Zeta-function and μ^*-Zariski pairs of surfaces

Christophe Eyral

joint work with Mutsuo Oka

Institute of Mathematics
Polish Academy of Sciences

Consider two polynomial functions

$$g_0 \colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0) \quad \text{and} \quad g_1 \colon (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$$

such that the corresponding surfaces

$$V(g_0) := \{g_0 = 0\} \quad \text{and} \quad V(g_1) := \{g_1 = 0\}$$

in $\mathbb{C}^3$ have an isolated singularity at 0

Theorem (Lê-Teissier)

$(\mathbb{C}^3, V(g_0)) \overset{\text{homeo}}{\simeq} (\mathbb{C}^3, V(g_1))$ near 0
(equivalently, $(\mathbb{S}^5_\varepsilon, K_{g_0}) \overset{\text{diffeo}}{\simeq} (\mathbb{S}^5_\varepsilon, K_{g_1})$ for ε small) $\Big| \Rightarrow \mu(g_0) = \mu(g_1)$

The converse is not true; however, in practice, given g_0 and g_1 with

$$\mu(g_0) = \mu(g_1) \quad \text{or even with} \quad \mu^*(g_0) = \mu^*(g_1),$$

it is difficult to determine whether $(\mathbb{S}^5_\varepsilon, K_{g_0})$ and $(\mathbb{S}^5_\varepsilon, K_{g_1})$ are diffeomorphic or not

☞ Today I will present a class of pairs of surface singularities with the same μ^*-invariant and which "are likely to" produce non-diffeomorphic embedded links

Zariski pair of projective curves

We say that a pair of projective curves C_0 and C_1 in $\mathbb{P}^2$ is a Zariski pair if there exist regular neighbourhoods N_0 and N_1 of C_0 and C_1 such that

$$(N_0, C_0) \overset{\text{homeo}}{\simeq} (N_1, C_1) \quad \text{while} \quad (\mathbb{P}^2, C_0) \overset{\text{homeo}}{\not\simeq} (\mathbb{P}^2, C_1)$$

μ^*-Zariski pair of surfaces

We start with a Zariski pair of projective curves C_0 and C_1 defined by reduced homogeneous polynomials $f_0(z_1, z_2, z_3)$ and $f_1(z_1, z_2, z_3)$ of degree d, and look at the affine surfaces in $\mathbb{C}^3$ defined by the polynomials

$$g_0 := f_0 + z_1^{d+m} \quad \text{and} \quad g_1 := f_1 + z_1^{d+m} \quad (m \geq 1)$$

We say that $(V(g_0), V(g_1))$, or simply (g_0, g_1), is a Zariski pair of surfaces if g_0 and g_1 have an isolated singularity at 0 and the same monodromy zeta-function; if, in addition, g_0 and g_1 have the same μ^*-invariant but lie in different path-connected components of the μ^*-constant stratum, then we say that (g_0, g_1) is a μ^*-Zariski pair of surfaces.

☞ Being a μ^*-Zariski pair of surfaces does not imply $(\mathbb{S}^5_\varepsilon, K_{g_0}) \overset{\text{diffeo}}{\not\simeq} (\mathbb{S}^5_\varepsilon, K_{g_1})$, but it is a necessary condition for that.

Again, consider a Zariski pair of projective curves C_0 and C_1 of degree d defined by reduced homogeneous polynomials

$$f_0(z_1, z_2, z_3) \quad \text{and} \quad f_1(z_1, z_2, z_3)$$

By a linear change of coordinates, we may assume that:

1. the singularities of the curves C_0 and C_1 are not on $z_1 z_2 z_3 = 0$
2. f_0 and f_1 are convenient and Newton non-degenerate on any face of the Newton diagram with non-maximal dimension

As above, let

$$g_0 := f_0 + z_1^{d+m} \quad \text{and} \quad g_1 := f_1 + z_1^{d+m}$$

Theorem (Oka) If the singularities of C_0 and C_1 are Newton non-degenerate in some suitable local coordinates, then $K_{g_0} \overset{\text{diffeo}}{\simeq} K_{g_1}$

☞ We expect that $(\mathbb{S}^5_\varepsilon, K_{g_0}) \overset{\text{diffeo}}{\not\simeq} (\mathbb{S}^5_\varepsilon, K_{g_1})$

Theorem (Oka and E.) Under the same assumptions, (g_0, g_1) is a μ^*-Zariski pair of surfaces

Sketch of the proof

1. (g_0, g_1) is a Zariski pair of surfaces

g_0, g_1 Newton non-degenerate $\overset{\text{Varchenko}}{\Rightarrow}$ $\zeta_{g_0,0}(t) = \zeta_{g_1,0}(t)$

In our situation, g_0, g_1 are not Newton non-degenerate. However they are almost Newton non-degenerate, i.e.,

- convenient;
- Newton non-degenerate on faces of non-maximal dimension;
- with a finite number of 1-dimensional critical loci on the face of maximal dimension;
- after blowing up, the singularities of their strict transforms in the exceptional divisor are Newton non-degenerate;

and in this case we can apply **Oka's formula**:

$$\zeta_{g,0}(t) = \underbrace{\zeta_{g_s,0}(t)}_{\text{with } s \neq 0} \times (1 - t^d)^{\mu^{\text{tot}}(C)} \times \prod_{\mathrm{p} \in \Sigma(C)} \zeta_{\pi^* g, \mathrm{p}}(t)$$

(by g we mean either g_0 or g_1; similarly for C)

Explanations of Oka's formula

$$\zeta_{g,0}(t) = \underbrace{\zeta_{g_s,0}(t)}_{\text{with } s \neq 0} \times (1-t^d)^{\mu^{\text{tot}}(C)} \times \prod_{\mathrm{p}\in\Sigma(C)} \zeta_{\pi^* g,\mathrm{p}}(t)$$

- Consider blowing-up $\pi\colon X \to \mathbb{C}^3$ at 0. Over $\mathbb{C}^3 \setminus V(g)$, π is a biholomorphism, so the Milnor fibration $g\colon B_\varepsilon(0) \cap g^{-1}(D \setminus \{0\}) \to D \setminus \{0\}$ can be "lifted" to X, so that

$$\pi^* g\colon \pi^{-1}(B_\varepsilon(0) \cap g^{-1}(D_\delta \setminus \{0\})) \to D_\delta \setminus \{0\}$$

is also a locally trivial fibration isomorphic to the Milnor fibration of g at 0

- Take the standard affine chart $U_1 := \mathbb{P}^2 \setminus \{Z_1 = 0\}$ of $\mathbb{P}^2$ with coordinates $(Z_2/Z_1, Z_3/Z_1)$, and in the corresponding chart $X \cap (\mathbb{C}^3 \times U_1)$ of X with coordinates $(z_1, Z_2/Z_1, Z_3/Z_1) =: (y_1, y_2, y_3)$,

$$\pi^* g = y_1^d (f(1, y_2, y_3) + y_1^m) \qquad \text{(again, } f \text{ is either } f_0 \text{ or } f_1\text{)}$$

- Decompose the lifted Milnor fibration $\pi^* g$ into the local Milnor fibrations at the singular points of C and the fibration on their complement

- The Oka formula says that $\zeta_{g,0}(t)$ is the product of the zeta-functions of these local Milnor fibrations and the fibration on the complement

Coming back to the proof

It suffices to show that the zeta-functions of the local Milnor fibrations at the singular points of C_0 and C_1 coincide

$\exists$ local coordinates $x \equiv (x_1, x_2, x_3)$ near $p_0 \in \Sigma(C_0)$ and $u \equiv (u_1, u_2, u_3)$ near $p_1 \in \Sigma(C_1)$ such that

$$\pi^* g_0 = x_1^d (h_0(x_2, x_3) + x_1^m) \quad \text{and} \quad \pi^* g_1 = u_1^d (h_1(u_2, u_3) + u_1^m)$$

and h_0 and h_1 are Newton non-degenerate; moreover, if $(C_0, p_0) \sim (C_1, p_1)$, we may assume that $\Gamma(h_0) = \Gamma(h_1)$; so $\pi^* g_0$ and $\pi^* g_1$ are Newton non-degenerate with the same Newton diagram, and by Varchenko's theorem,

$$\zeta_{\pi^* g_0, \mathrm{p}_0}(t) = \zeta_{\pi^* g_1, \mathrm{p}_1}(t)$$

2. g_0 and g_1 have the same μ^*-invariant; it is given by

$$(\underbrace{(d-1)^3 + m\mu^{\text{tot}}(C)}_{\text{Milnor number}}, \underbrace{(d-1)^2}_{\text{Milnor number of generic plane section}}, \underbrace{d-1}_{\text{multiplicity } -1})$$

3. g_0 and g_1 are in different path-connected components of μ^*-const stratum

We argue by contradiction. Assume they are in the same component.

Step 1 There exists a μ^*-constant piecewise complex-analytic family $\{g_s\}_{0\leq s\leq 1}$ connecting g_0 and g_1

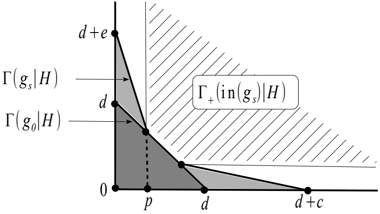
In particular, $\mathrm{mult}_0(g_s)$ is constant, and $\mathrm{in}(g_s)$ has degree d; moreover, $\mathrm{in}(g_s)$ is reduced, and so the corresponding curve C_s has only isolated singularities

- Suppose $\mathrm{in}(g_s)$ is not reduced, and take generic plane H and coordinates (x,y) for H such that

$$\mathrm{in}(g_s)|_H = \ell_1^{p_1}(x,y)\cdots\ell_q^{p_q}(x,y) = x^{p_1}\ell_2^{p_2}(x,y)\cdots\ell_q^{p_q}(x,y) \quad \text{with} \quad p_1 \geq 2$$

Since $p_1 \geq 2$, we have

$$\underbrace{\nu(g_s|_H)}_{\substack{2\,(\text{area dark+light gray})\\ -(d+c)-(d+e)+1}} > \underbrace{\nu(g_0|_H)}_{\substack{2\,(\text{area dark gray})\\ -2d+1}} = (d-1)^2$$



It follows that

$$\mu^{(2)}(g_s) := \mu(g_s|_H) \geq \nu(g_s|_H) > \nu(g_0|_H) = (d-1)^2 = \mu^{(2)}(g_0) \text{ – a contradiction}$$

Step 2 $\mu^{\text{tot}}(C_s)$ is independent of s

Indeed, by A'Campo formula, $\zeta_{g_s,0}(t)$ is uniquely written as

$$\zeta_{g_s,0}(t) = \prod_{i=1}^{\ell}(1-t^{d_i})^{\nu_i}$$

where $d_1, \ldots, d_\ell$ are mutually disjoint.

- $\min\{d_1, \ldots, d_\ell\}$ is called the zeta-multiplicity
- the factor $(1-t^{d_i})^{\nu_i}$ corresponding to d_i = zeta-multiplicity is called the zeta-multiplicity factor

1. $\zeta_{g_s,0}(t)$ is independent of s (Teissier); in particular, the zeta-multiplicity and the zeta-multiplicity factor are independent of s
2. The zeta-multiplicity is d and the zeta-multiplicity factor is

$$(1-t^d)^{-d^2+3d-3+\mu^{\text{tot}}(C_s)}$$

Step 3 We conclude thanks to two theorems of Lê

Theorem 1 (Lê) Assume that at $s = s_0$, the family $\{g_s\}$ has a bifurcation of singularities in a small ball B centred at a singular point p_0 of C_{s_0}. Then for $s \neq s_0$ near s_0,

$$\sum_{p \in B \cap \Sigma(C_s)} \mu(C_s, p) < \mu(C_{s_0}, p_0)$$

☞ So, if such an s_0 exists, then $\mu^{\text{tot}}(C_s) < \mu^{\text{tot}}(C_{s_0})$ – a contradiction

Theorem 2 (Lê)

No bifurcation of singularities $\Rightarrow$ topological type of $(\mathbb{P}^2, C_s)$ independent of s

☞ In particular $(\mathbb{P}^2, C_0) \overset{\text{homeo}}{\simeq} (\mathbb{P}^2, C_1)$, and therefore (C_0, C_1) is not a Zariski pair of curves – a contradiction

Thank you for your attention!