On the Fukui-Kurdyka-Paunescu Conjecture

A. Fernandes, Z. Jelonek, E. Sampaio

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1. Zariski's famous Multiplicity Conjecture, stated by Zariski in 1971, is formulated as follows:

   Zariski's Multiplicity Conjecture *Let $f, g\colon (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be two reduced complex analytic functions. If there is a homeomorphism $\varphi\colon (\mathbb{C}^n, V(f), 0) \to (\mathbb{C}^n, V(g), 0)$, then $m(V(f), 0) = m(V(g), 0)$.*

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In the real case, of course, Zariski's Multiplicity Conjecture does not hold in the same form as in the complex case. However, we have the following conjecture, stated by Fukui, Kurdyka and Paunescu in 2004:

Fukui-Kurdyka-Paunescu's Conjecture *Let $X, Y \subset \mathbb{R}^n$ be two germs at the origin of irreducible real analytic subsets. If $h\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ is the germ of a subanalytic, arc-analytic and bi-Lipschitz homeomorphism such that $h(X) = Y$, then $m(X, 0) \equiv m(Y, 0) \bmod 2$.*

Let us recal that in the real case $m(X, 0) = m(X_{\mathbb{C}}, 0)$ where $X_{\mathbb{C}}$ is a complexification of X. Similarly we define the real degree of real algebraic set.

1. Several authors approached this conjecture: For example, J.-J. Risler proved that multiplicity $\mod 2$ of a real analytic curve is invariant under bi-Lipschitz homeomorphisms; T. Fukui, K. Kurdyka and L. Paunescu also confirmed the conjecture in the case that X and Y are real analytic curves.

2. G. Valette in 2010 showed that multiplicity $\mod 2$ of real analytic hypersurfaces is invariant under arc-analytic bi-Lipschitz homeomorphisms. Sampaio proved in the real version of Gau-Lipman's theorem: i.e., multiplicity $\mod 2$ of real analytic sets is invariant under homeomorphisms $\varphi\colon (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ such that φ and φ^{-1} have a derivative at the origin.

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Coming back to the complex case, let us list some contributions to Zariski's Multiplicity Conjecture from the Lipschitz point of view. For instance, Neumann and Pichon, with previous contributions of Pham and Teissier and Fernandes, proved that the bi-Lipschitz geometry of plane curves determines the Puiseux pairs, and as a consequence if two germs of complex analytic curves with any codimension are bi-Lipschitz homeomorphic (with respect to the outer metric), then they have the same multiplicity.

1. Comte in 1998 proved that multiplicity of complex analytic germs (not necessarily codimension 1 sets) is invariant under bi-Lipschitz homeomorphisms with the severe assumption that the Lipschitz constants are close enough to 1. This motivated the following conjecture (Bobadila, Fernandes, Sampaio):

2. **Conjecture 1** *Let* $X \subset \mathbb{C}^n$ *and* $Y \subset \mathbb{C}^m$ *be two complex analytic sets with* $\dim X = \dim Y = d$. *If their germs at zero are bi-Lipschitz homeomorphic, then their multiplicities* $m(X, 0)$ *and* $m(Y, 0)$ *are equal.*

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2. **Conjecture 2** *Let* $X \subset \mathbb{C}^n$ *and* $Y \subset \mathbb{C}^m$ *be two complex algebraic sets with* $\dim X = \dim Y = d$*. If* X *and* Y *are bi-Lipschitz homeomorphic at infinity, then* $\deg(X) = \deg(Y)$*.*

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1. Conjectures 1 and 2 are equivalent and, moreover, have positive answers for $d = 1$ and $d = 2$.
2. However, Birbrair, Rernandes, Sampaio and Verbitsky disproved these conjectures when $d \geq 3$, by showing explicit counter-examples. More precisely, it was shown that we have two different embeddings of $\mathbb{P}^1(\mathbb{C}) \times \mathbb{P}^1(\mathbb{C})$ into $\mathbb{P}^5(\mathbb{C})$, say X and Y, such that their affine cones $\text{Cone}(X), \text{Cone}(Y) \subset \mathbb{C}^6$ are bi-Lipschitz equivalent, but they have different degrees.

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2. In this paper, we prove some generalizations of the results proved by Jelonek. For instance, we show that the multiplicity of complex analytic sets is invariant under semi-bi-Lipschitz homeomorphisms which have analytic graph and the degree of complex algebraic sets is invariant under semi-bi-Lipschitz homeomorphisms at infinity which have algebraic graph.

3. We also prove that degree of a complex algebraic set is invariant under semialgebraic semi-bi-Lipschitz homeomorphisms at infinity such that the closure of their graphs are orientable homological cycles.

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1. **Definition** Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be two sets and let $h\colon X \to Y$.
2. We say that h is **Lipschitz** if there exists a positive constant C such that

$$\|h(x) - h(y)\| \leq C\|x - y\|, \quad \forall x, y \in X.$$

3. We say that h is **bi-Lipschitz** if h is a homeomorphism, it is Lipschitz and its inverse is also Lipschitz.
4. We say that h is **bi-Lipschitz at infinity** (resp. **a homeomorphism at infinity**) if there exist compact subsets $K \subset \mathbb{R}^n$ and $K' \subset \mathbb{R}^m$ such that $h|_{X \setminus K}\colon X \setminus K \to Y \setminus K'$ is bi-Lipschitz (resp. a homeomorphism).

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1. Now we give a geometric characterization of semi-bi-Lipschitz mappings.

2. **Definition** *Let L^s, H^{n-s-1} be two disjoint linear subspaces of $\mathbb{P}^n(\mathbb{C})$. Let π_∞ be a hyperplane (a hyperplane at infinity) and assume that $L^s \subset \pi_\infty$. The projection π_L with center L^s is the mapping*

$$\pi_L : \mathbb{C}^n = \mathbb{P}^n(\mathbb{C}) \setminus \pi_\infty \ni x \mapsto \langle L^s, x\rangle \cap H^{n-s-1} \in H^{n-s-1} \setminus \pi_\infty = \mathbb{C}^{n-s-1}$$

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Here $\langle L, x \rangle$ we mean the linear projective subspace spanned by L and $\{x\}$.

1. **Lemma 1** *Let* X *be a closed subset of* $\mathbb{C}^n$. *Denote by* $\Lambda_0 \subset \pi_\infty$ *the set of directions of all secants of* X *which contain* x_0 *and let* $\Sigma_0 = \overline{\Lambda_0}$, *where* π_∞ *is the hyperplane at infinity and we consider the euclidean closure. Let* $\pi_L : \mathbb{C}^n \to \mathbb{C}^l$ *be the projection with center* L. *Then* $\pi_L|_X$ *is semi-bi-Lipschitz at* x_0 *if and only if* $L \cap \Sigma_0 = \emptyset$.

2. **Lemma 2** *Let* $X \subset \mathbb{C}^n$ *be a closed set and let* $f : X \to \mathbb{C}^m$ *be a semi-Lipschitz homeomorphism. Let* $Y := \mathrm{graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^m$. *Then the mapping* $\phi : X \ni x \mapsto (x, f(x)) \in Y$ *is a semi-bi-Lipschitz homeomorphism.*

3. **Remark** It is easy to note that Lemmas 1 and 2 hold in the real case also.

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2. **Lemma 2** *Let $X \subset \mathbb{C}^n$ be a closed set and let $f : X \to \mathbb{C}^m$ be a semi-Lipschitz homeomorphism. Let $Y := \text{graph}(f) \subset \mathbb{C}^n \times \mathbb{C}^m$. Then the mapping $\phi : X \ni x \mapsto (x, f(x)) \in Y$ is a semi-bi-Lipschitz homeomorphism.*

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Definition

Let $A \subset \mathbb{R}^n$ be a subset. We say that $v \in \mathbb{R}^n$ is a **tangent vector to** A **at** $p \in \overline{A}$ *(resp.* **at infinity***) if there is a sequence of points $\{x_i\}_{i\in\mathbb{N}} \subset A$ such that $\lim_{i\to\infty} \|x_i - p\| = 0$ (resp. $\lim_{i\to\infty} \|x_i\| = +\infty$) and there is a sequence of positive numbers $\{t_i\}_{i\in\mathbb{N}} \subset \mathbb{R}^+$ such that*

$$\lim_{i\to\infty} \frac{1}{t_i}(x_i - p) = v \quad \textit{(resp. } \lim_{i\to\infty} \frac{1}{t_i}x_i = v).$$

Let $C(A,p)$ (resp. $C_\infty(A)$) denote the set of all tangent vectors to A at p (resp. at infinity). The subset $C(A,p)$ (resp. $C_\infty(A)$) is called **the tangent cone of** A **at** p *(resp.* **at infinity***).*

Definition

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be subanalytic sets with $0 \in X$ and $0 \in Y$ and let $h\colon (X,0) \to (Y,0)$ be a subanalytic Lipschitz mapping. We define the **pseudo-derivative of h at 0**, $d_0 h\colon C(X,0) \to C(Y,0)$, *by* $d_0 h(v) = \lim\limits_{t \to 0^+} \frac{h(\gamma(t))}{t}$, *where* $\gamma\colon [0,+\varepsilon) \to X$ *satisfies* $\lim\limits_{t \to 0^+} \frac{\gamma(t)}{t} = v$.

Definition

Let $X \subset \mathbb{R}^n$ and $Y \subset \mathbb{R}^m$ be semialgebraic sets and let $h\colon X \to Y$ be a semialgebraic Lipschitz mapping. We define the **pseudo-derivative of h at infinity** $d_\infty h\colon C(X,\infty) \to C(Y,\infty)$ *by* $d_\infty h(v) = \lim_{t\to+\infty} \frac{h(\gamma(t))}{t}$*, where* $\gamma\colon (r,+\infty) \to X$ *satisfies* $\lim_{t\to+\infty} \frac{\gamma(t)}{t} = v$.

1. **Homological cycles.**

2. Let M be a smooth compact manifold of (real) dimension n. Given homology classes $\alpha \in H_k(M)$ and $\beta \in H_{n-k}(M)$, we choose representative cycles $\tilde{\alpha}$ and $\tilde{\beta}$, respectively.

3. We can assume that every singular simplex appearing in each of these cycles is a smooth mapping and also that any two simplices meet transversally. This means that the only points of intersection are where the interior of a k-simplex in $\tilde{\alpha}$ meets the interior of an $(n-k)$-simplex in $\tilde{\beta}$.

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1. At every such point x of intersection both $\tilde{\alpha}$ and $\tilde{\beta}$ are local embeddings and their tangent spaces are complementary in T_xM. We assign a sign to each point of intersection by comparing the direct sum of the orientations of the tangent spaces of $\tilde{\alpha}$ and of $\tilde{\beta}$ with the ambient orientation of the tangent space of M. The sum of the signs over the (finitely many) points of intersection gives the intersection pairing applied to (α, β).
2. If $M = \mathbb{P}^n(\mathbb{C})$, then $H_{2i}(M, \mathbb{Z}) = \mathbb{Z}$ for $i = 0, 1, \ldots, n$ and $H_{2i-1}(M, \mathbb{Z}) = 0$. The space $H_{2i}(M, \mathbb{Z})$ is generated by the class L^{n-i} where L is a hyperplane, and we consider it as an algebraic cycle. Hence every $2i$-dimensional homological cycle α can be described as dL^{n-i}. We say that the number $|d|$ is **the topological degree of** α. Note that if $X \subset M$ is an i-dimensional projective subvariety, then the algebraic degree of X coincides with the topological degree.

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1. Similarly, if $M = \mathbb{P}^n(\mathbb{R})$, then $H_i(M, \mathbb{Z}/(2)) = \mathbb{Z}/(2)$ for $i = 0, 1, \dots, n$. The space $H_i(M, \mathbb{Z}/(2))$ is generated by the class L^{n-i} where L is a hyperplane and we consider it as an algebraic cycle. Hence every i-dimensional homological cycle α can be described as dL^{n-i}. We say that the number d is **the topological degree mod 2 of α**. Note that if $X \subset M$ is an i-dimensional projective subvariety, then the algebraic degree mod 2 coincides with the topological degree.
2. Let $R = \mathbb{Z}$ or $R = \mathbb{Z}/(2)$. Let X be a compact semi-algebraic set of dimension d. We say that X is a homological cycle over R, if there exists a stratification $\mathcal{S}$ of X such that it gives on X a structure of a R-homological d-cycle α. We say that this cycle is orientable if $R = \mathbb{Z}$ and $[\alpha] \neq 0$ in $H_d(X, \mathbb{Z})$. It is well known that if $X \subset \mathbb{P}^n(\mathbb{C})$ is an irreducible algebraic variety, then it is an orientable homological cycle.

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2. Let $R = \mathbb{Z}$ or $R = \mathbb{Z}/(2)$. Let X be a compact semi-algebraic set of dimension d. We say that X is a homological cycle over R, if there exists a stratification $\mathcal{S}$ of X such that it gives on X a structure of a R-homological d-cycle α. We say that this cycle is orientable if $R = \mathbb{Z}$ and $[\alpha] \neq 0$ in $H_d(X, \mathbb{Z})$. It is well known that if $X \subset \mathbb{P}^n(\mathbb{C})$ is an irreducible algebraic variety, then it is an orientable homological cycle.

1. **Theorem 1** *Let* $X \subset \mathbb{C}^n, Y \subset \mathbb{C}^m$ *be complex algebraic varieties of dimension* d *and let* $h\colon X \to Y$ *be a semialgebraic and semi-bi-Lipschitz homeomorphism. Assume that the closure of* $Graph(h)$ *in* $\mathbb{P}^{n+m}(\mathbb{C})$ *is an orientable homological cycle. Then* $\deg(X) = \deg(Y)$.

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1. Theorem 2 holds if X, Y are subanalytic sets with subanalytic projective closures, which are homological cycles, e.g, for subanalytic cones with Euler links.

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1. **Definition**(Parusiński) We say that $E \subset \mathbb{P}^N(\mathbb{R})$ is **arc-symmetric** if for any analytic arc $\gamma\colon (-1,1) \to \mathbb{P}^N(\mathbb{R})$ such that $\gamma((-1,0)) \subset E$, we have $\gamma((0,\epsilon)) \subset E$, for some $\epsilon > 0$.

2. **Theorem** Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ be real algebraic d-dimensional sets and let $h\colon A \to B$ be a semialgebraic and semi-bi-Lipschitz homeomorphism. If the graph of h is arc-symmetric, then $\deg(A) \equiv \deg(B) \bmod 2$.

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2. The global version of Fukui-Kurdyka-Paunescu's Conjecture Let $A \subset \mathbb{R}^n, B \subset \mathbb{R}^m$ be real algebraic $d-$dimensional sets, let $\overline{A}$ be the closure of A in $\mathbb{P}^n(\mathbb{R})$ and let $h\colon A \to B$ be a semialgebraic and semi-bi-Lipschitz homeomorphism. Assume that h is arc-analytic at $\overline{A}$. Then $\deg(A) \equiv \deg(B) \bmod 2$.

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1. **Definition** The mapping $\beta_n : \mathbb{S}^{n-1} \times \mathbb{R}^+ \to \mathbb{R}^n$ given by $\beta_n(x, r) = rx$ is called the **spherical blowing-up** (at the origin) of $\mathbb{R}^n$.
2. Note that $\beta_n : \mathbb{S}^{n-1} \times (0, +\infty) \to \mathbb{R}^n \setminus \{0\}$ is a homeomorphism with inverse $\beta_n^{-1} : \mathbb{R}^n \setminus \{0\} \to \mathbb{S}^{n-1} \times (0, +\infty)$ given by $\beta_n^{-1}(x) = (\frac{x}{\|x\|}, \|x\|)$.
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a) the connected components of $(X' \cap U) \setminus \partial X'$, say $X_1, \ldots, X_r$, are C^1 manifolds with $\dim X_i = \dim X$, $i = 1, \ldots, r$;

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2. **Remark.** It is clear that the function k_X is locally constant. In fact, k_X is constant in each connected component C_j of $\mathrm{Smp}(\partial X')$. Then, we define $k_X(C_j) := k_X(x)$ with $x \in C_j$.

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2. **Definition.** *Let* $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^d$ *and* $C \subset A$ *be subanalytic sets and* $\pi : A \to B$ *be a continuous mapping. If* $\#(\pi^{-1}(x) \cap C)$ *is constant* $\bmod 2$ *for a generic* $x \in B$, *we define the* **degree of** C **with respect to** π *to be* $\deg_\pi(C) := \#(\pi^{-1}(x) \cap C) \bmod 2$, *for a generic* $x \in B$.

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2. **Proposition Valette-Sampaio.** *Let $X \subset \mathbb{R}^n$ be a d-dimensional real analytic set with $0 \in X$ and $\pi : \mathbb{C}^n \to \mathbb{C}^d$ be a projection such that $\pi^{-1}(0) \cap C(X_{\mathbb{C}}, 0) = \{0\}$. Let $\pi' : \mathbb{S}^{n-1} \setminus \pi^{-1}(0) \to \mathbb{S}^{d-1}$ be the mapping given by $\pi'(u) = \frac{\pi(u)}{\|\pi(u)\|}$. Then $\deg_{\pi'}(C'_X)$ is defined and satisfies $\deg_{\pi'}(C'_X) \equiv m(X, 0) \mod 2$.*

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1. **Definition** An $(n-1)$-dimensional subanalytic set C is said to be an **Euler cycle** if it is a closed set and if, for a stratification of C (and hence for any that refines it), the number of $(n-1)$-dimensional strata containing a given $(n-2)$-dimensional stratum in their closure is even.

2. We say that a set $C \subset \mathbb{R}^n$ is a**-invariant** if it is preserved by the antipodal mapping (i.e. $a(C) = C$, with $a(x) = -x$).

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1. **Proof.**
2. By previous Sampaio results we can assume that $C'_X \neq \emptyset$.
3. The degree of C'_X with respect to π', $\deg_{\pi'}(C'_X)$, is well defined and $\deg_{\pi'}(C'_X) \equiv m(X,0) \bmod 2$, where $\pi = p|_{\mathbb{R}^n} : \mathbb{R}^n \to \mathbb{R}^d$ and $\pi' : \mathbb{S}^{n-1} \setminus \pi^{-1}(0) \to \mathbb{S}^{d-1}$ is given by $\pi'(u) = \frac{\pi(u)}{\|\pi(u)\|}$.
4. C'_X is a-invariant and it is an Euler cycle.
5. If $C'(X,0)$ is the cone over C'_X, then the topological degree of $\overline{C'(X,0)}$ is well defined and

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1. **Theorem** Let $(X,0) \subset (\mathbb{C}^n,0), (Y,0) \subset (\mathbb{C}^m,0)$ be germs of complex analytic sets and let $h\colon (X,0) \to (Y,0)$ be a germ of homeomorphism which is also semi-bi-Lipschitz at 0. Assume that the graph of h is a complex analytic set. Then $m(X,0) = m(Y,0)$.

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1. We cannot expect invariance of multiplicity without mod 2 in Fukui-Kurdyka-Paunescu's Conjecture:
2. **Example.** Consider $X = \{(x, y, z) \in \mathbb{R}^3; z(x^2 + y^2) = y^3\}$ and $Y = \{(x, y, z) \in \mathbb{R}^3; z(x^4 + y^4) = y^5\}$. Let $h\colon (\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ be the mapping given by

$$h(x,y,z) = \begin{cases} \left(x, y, z - \frac{y^3}{x^2+y^2} + \frac{y^5}{x^4+y^4}\right) & \text{if } x^2 + y^2 \neq 0 \\ (0, 0, z) & \text{if } x^2 + y^2 = 0. \end{cases}$$

   Then X and Y are irreducible real analytic sets such that $m(X, 0) = 3$ and $m(Y, 0) = 5$. Moreover, h is a semialgebraic arc-analytic bi-Lipschitz homeomorphism such that $h(X) = Y$.

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THANK YOU FOR ATTENTION!