies$ lift of $\frac{\partial}{\partial t}$ to X is Lipschitz

VII Families of surface singularities in $\mathbb{C}^3$.

$$f_t(x,y,z) = z^d + \sum a_i(x,y,t)\, z^{d-i}$$

Definition

$X_t := f_t^{-1}(0)$ is Zariski equisingular (ZE) if $\Delta_f(x,y,t)=0$ defines an equisingular family.

First reflex: Everything goes wrong.

Suppose for simplicity $T=\{x=y=z=0\} = \operatorname{Sing} X$, $X = f^{-1}(0)$.

$$\begin{array}{lll}
 & \not\Rightarrow & (X\setminus T, T) \text{ Whitney} \\
 & & \quad \not\Uparrow \ \not\Downarrow \\
 & \not\Rightarrow & \text{ZE in a coord. system} \\
\text{Topological triviality} & & \quad \not\Downarrow \\
 & \not\Rightarrow & \text{ZE in a } \pitchfork \text{ coord system} \\
 & & \quad \not\Downarrow \\
 & \not\Rightarrow & \text{ZE in a generic coord system}
\end{array}$$

VIII Second thought: some things go right.

ZE $\Longrightarrow$ topological triviality (Varchenko)

ZE $\Longrightarrow$ Whitney (Speder for generic ZE)
P-P for transverse ZE

(All these hold in arbitrary dimension).

Theorem (P-Paunescu)

If a family of surface singularities in $\mathbb{C}^3$ is generically ZE then
then $(X \setminus C \cup \Sigma, C \cup \Sigma \setminus T, T)$
is Lipschitz stratification in the sense of Mostowski

- conjectured by J.P. Henry & T. Mostowski
- Under same assumptions A. Pichon & W. Neumann showed bi-Lipschitz triviality (preprint 2013)

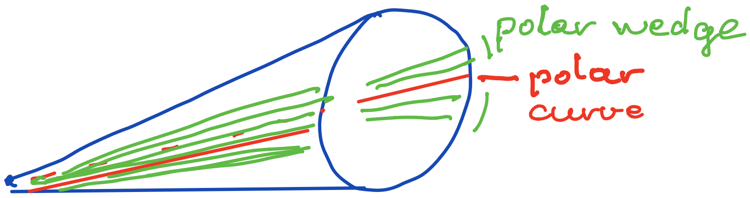
IX. Polar curves and polar wedges

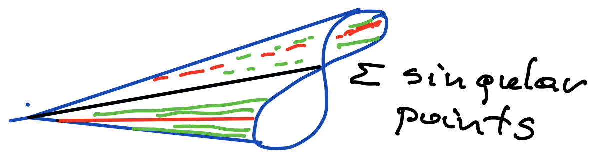
Given $X_0 = \{ f(x,y,z) = 0 \} \subset (\mathbb{C}^3, 0)$

$\Sigma = X_{sing} : f = f'_x = f'_y = f'_z = 0$ singular points of X

$\mathcal{C} : \quad f = f'_z = 0$ contour curve

$C = \overline{\mathcal{C} \setminus \Sigma}$ polar curve





X

Let $\pi_b(x,y,z) = (x, y - zb)$, b parameter.

$$\mathcal{C}_b : \; f = f'_z + b f'_y = 0, \quad C_b = \overline{\mathcal{C}_b \setminus \Sigma}$$

Polar wedge: $\displaystyle\bigcup_{|b|<\varepsilon} \mathcal{C}_b = \Sigma \cup \bigcup_{|b|<\varepsilon} C_b = PW = \Sigma \cup PW_i$

$\pi_b(\mathcal{C}_b) = \Delta_b$ the discriminant locus.

(In what follows we suppose the coord system sufficiently generic, giving precise conditions)

In particular we suppose Briançon – Henry Condition

$b \longrightarrow \Delta_b$ equisingular family of plane curve singularities

(after a breakthrough paper of J. Briançon & J.-P. Henry
Bull. Soc. Math. France 1980)

XI Parameterization of polar wedges

Theorem
If the system is sufficiently generic then each polar wedge PW_i can be parameterized

$$p(u,b) = \begin{cases} x = u^n \\ y = y_i(u,0) + u^{m_i} b^2 \operatorname{unit}(u,b) \\ z = z_i(u,0) + u^{m_i} b \operatorname{unit}(u,b) \end{cases}$$

For two PW_i, PW_j: $\begin{cases} y_i - y_j = u^{k_{ij}} \operatorname{unit}(u,b) \\ z_i - z_j = O(u^{k_{ij}}) \end{cases}$

This was proven (in pameterized case) by Pichon-Neumann using the key lemma of Briançon-Henry.

XII Key Lemma (Briançon - Henry)

Under the assumption $b \to \Delta_b$ equisingular

$$b \frac{\partial z_i}{\partial b} = \frac{\partial y_i}{\partial b}$$

proof

$f(u^n, y(u,b), z(u,b)) = f_z' + b f_y' \equiv 0$

$\rightsquigarrow \quad 0 = \frac{\partial y}{\partial b} \cdot f_y' + \frac{\partial z}{\partial b} \cdot f_z' = f_y' \left(\frac{\partial y}{\partial b} - b \frac{\partial z}{\partial b} \right) \qquad \square$

Then if there is $b(u)$ s.t. $\frac{\partial z}{\partial b}(u,b) \equiv 0$ then $\frac{\partial y}{\partial b} \equiv 0$ and on $u \to (u^n, y_i(u^n, b(u)), z_i(u^n, b(u)))$

$0 \equiv \frac{\partial}{\partial b}(f_z' + b f_y') = f_{zy}'' \frac{\partial y}{\partial b} + f_{zz}'' \frac{\partial z}{\partial b} + b\left(f_{yy}'' \frac{\partial y}{\partial b} + f_{yz}'' \frac{\partial z}{\partial b}\right) + f_y' \equiv f_y'$

impossible. Hence $\left(\frac{\partial z_i}{\partial b} = 0\right) \subset \{u = 0\} \Rightarrow z_i(u,b) = z_i(u,0) + u^{m_i} b \operatorname{unit}$

XIII Polar wedges in families of surfaces

We suppose $X_t = f_t^{-1}(0)$ generically ZE. Then, in a generic coord. system $(b,t) \rightsquigarrow \Delta(x,y,b,t)$ is ZE family of curves and the polar wedges can be parameterized by

$$p(u,b) = \begin{cases} x = u^n \\ y = y_i(u,0) & u^{m_i} b^2 \operatorname{unit}(u,b) \\ z = z_i(u,0) & u^{m_i} b \operatorname{unit}(u,b) \end{cases}$$

For two PW_i, PW_j: $\begin{cases} y_i - y_j = u^{k_{ij}} \operatorname{unit}(u,b) \\ z_i - z_j = O(u^{k_{ij}}) \end{cases}$

This can be used to extend Lipschitz vector fields.

XIV What is Lipschitz stratification?

Let $X = \bigcup S_i$ be a stratification.

- Lipschitz stratification is given by conditions on angles between tangent spaces to different strata.
- Equivalently, it can be expressed by the following property of extension of Lipschitz vector fields tangent to strata:

$$X = X^k \supset X^{k-1} \supset \dots \supset X^l \neq \emptyset$$

defines Lipschitz stratification by taking connected components of $X^i \setminus X^{i-1}$ as i-dimensional strata if and only if $\forall i, W$

$$X^{i-1} \subset W \subset X^i$$

a Lipschitz v.f on W can be extended to a Lipschitz v.f. on X^i.

XV Why $\frac{\partial}{\partial t}$ can be extended from T to a Lipschitz v.f. on X.

$$(u, b, t) \longrightarrow \left(0, \frac{\partial y_i}{\partial t}, \frac{\partial z_i}{\partial t}, 1\right)$$

Then we bound $\frac{\partial^2 y_i}{\partial t \partial x}$, $\frac{\partial^2 y_i}{\partial t \partial b} \cdot u^{-m_i}$ by a constant

$$\left| x \frac{\partial^2 y_i}{\partial t \partial x} \right| \sim \left| u \frac{\partial^2 y_i}{\partial t \partial u} \right| \leq C_1 |u^n| = C_1 |x|$$

$$\left| u^{-m_i} \frac{\partial y_i}{\partial t \partial b} \right| \leq \text{const}$$, (and $u^{-m_i} \frac{\partial}{\partial b}$ is like $\frac{\partial}{\partial z}$)

similarly for $\left| \frac{\partial z_i}{\partial t \partial x} \right|$, $\left| \frac{\partial z_i}{\partial t \partial b} \cdot u^{-m_i} \right|$

For two polar wedges (u^n, y_i, z_i), (u^n, y_j, z_j)

$$\left| \frac{\partial (y_i - y_j)}{\partial t} \right| \leq C_1 |y_i - y_j|$$

and similar for $z_i - z_j$. This shows that $\frac{\partial}{\partial t}$ can be extended from T to a Lipschitz v.f on the union of polar wedges.